Solutions

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1 Flow lines

Let us introduce a more compact notation:

$$Z[J_B, J_L] \sim \int \mathcal{D}L\mathcal{D}B\, \exp\Big[-\Big(L_{t,x}(\partial_t - \Box_x)B_{t,x} - J_{B,t,x}B_{t,x} - L_{t,x}J_{L,t,x}\Big)\Big]. \tag{1}$$

Now do the substitutions in the exponent:

$$\begin{aligned}
&(L_{t,x} + J_{B,s,y}P_{s-t,y-x})(\partial_t - \Box_x)(B_{t,x} + P_{t-s,x-y}J_{L,s,y})\\
&- J_{B,t,x}(B_{t,x} + P_{t-s,x-y}J_{L,s,y'}) - (L_{t,x} + J_{B,s,y}P_{s-t,y-x})J_{L,t,x} =\\
&= L_{t,x}(\partial_t - \Box_x)B_{t,x} + L_{t,x}(\partial_t - \Box_x)P_{t-s,x-y}J_{L,s,y}\\
&+ J_{B,s,y}P_{s-t,y-x}(\partial_t - \Box_x)B_{t,x} + J_{B,s,y}P_{s-t,y-x}(\partial_t - \Box_x)P_{t-s',x-y'}J_{L,s',y'}\\
&- J_{B,t,x}B_{t,x} - J_{B,t,x}P_{t-s,x-y}J_{L,s,y} - L_{t,x}J_{L,t,x} - J_{B,s,y}P_{s-t,y-x}J_{L,t,x} =\\
&= L_{t,x}(\partial_t - \Box_x)B_{t,x} + L_{t,x}J_{L,t,x}\\
&+ J_{B,s,y}P_{s-t,y-x}(\partial_t - \Box_x)B_{t,x} + J_{B,s,y}P_{s-s',y-y'}J_{L,s',y'}\\
&- J_{B,t,x}B_{t,x} - J_{B,t,x}P_{t-s',x-y'}J_{L,s',y'} - L_{t,x}J_{L,t,x} - J_{B,s',y'}P_{s'-t,y'-x}J_{L,t,x}
\end{aligned} \tag{2}$$

For the third term, we need to do two partial integrations in x (which does not affect the sign), and one in t. The latter changes the sign, but

$$(-\partial_t - \Box_x)P_{s-t,y-x} = (\partial_s - \Box_x)P_{s-t,y-x} = \delta(t-s)\delta(x-y)\,. \tag{3}$$

Collecting all the terms gives

$$\begin{aligned}Z[J_B, J_L] &\sim \int \mathcal{D}L\mathcal{D}B \; \exp\Big[-\Big(L_{t,x}(\partial_t - \Box_x)B_{t,x} - J_{B,t,x}P_{t-s,x-y}J_{L,s,y}\Big)\Big] = \\ &= \exp\Big[-J_{B,t,x}P_{t-s,x-y}J_{L,s,y}\Big]\int \mathcal{D}L\mathcal{D}B \; \exp\Big[-L_{t,x}(\partial_t - \Box_x)B_{t,x}\Big] \\ &\sim \exp\Big[-J_{B,t,x}P_{t-s,x-y}J_{L,s,y}\Big]\end{aligned} \tag{4}$$

since the integral is now independent of the currents. Now we can simply evaluate the two-point function by differentiating w.r.t. the currents, which gives the desired result.

2 Asymptotic expansion of flow-time integrals

The task is to solve the integral

$$I = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} \tag{5}$$

in the limit $tq^2 \ll 1$, or equivalently $q^2 \ll 1/t$, using the strategy of regions [1]. We need to consider two integration regions:

(i) $k^2 \ll 1/t$. In this case, all terms in the exponent are small, and we can expand the exponential. On the other hand, there is no large quantity in the denominators, so we cannot expand them:

$$\begin{aligned}I^{(i)} &= \int \frac{\mathrm{d}^D k}{(2\pi)^D}\left[\frac{1}{k^2(k-q)^2} - t\frac{k^2+(k-q)^2}{k^2(k-q)^2} + \cdots\right] \\ &= \int \frac{\mathrm{d}^D k}{(2\pi)^D}\left[\frac{1}{k^2(k-q)^2} - t\frac{1}{(k-q)^2} - t\frac{1}{k^2} + \cdots\cdots\right]\end{aligned} \tag{6}$$

The second and third (and all higher terms!) lead to scaleless integrals (after appropriate shifts in the integration momentum) and vanish. The first one is just the regular massless two-point function. It can be calculated using Feynman parameters, for example:

$$I^{(i)} = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + 2 - \ln q^2 + \mathcal{O}(\epsilon)\right]. \tag{7}$$

(ii) $k^2 \gtrsim 1/t$.

$$\begin{aligned}I^{(ii)} &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{-2tk^2}e^{2tk\cdot q - tq^2}}{k^4\left(1 - \frac{2k\cdot q - q^2}{k^2}\right)} = \\ &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{-2tk^2}}{k^4}\left[1 + t(2k\cdot q - q^2) + \cdots\right]\left[1 + \frac{2k\cdot q - q^2}{k^2} + \cdots\right].\end{aligned} \tag{8}$$

Let us just keep the first term:

$$
\begin{aligned}
I^{(ii)} &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{-2tk^2}}{k^4} = \frac{\Omega_D}{(2\pi)^D} \int_0^\infty \mathrm{d}k\, k^{D-5}\, e^{-2tk^2} = \\
&= \frac{1}{(2\pi)^D} \cdot \frac{2\pi^{D/2}}{\Gamma(D/2)} \cdot t(2t)^{1-D/2} \Gamma(D/2-2) = \frac{1}{16\pi^2} \left[-\frac{1}{\epsilon} - 1 - \ln 8\pi t + \mathcal{O}(\epsilon) \right] .
\end{aligned} \tag{9}
$$

Adding the results of the two regions, we find

$$
I = I^{(i)} + I^{(ii)} = \frac{1}{16\pi^2} \left[1 - \ln 8\pi t q^2 + \cdots \right] , \tag{10}
$$

where higher orders in tq^2 and ϵ are neglected.

Remark: The beauty of the strategy of regions is that, even though we expand in a particular region, we can still extend the integration in $I^{(1)}$ and $I^{(2)}$ over all k. The reason is the following (see also Ref. [2]). Let us denote by $\mathcal{T}_x$ the operator which performs a Taylor expansion in the variables x (all integrals are over $\mathrm{d}^D k$):

$$
\begin{aligned}
I &= \int_{(i)} \mathcal{T}_{q,k} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} + \int_{(ii)} \mathcal{T}_q \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} = \\
&= \left(\int - \int_{(ii)} \right) \mathcal{T}_{q,k} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} + \left(\int - \int_{(i)} \right) \mathcal{T}_q \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} = \\
&= I^{(i)} + I^{(ii)} - \int_{(ii)} \mathcal{T}_{q,k} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} - \int_{(i)} \mathcal{T}_q \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} .
\end{aligned} \tag{11}
$$

Since in region (i) we can apply $\mathcal{T}_{q,k}$, and $\mathcal{T}_q$ in region (ii), we get for the last two terms

$$
\Delta \equiv \int_{(ii)} \mathcal{T}_q \mathcal{T}_{q,k} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} + \int_{(i)} \mathcal{T}_{q,k} \mathcal{T}_q \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} . \tag{12}
$$

We can interchange the order of the Taylor expansions, which allows us to combine the two integrals again:

$$
\Delta = \int \mathcal{T}_q \mathcal{T}_{q,k} \frac{e^{-t[k^2+(k-q)^2]}}{k^2(k-q)^2} = \int \mathcal{T}_q \frac{1}{k^2(k-q)^2} = \int \frac{1}{k^4} \sum_{n=0}^{\infty} \left(\frac{2k \cdot q - q^2}{k^2} \right)^n , \tag{13}
$$

where in the first step, we have used the result of Eq. (6). The final term only involves scaleless integrals, and thus $\Delta = 0$.

References

[1] M. Beneke and V. A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, *Nucl. Phys. B* **522** (1998) 321–344, `arXiv:hep-ph/9711391`.

[2] R. Harlander, *Asymptotic expansions: Methods and applications*, *Acta Phys. Polon. B* **30** (1999) 3443–3462, `arXiv:hep-ph/9910496`.