Problem set 9

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Problem 20: Misalignment mechanism

Consider a complex scalar field Φ with Lagrangian

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - \lambda(\Phi^\dagger \Phi - f_a^2)^2 \, , \tag{1}$$

where λ and f_a are parameters of the model. By introducing two real scalar fields φ and α, we can write $\Phi = e^{i\alpha}(\varphi + f_a)$.

a) Show that the resulting Lagrangian is invariant under the field shift $\alpha \to \alpha + \epsilon$. In particular, there is no mass term for α (i.e. no term proportional to α^2) and hence the field remains massless (a so-called Goldstone Boson).

b) The QCD phase transition introduces additional terms in the Lagrangian that depend on $\Phi + \Phi^\dagger$. Show that such terms are no longer invariant under the continuous shift symmetry, but only under a discrete shift symmetry $\alpha \to \alpha + 2\pi$.

In particular, the QCD phase transition introduces an additional term of the form

$$V(\Phi, \Phi^\dagger) \to V(\Phi, \Phi^\dagger) + \frac{m_\pi^2 f_\pi^2}{4} \left[1 - \cos(\alpha - \theta)\right] \tag{2}$$

such that the potential is minimised for $\alpha = \theta$.

c) By writing $\alpha = \theta + a/f_a$, expand the new term in the potential around the minimum $a \approx 0$. Show that the field a now has a mass term of the form $m_a^2 a^2/2$ and determine m_a. Since m_a is small but non-zero, the field a is called a Pseudo-Goldstone Boson.

d) It can be shown that the cosmological abundance of axions is proportional to the square of the initial misalignment angle $\Delta\theta_0 = \alpha_0 - \theta$. Assuming that $\Delta\theta_0$ takes random values in the range $[-\pi, \pi]$ in different patches of the observable Universe, show that the average axion abundance from the misalignment mechanism can be estimated by replacing $\Delta\theta_0 \to \pi/\sqrt{3}$.

Problem 21: Coherent oscillations in quantum mechanics

Let us consider the quantum harmonic oscillator

$$H = a^\dagger a + \frac{1}{2} \,, \tag{3}$$

where in position space (and setting $\hbar = 1$)

$$a = \frac{1}{\sqrt{2\, m\, \omega}} \left(m\, \omega\, x + \frac{\mathrm{d}}{\mathrm{d}x} \right) \,. \tag{4}$$

Again in position space, the ground state is given by

$$|0\rangle = C_0 \exp\left(-\frac{m\, \omega\, x^2}{2} \right) \,, \tag{5}$$

from which higher states are obtained via $|n\rangle = \frac{1}{\sqrt{n}} a^\dagger |n-1\rangle$.

Let us now consider a state that is obtained from the ground state by applying a displacement x_0

$$|x_0\rangle = C_0 \exp\left(-\frac{m\, \omega (x - x_0)^2}{2} \right) \,. \tag{6}$$

a) Show that $|x_0\rangle$ is an eigenstate of the lowering operator a with eigenvalue $\sqrt{\frac{m\,\omega}{2}} x_0$ and thus the expectation value of the position operator is given by

$$\langle x_0 | x | x_0 \rangle = \frac{1}{\sqrt{2\, m\, \omega}} \langle x_0 | (a + a^\dagger) | x_0 \rangle = \mathrm{Re}\, x_0 \,. \tag{7}$$

A state with this property is called a *coherent state.*

b) Use this result to show that

$$|x_0\rangle = c_0 \sum_{n=0}^{\infty} \frac{\left(\sqrt{\frac{m\omega}{2}} x_0 \right)^n}{\sqrt{n!}} |n\rangle \,, \tag{8}$$

where c_0 is a normalisation constant.

The time evolution of the eigenstates of the harmonic oscillator is given by

$$|n(t)\rangle = e^{i\,\omega \left(n + \frac{1}{2} \right) t} |n\rangle \,. \tag{9}$$

c) Use this property to show that $|x_0(t)\rangle$ remains a coherent state with $x_0(t) = x_0 e^{i\omega t}$, such that the expectation value of the position operator is given by $\langle x_0(t) | x | x_0(t) \rangle = x_0 \cos(\omega t)$. We can therefore interpret a coherent state as a displaced ground state with oscillation frequency ω.

d) Calculate finally the expectation value of the number operator $N = a^\dagger a$ and show that the average number of excited quanta is proportional to x_0^2.