Problem set 3

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Problem 6: The Lee-Weinberg bound

We assume that in addition to the three well-known neutrino species there is a fourth Dirac neutrino ν with mass $1\,\mathrm{MeV} \ll m_\nu$ and interactions

$$\mathcal{L} = \frac{g}{2\cos\theta_\mathrm{W}} \bar{\nu}\gamma_\mu(1-\gamma^5)\nu Z^\mu \,, \tag{1}$$

where $g \approx 0.65$ is the weak gauge coupling and $\theta_\mathrm{W} \approx 0.50$ is the Weinberg angle. This interaction induces the annihilation processes $\nu\bar{\nu} \to f\bar{f}$, where f denotes Standard Model fermions. The corresponding cross section (assuming $m_\nu \ll M_Z$) is given by

$$\langle \sigma v \rangle = \frac{g^4\, m_\nu^2}{16\pi\, \cos^4\theta_\mathrm{W}\, M_Z^4} \times C \,, \tag{2}$$

where $M_Z = 91\,\mathrm{GeV}$ is the Z-boson mass and C is a numerical constant that depends on the number of fermions that are kinematically accessible. For the mass range of interest one finds $C \sim 6$.

a) Check explicitly that the additional neutrino would be in thermal equilibrium with the Standard Model particles for $T \approx m_\nu$. You can assume $g_\nu = 4$ and $g_* \approx 80$.

b) Estimate the relic abundance $\Omega_\nu h^2$ from thermal freeze-out.

c) To be consistent with observations, $\Omega_\nu h^2$ must not exceed the total amount of DM in the Universe, $\Omega_\mathrm{DM} h^2 = 0.12$. Use this requirement to obtain a lower bound on m_ν.

d) If the mass of the fourth neutrino were to saturate this bound, it would account for all of the DM in the Universe. What independent measurement excludes this possibility?

Problem 7: The unitarity bound

The velocity distribution of non-relativistic dark matter particles in the Early Universe is given by the Boltzmann distribution $f(v) \propto \exp(-E/T)$, where $E \approx m_{\rm DM}(1 + \frac{1}{2}v^2)$.

a) Let $v_{\rm rel}$ be the relative velocity between two dark matter particles. Show that the distribution function for $v_{\rm rel}$ is given by

$$f(v_{\rm rel}) = \left(\frac{x}{4\pi}\right)^{3/2} \exp\left(-\frac{m_{\rm DM}\, v_{\rm rel}^2}{4T}\right) . \tag{3}$$

b) Assuming that the dark matter annihilation cross section can be written as

$$\sigma = \frac{a}{v_{\rm rel}} + b\, v_{\rm rel} + \mathcal{O}(v_{\rm rel}^3) , \tag{4}$$

show that

$$\langle \sigma v_{\rm rel} \rangle \approx a + \frac{6b}{x} , \tag{5}$$

where $x = m_{\rm DM}/T$ is the dimensionless temperature.

c) It can be shown in a very general way from the requirement of unitarity (i.e. the conservation of probability) that the annihilation cross section for a Majorana fermion must satisfy the inequality

$$\sigma < \frac{\pi}{m_{\rm DM}^2\, v_{\rm rel}^2} . \tag{6}$$

Use this relation to derive an upper bound on $\langle \sigma v_{\rm rel} \rangle$ as a function of $m_{\rm DM}$.

d) Calculate a lower bound on the thermal abundance of a dark matter particle with mass $m_{\rm DM}$. Combine this result with the measured value $\Omega_{\rm DM} h^2 = 0.12$ to derive the so-called unitarity bound on the WIMP mass.