Problem set 3

Tutorial: 2 May 2018, 12:15

Problem 6: The Lee-Weinberg bound

We assume that in addition to the three well-known neutrino species there is a fourth Dirac neutrino $\nu$ with mass $1\,\text{MeV} \ll m_\nu$ and interactions

$$\mathcal{L} = \frac{g}{2\cos\theta_W} \bar{\nu}\gamma_\mu(1 - \gamma^5)\nu Z^\mu, \tag{1}$$

where $g \approx 0.65$ is the weak gauge coupling and $\theta_W \approx 0.50$ is the Weinberg angle. This interaction induces the annihilation processes $\nu\bar{\nu} \rightarrow f\bar{f}$, where $f$ denotes Standard Model fermions. The corresponding cross section (assuming $m_\nu \ll M_Z$) is given by

$$\langle\sigma v\rangle = \frac{g^4 m_\nu^2}{16\pi \cos^4 \theta_W M_Z^4} \times C, \tag{2}$$

where $M_Z = 91\,\text{GeV}$ is the $Z$-boson mass and $C$ is a numerical constant that depends on the number of fermions that are kinematically accessible. For the mass range of interest one finds $C \sim 6$.

a) Check explicitly that the additional neutrino would be in thermal equilibrium with the Standard Model particles for $T \approx m_\nu$. You can assume $g_\nu = 4$ and $g_* \approx 80$.

b) Estimate the relic abundance $\Omega_\nu h^2$ from thermal freeze-out.

c) To be consistent with observations, $\Omega_\nu h^2$ must not exceed the total amount of DM in the Universe, $\Omega_{\text{DM}} h^2 = 0.12$. Use this requirement to obtain a lower bound on $m_\nu$.

d) If the mass of the fourth neutrino were to saturate this bound, it would account for all of the DM in the Universe. What independent measurement excludes this possibility?
Problem 7: The unitarity bound

The velocity distribution of non-relativistic dark matter particles in the Early Universe is given by the Boltzmann distribution \( f(v) \propto \exp(-E/T) \), where \( E \approx m_{\text{DM}}(1 + \frac{1}{2}v^2) \).

a) Let \( v_{\text{rel}} \) be the relative velocity between two dark matter particles. Show that the distribution function for \( v_{\text{rel}} \) is given by

\[
    f(v_{\text{rel}}) = \left( \frac{x}{4\pi} \right)^{3/2} \exp \left( -\frac{m_{\text{DM}} v_{\text{rel}}^2}{4T} \right) .
\]

b) Assuming that the dark matter annihilation cross section can be written as

\[
    \sigma = \frac{a}{v_{\text{rel}}} + b v_{\text{rel}} + O(v_{\text{rel}}^3) ,
\]

show that

\[
    \langle \sigma v_{\text{rel}} \rangle \approx a + \frac{6b}{x} ,
\]

where \( x = m_{\text{DM}}/T \) is the dimensionless temperature.

c) It can be shown in a very general way from the requirement of unitarity (i.e. the conservation of probability) that the annihilation cross section for a Majorana fermion must satisfy the inequality

\[
    \sigma < \frac{\pi}{m_{\text{DM}}^2 v_{\text{rel}}^2} .
\]

Use this relation to derive an upper bound on \( \langle \sigma v_{\text{rel}} \rangle \) as a function of \( m_{\text{DM}} \).

d) Calculate a lower bound on the thermal abundance of a dark matter particle with mass \( m_{\text{DM}} \). Combine this result with the measured value \( \Omega_{\text{DM}} h^2 = 0.12 \) to derive the so-called unitarity bound on the WIMP mass.