Problem set 6

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Problem 12: Inelastic dark matter

Consider a DM particle with mass $m_\chi$ and velocity $v$ scattering off a nucleus with mass $m_N$, which is initially at rest in the laboratory frame.

a) Using energy and momentum conservation, show that the recoil energy $E_R$ of the nucleus is given by

$$E_R = v^2 \frac{2\mu^2}{m_N} \cos^2 \theta, \quad (1)$$

where $\mu = m_N m_\chi/(m_N + m_\chi)$ is the reduced mass and $\theta$ is the angle between the velocities of the recoiling nucleus and the incoming DM particle.

b) Use this relation to obtain the standard expression for the minimum velocity needed for a given recoil energy

$$v_{\text{min}} = \sqrt{\frac{m_N E_R}{2\mu^2}}. \quad (2)$$

Let us now assume that DM particles cannot scatter elastically. Instead, in each scattering process, the DM particle will absorb some of its kinetic energy in order to make a transition to an excited state of slightly larger mass $m_\chi^* = m_\chi + \delta$ with $\delta \ll m_\chi$.

c) Repeat the calculation from above with the extra parameter $\delta$. Show that

$$v_{\text{min}} = \frac{1}{\sqrt{2m_N E_R}} \left( \delta + \frac{E_R m_N}{\mu} \right). \quad (3)$$

d) Assume that there are no DM particles with velocity $v > v_{\text{max}}$. Show that scattering will only be possible if $2\delta < \mu v_{\text{max}}$.

e) Show that in this case there is a a minimum energy $E_{\text{min}}$, below which there will be will be no nuclear recoils.

f) Qualitatively sketch the recoil spectrum expected for inelastic DM.
Problem 13: Velocity integrals

Consider a DM velocity distribution of the Maxwell-Boltzmann form

\[ f(v) = \frac{1}{(\pi v_0^2)^{3/2}} \exp \left( -\frac{v^2}{v_0^2} \right), \tag{4} \]

neglecting the effects from a finite escape velocity. The velocity distribution of DM particles in an Earth-based laboratory is then given by \( f_{\text{lab}}(v) = f(v + v_{\text{Earth}}) \).

a) Show that the velocity integral is given by

\[ \int_{v_{\min}} v f_{\text{lab}}(v) \, dv = \frac{1}{2v_{\text{Earth}}} \left[ \text{Erf} \left( \frac{v_{\min} + v_{\text{Earth}}}{v_0} \right) - \text{Erf} \left( \frac{v_{\min} - v_{\text{Earth}}}{v_0} \right) \right], \tag{5} \]

where \( v_{\text{Earth}} = |v_{\text{Earth}}| \).

The speed of the Earth relative to the Galactic rest frame changes over the course of the year due to the rotation of the Earth around the sun: \( v_{\text{Earth}} \approx v_{\text{Sun}} + \Delta v \cos(2\pi t/t_0 + \phi_0) \) with \( t_0 = 1 \text{yr} \) and an appropriate phase \( \phi_0 \) such that \( v_{\text{Earth}} \) is largest in summer and smallest in winter. It is a good approximation to assume \( v_{\text{Sun}} \approx v_0 \) and \( \Delta v \ll v_0 \).

b) Making use of these approximations, determine the velocity integral as a function of time to linear order in \( \Delta v \).

c) Show (e.g. by explicit evaluation) that the differential event rate for the scattering of DM particles is expected to be smaller in summer than in winter for small recoil energies \( (v_{\min} \ll v_0) \) but larger in summer than in winter for large recoil energies \( (v_{\min} \gg v_0) \).

Problem 14: Nuclear form factors

The electromagnetic form factor \( F(q^2) \) for a nucleus with charge distribution \( \rho(x) \) (in units of the proton charge) is given by

\[ F(q^2) = \int e^{-iq \cdot x} \rho(x) \, d^3x. \tag{6} \]

a) Show that for a spherically symmetric charge distribution \( \rho(x) = \rho(r) \) the form factor can be written as

\[ F(q^2) = \frac{4\pi}{q} \int r \rho(r) \sin(qr) \, dr. \tag{7} \]

Let us approximate the charge distribution of the nucleus as uniform, i.e.

\[ \rho(r) = \frac{3Z}{4\pi R^3} \tag{8} \]

for \( r < R \) and 0 otherwise.

b) Calculate the electromagnetic form factor. Convince yourself that your result has the correct behaviour for \( qR \to 0 \).

c) What is the corresponding form factor for the scattering of a DM particle that couples to the total mass of the nucleus, assuming that charge and mass have the same distribution?