Problem set 6

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Problem 12: Inelastic dark matter

Consider a DM particle with mass m_χ and velocity v scattering off a nucleus with mass m_N, which is initially at rest in the laboratory frame.

a) Using energy and momentum conservation, show that the recoil energy E_R of the nucleus is given by

$$E_\mathrm{R} = v^2 \frac{2\mu^2}{m_N} \cos^2\theta \,, \tag{1}$$

where $\mu = m_N\, m_\chi/(m_N+m_\chi)$ is the reduced mass and θ is the angle between the velocities of the recoiling nucleus and the incoming DM particle.

b) Use this relation to obtain the standard expression for the minimum velocity needed for a given recoil energy

$$v_\mathrm{min} = \sqrt{\frac{m_N E_\mathrm{R}}{2\mu^2}} \,. \tag{2}$$

Let us now assume that DM particles cannot scatter elastically. Instead, in each scattering process, the DM particle will absorb some of its kinetic energy in order to make a transition to an excited state of slightly larger mass $m_{\chi^*} = m_\chi + \delta$ with $\delta \ll m_\chi$.

c) Repeat the calculation from above with the extra parameter δ. Show that

$$v_\mathrm{min} = \frac{1}{\sqrt{2\, m_N\, E_\mathrm{R}}} \left(\delta + \frac{E_\mathrm{R}\, m_N}{\mu} \right) \,. \tag{3}$$

d) Assume that there are no DM particles with velocity $v > v_\mathrm{max}$. Show that scattering will only be possible if $2\delta < \mu v_\mathrm{max}$.

e) Show that in this case there is a a minimum energy E_min, below which there will be will be no nuclear recoils.

f) Qualitatively sketch the recoil spectrum expected for inelastic DM.

Problem 13: Velocity integrals

Consider a DM velocity distribution of the Maxwell-Boltzmann form

$$f(\mathbf{v}) = \frac{1}{(\pi\, v_0^2)^{3/2}} \exp\left(-\frac{v^2}{v_0^2}\right) , \tag{4}$$

neglecting the effects from a finite escape velocity. The velocity distribution of DM particles in an Earth-based laboratory is then given by $f_{\text{lab}}(\mathbf{v}) = f(\mathbf{v} + \mathbf{v}_{\text{Earth}})$.

a) Show that the velocity integral is given by

$$\int_{v_{\text{min}}} \frac{f_{\text{lab}}(\mathbf{v})}{v} \mathrm{d}^3 v = \frac{1}{2 v_{\text{Earth}}} \left[\text{Erf}\left(\frac{v_{\text{min}} + v_{\text{Earth}}}{v_0}\right) - \text{Erf}\left(\frac{v_{\text{min}} - v_{\text{Earth}}}{v_0}\right) \right] , \tag{5}$$

where $v_{\text{Earth}} = |\mathbf{v}_{\text{Earth}}|$.

The speed of the Earth relative to the Galactic rest frame changes over the course of the year due to the rotation of the Earth around the sun: $v_{\text{Earth}} \approx v_{\text{Sun}} + \Delta v \cos(2\pi t/t_0 + \phi_0)$ with $t_0 = 1\,\text{yr}$ and an appropriate phase ϕ_0 such that v_{Earth} is largest in summer and smallest in winter. It is a good approximation to assume $v_{\text{Sun}} \approx v_0$ and $\Delta v \ll v_0$.

b) Making use of these approximations, determine the velocity integral as a function of time to linear order in Δv.

c) Show (e.g. by explicit evaluation) that the differential event rate for the scattering of DM particles is expected to be *smaller* in summer than in winter for small recoil energies ($v_{\text{min}} \ll v_0$) but *larger* in summer than in winter for large recoil energies ($v_{\text{min}} \gg v_0$).

Problem 14: Nuclear form factors

The electromagnetic form factor $F(q^2)$ for a nucleus with charge distribution $\rho(\mathbf{x})$ (in units of the proton charge) is given by

$$F(q^2) = \int e^{-i\mathbf{q}\cdot\mathbf{x}} \rho(\mathbf{x}) \mathrm{d}^3 x \, . \tag{6}$$

a) Show that for a spherically symmetric charge distribution $\rho(\mathbf{x}) = \rho(r)$ the form factor can be written as

$$F(q^2) = \frac{4\pi}{q} \int r \rho(r) \sin(q\, r) \mathrm{d}r \, . \tag{7}$$

Let us approximate the charge distribution of the nucleus as uniform, i.e.

$$\rho(r) = \frac{3\,Z}{4\pi R^3} \tag{8}$$

for $r < R$ and 0 otherwise.

b) Calculate the electromagnetic form factor. Convince yourself that your result has the correct behaviour for $qR \to 0$.

c) What is the corresponding form factor for the scattering of a DM particle that couples to the total mass of the nucleus, assuming that charge and mass have the same distribution?