Problem set 1

Tutorial: 18 April 2018, 14:15

Problem 1: Dark matter distribution functions

The distribution of dark matter in an astrophysical system (for example a galaxy) is described by a distribution function $f(x, v)$, which is related to the density profile $\rho(x)$ via $\rho(x) = \int d^3v f(x, v)$. The Jeans theorem states that any self-consistent steady-state distribution function can depend on $x$ and $v$ only via so-called integrals of motion, meaning quantities that are conserved under gravitational interactions. The simplest such quantity is the total energy of a dark matter particle with mass $m$:

$$E = m v^2/2 + m \Phi(r) \tag{1}$$

where $\Phi(r)$ denotes the gravitational potential.

Consider the distribution function given by

$$f(x, v) = \frac{\rho_1}{(2\pi \sigma^2)^{3/2}} \exp\left(-\frac{E}{m \sigma^2}\right), \tag{2}$$

where $\rho_1$ and $\sigma$ denote free parameters of the solution.

a) Obtain an expression for the gravitational potential $\Phi(r)$ in terms of the density $\rho(r)$.

b) Substitute this expression into Poisson’s equation $\Delta \Phi(r) = 4\pi G \rho(r)$ to find a self-consistent solution for $\rho(r)$.

c) Compare this solution to the one obtained from hydrostatic equilibrium.

d) Calculate the velocity dispersion $\langle v^2 \rangle$ and the circular velocity $v_c$ as a function of radius $r$. What is the physical interpretation of $\sigma$?

e) The circular velocity of the sun is about $v_c \approx 220 \text{ km/s}$. Use this information to calculate the local dark matter density in the solar neighbourhood, $\rho_0 \equiv \rho(r \approx 8 \text{ kpc})$.

f) Measurements of the local dark matter density give $\rho_0 \approx 0.3 \text{ GeV/cm}^3$. Why does this number differ from the one calculated above?
Problem 2: Modified Newtonian Dynamics

An alternative proposal to explain galactic rotation curves is to assume that Newton’s second law of motion is modified for very small acceleration $a \ll a_0$. Rather than $F = m a$, the relation between force and acceleration is then given by

$$F = m \frac{a^2}{a_0}.$$  \hspace{1cm} (3)

a) Show that this modification implies $v_c(r) = (a_0 GM(r))^{1/4}$.

b) Why would this explain galactic rotation curves without the need to introduce dark matter?

c) For the Milky Way one measures a circular velocity of $v_c \approx 220 \text{ km/s}$ and a total stellar mass of $M \approx 7 \cdot 10^{10} M_\odot$. Obtain an estimate for $a_0$.

d) Why is it impossible to test Modified Newtonian Dynamics with terrestrial experiments?