Problem set 1

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Problem 1: Dark matter distribution functions

The distribution of dark matter in an astrophysical system (for example a galaxy) is described by a distribution function $f(\mathbf{x}, \mathbf{v})$, which is related to the density profile $\rho(\mathbf{x})$ via $\rho(\mathbf{x}) = \int \mathrm{d}^3 v\, f(\mathbf{x}, \mathbf{v})$. The *Jeans theorem* states that any self-consistent steady-state distribution function can depend on $\mathbf{x}$ and $\mathbf{v}$ only via so-called *integrals of motion*, meaning quantities that are conserved under gravitational interactions. The simplest such quantity is the total energy of a dark matter particle with mass m:

$$E = m\,\mathbf{v}^2/2 + m\,\Phi(r)\ , \tag{1}$$

where $\Phi(r)$ denotes the gravitational potential.

Consider the distribution function given by

$$f(\mathbf{x}, \mathbf{v}) = \frac{\rho_1}{(2\pi\,\sigma^2)^{3/2}} \exp\left(-\frac{E}{m\,\sigma^2}\right)\ , \tag{2}$$

where ρ_1 and σ denote free parameters of the solution.

a) Obtain an expression for the gravitational potential $\Phi(r)$ in terms of the density $\rho(r)$.

b) Substitute this expression into Poisson's equation $\triangle\Phi(r) = 4\pi\,G\,\rho(r)$ to find a self-consistent solution for $\rho(r)$.

c) Compare this solution to the one obtained from hydrostatic equilibrium.

d) Calculate the velocity dispersion $\langle v^2 \rangle$ and the circular velocity v_c as a function of radius r. What is the physical interpretation of σ?

e) The circular velocity of the sun is about $v_\mathrm{c} \approx 220\,\mathrm{km/s}$. Use this information to calculate the local dark matter density in the solar neighbourhood, $\rho_0 \equiv \rho(r \approx 8\,\mathrm{kpc})$.

f) Measurements of the local dark matter density give $\rho_0 \approx 0.3\,\mathrm{GeV/cm^3}$. Why does this number differ from the one calculated above?

Problem 2: Modified Newtonian Dynamics

An alternative proposal to explain galactic rotation curves is to assume that Newton's second law of motion is modified for very small acceleration $a \ll a_0$. Rather than $F = m\, a$, the relation between force and acceleration is then given by

$$F = m\, \frac{a^2}{a_0}\,. \tag{3}$$

a) Show that this modification implies $v_c(r) = (a_0\, G\, M(r))^{1/4}$.

b) Why would this explain galactic rotation curves without the need to introduce dark matter?

c) For the Milky Way one measures a circular velocity of $v_\mathrm{c} \approx 220\,\mathrm{km/s}$ and a total stellar mass of $M \approx 7 \cdot 10^{10}\, M_\odot$. Obtain an estimate for a_0.

d) Why is it impossible to test Modified Newtonian Dynamics with terrestrial experiments?