The perturbative gradient flow

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What is the gradient flow?
Perturbative solution
Effective Field Theories
Calculational techniques

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Part 2

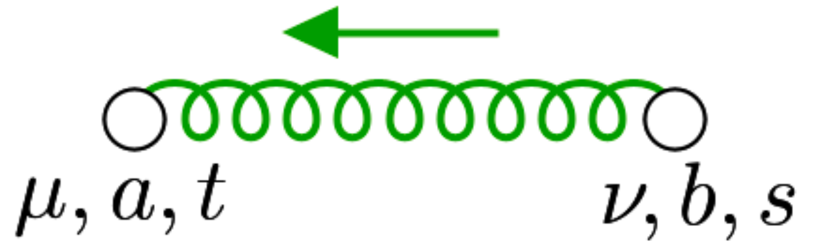
Feynman rules

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

L_μ Lagrange multiplier field

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$



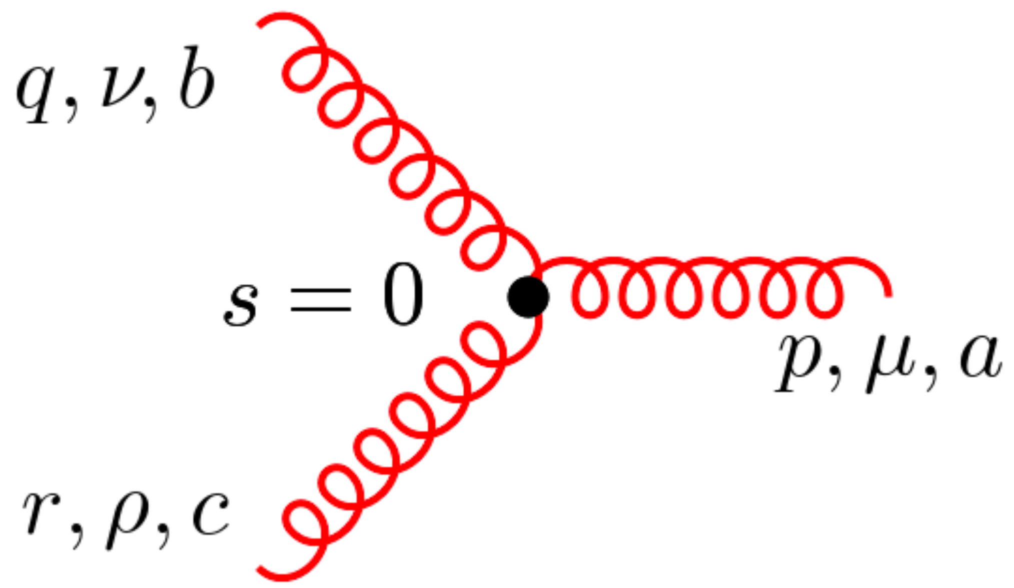
$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

"gluon flow line"

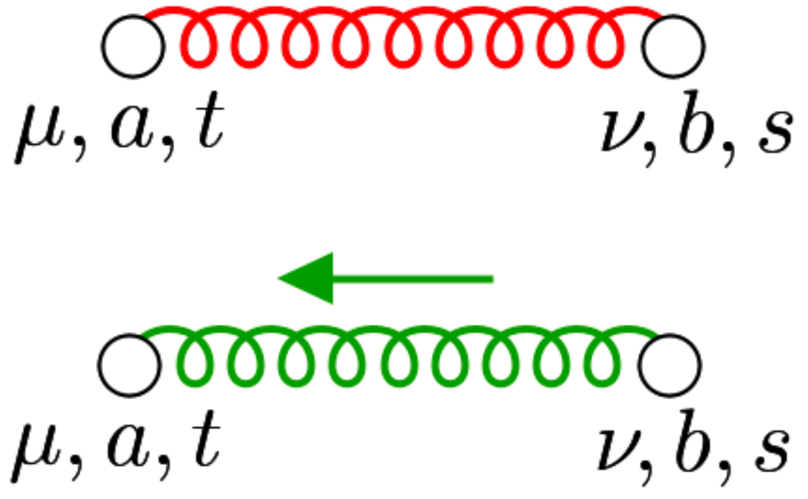
analogously for quarks:

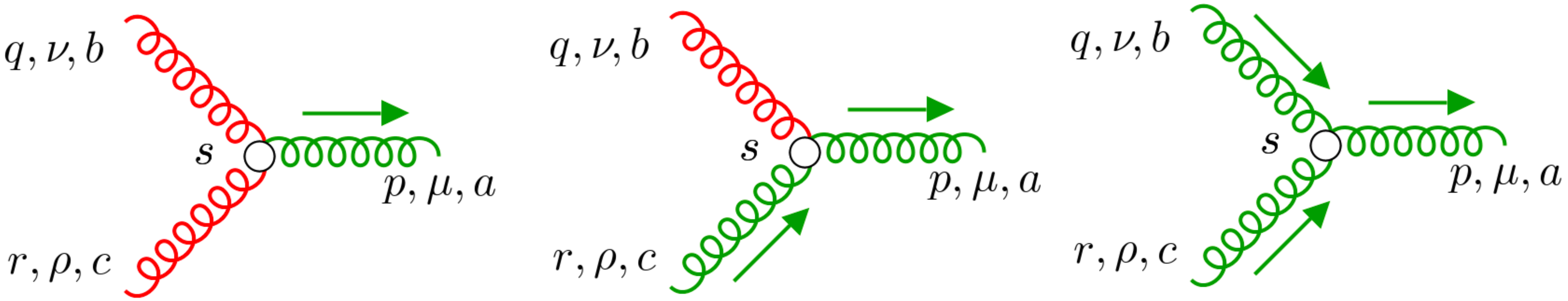
$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda}\left(\partial_t - \Delta\right)\chi + \mathrm{h.c.}$$

Vertices



regular 3-gluon vertex

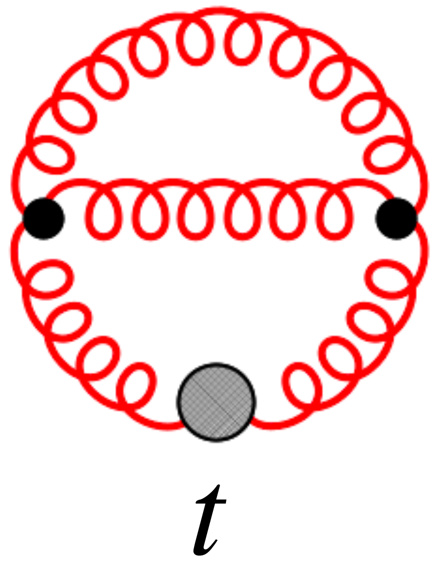


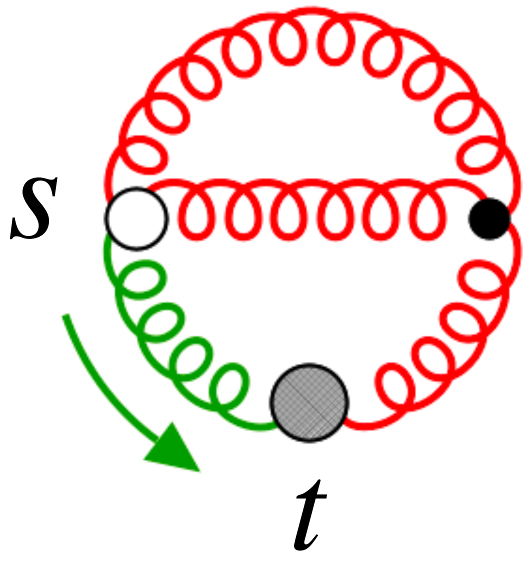


$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu \\ + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

analogously for 4-gluon vertex and quarks

Higher orders





$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4 k^2 (p-k)^2}$$

$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

- more loop integrals
- integration over flow-time parameters
- renormalization: same as fundamental QCD!

Integration-by-parts relations

- After tensor reduction, we end up with many scalar integrals of the form

$$I(\{t_f^{\text{up}}\},\{T_i\},\{a_i\}) = \left(\prod_{f=1}^{F}\int_0^{t_f^{\text{up}}} \mathrm{d}t_f\right)\int_{k_1,\dots,k_L} \frac{\exp[-(T_1 q_1^2+\cdots+T_N q_N^2)]}{q_1^{2a_1}\cdots q_N^{2a_N}}$$

with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$

- Chetyrkin and Tkachov observed [Tkachov 1981; Chetyrkin, Tkachov 1981]

$$\int_{k_1,\dots k_L}\frac{\partial}{\partial k_i^\mu}\left(\tilde{q}_j^\mu \frac{1}{P_1^{a_1}\dots P_N^{a_N}}\right)=0$$

⇒ Linear relations between Feynman integrals

- Can easily be adopted to gradient-flow integrals
- Additional new relations for gradient-flow integrals: [Artz, RH, Lange, Neumann, Prausa '19]

$$\int_0^{t_f^{\text{up}}} \mathrm{d}t_f\,\partial_{t_f} F(t_f,\dots) = F(t_f^{\text{up}},\dots) - F(0,\dots)$$

Laporta algorithm

- Schematically integration-by-parts read

$$0 = (d - a_1)I(a_1, a_2, a_3) + (a_1 - a_2)I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2)I(a_1 + 1, a_2, a_3 - 1)$$

- Rarely possible to find general solution like

$$I(a_1, a_2, a_3) = a_1 I(a_1 - 1, a_2, a_3) + (d + a_1 - a_2)I(a_1, a_2 - 1, a_3) + 2a_3 I(a_1, a_2, a_3 - 1)$$

- Instead set up system of equations and solve it [Laporta 2000] :
  - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$\begin{aligned} 0 &= (d - 1)I(1, 1, 1) + I(2, 1, 0), \\ 0 &= (d - 2)I(2, 1, 1) + I(3, 0, 1) - I(3, 1, 0), \\ &\vdots \end{aligned}$$

  - Solve with Gaussian elimination

⇒ Express integrals through significantly smaller number of master integrals

Laporta algorithm

- Schematically integration-by-parts read

$$0 = (d - a_1) I(a_1, a_2, a_3) + (a_1 - a_2) I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2) I(a_1 + 1, a_2, a_3 - 1)$$

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- Instead set up system of equations and solve it [Laporta 2000] :
  - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$0 = (d - 1) I(1, 1, 1) + I(2, 1, 0),$$
$$0 = (d - 2) I(2, 1, 1) + I(3, 0, 1) - I(3, 1, 0),$$

e.g. NNLO chromo-magnetic dipole operator:
O(4000) integrals reduced to 13 master integrals

  - Solve with Gaussian elimination

⇒ Express integrals through significantly smaller number of master integrals

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\,(p_1-p_2)^2\,(p_1-p_3)^2\,(p_2-p_3)^2}$$

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \;\cdots\; \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int\!\!\int_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2}$$

Schwinger parameters:

$$\frac{1}{p^2} = \int_0^\infty \mathrm{d}x\, e^{-xp^2}$$

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2}$$

Schwinger parameters:

$$\frac{1}{p^2} = \int_0^\infty \mathrm{d}x\, e^{-xp^2}$$

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int_0^\infty \mathrm{d}x_1 \cdots \int_0^\infty \mathrm{d}x_6 \iint_{p_1,p_2,p_3} \exp\left(-\mathbf{p}^T B(u_1,\ldots,u_f,x_1,\cdots,x_6)\,\mathbf{p}\right)$$

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \;\cdots\; \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\,(p_1-p_2)^2\,(p_1-p_3)^2\,(p_2-p_3)^2}$$

Schwinger parameters:

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$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \;\cdots\; \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int_0^\infty \mathrm{d}x_1 \;\cdots\; \int_0^\infty \mathrm{d}x_6 \iint_{p_1,p_2,p_3} \exp\left(-\mathbf{p}^T B(u_1,\ldots,u_f,x_1,\cdots,x_6)\,\mathbf{p}\right)$$

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \;\cdots\; \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int_0^\infty \mathrm{d}x_1 \;\cdots\; \int_0^\infty \mathrm{d}x_6\, [\det B(u_1,\cdots,u_f,x_1,\cdots,x_6)]^{-D/2}$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1 - p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1 - p_2)^2\, (p_1 - p_3)^2\, (p_2 - p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\begin{aligned}
\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \quad & u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
& + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
& + 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
& + u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
& + 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
& + 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 + \\
& + u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
& + u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 + \\
& + u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 + \\
& + u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
& + x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
& + 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 + \\
& + 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
& + 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 + \\
& + x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 + \\
& + x_2\, x_4\, x_5 + x_2\, x_5\, x_6 + x_3\, x_4\, x_5 + x_3\, x_4\, x_6 + x_4\, x_5\, x_6)^{\epsilon-2}
\end{aligned}$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \quad \longrightarrow$$

$$\begin{aligned}
&\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \quad u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
&+ u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
&+ 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 + \\
&+ u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
&+ x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
&+ 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 + \\
&+ 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
&+ 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 + \\
&+ x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 + \\
&+ x_2\, x_4\, x_5 + x_2\, x_5\, x_6 + x_3\, x_4\, x_5 + x_3\, x_4\, x_6 + x_4\, x_5\, x_6)^{\epsilon-2}
\end{aligned}$$

$$y = \frac{1}{1+x}$$

$$\int_0^\infty \mathrm{d}x\, f(x) = \int_0^1 \frac{\mathrm{d}y}{y^2} f(x(y))$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\begin{aligned}
&\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \;\; u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
&+ u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
&+ 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 + \\
&+ u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
&+ x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
&+ 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 + \\
&+ 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
&+ 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 + \\
&+ x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 + \\
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\end{aligned}$$

$$y = \frac{1}{1+x}$$

$$\int_0^\infty \mathrm{d}x\, f(x) = \int_0^1 \frac{\mathrm{d}y}{y^2} f(x(y))$$

overlapping singularities as $x_i, u_j \to 0$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\begin{aligned}
&\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \;\; u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
&+ u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
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&+ u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
&+ x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
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&+ 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
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overlapping singularities
as $x_i, u_j \to 0$

→ sector decomposition

example:

[Heinrich ‘08]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\,y\Big)^{-1}$$

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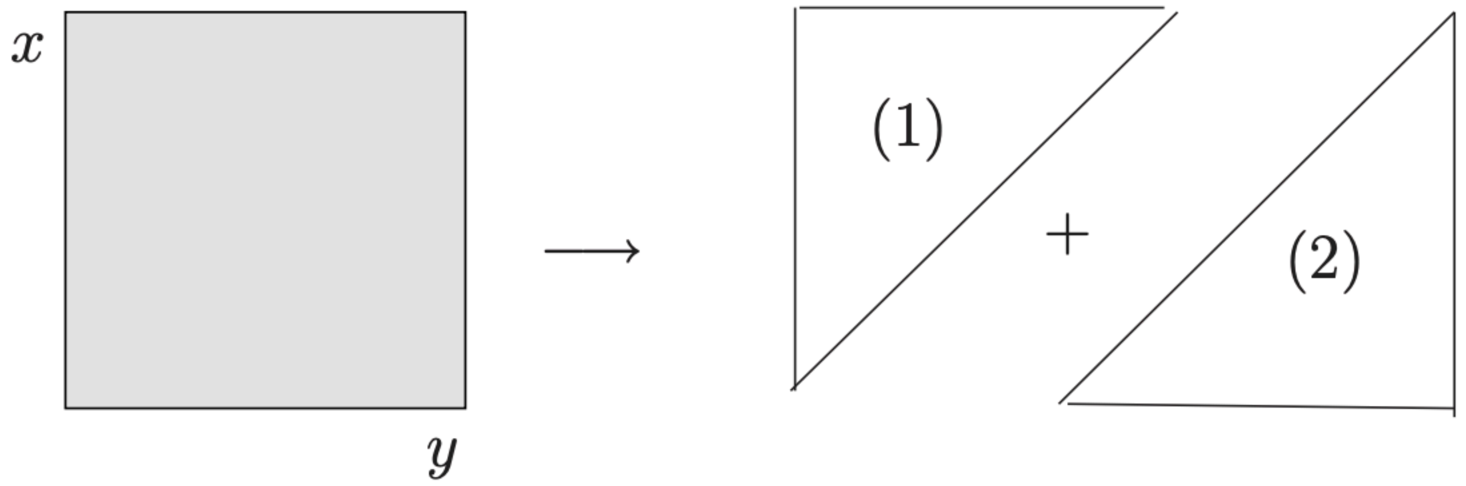

example: [Heinrich ‘08]

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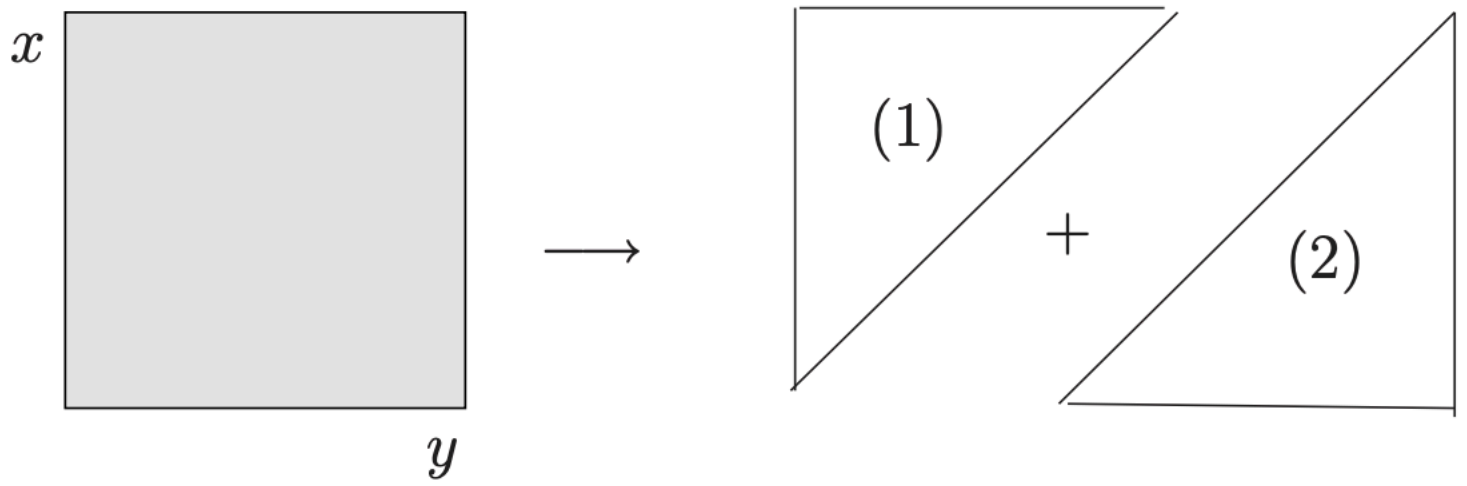

$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \left(1 + (1-x)\,t\right)^{-1} + \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \left(1 + (1-y)\,t\right)^{-1}$$

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$$= -\frac{1}{(a+b)\,\epsilon} \int_0^1 dt\, \frac{t^{-b\epsilon}}{1+t}$$

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$$= -\frac{1}{(a+b)\,\epsilon} \int_0^1 dt\, \frac{t^{-b\epsilon}}{1+t} - \frac{1}{(a+b)\,\epsilon} \int_0^1 \frac{dt}{1+t} \left[-\frac{1}{a\,\epsilon}\delta(t) - \left(\frac{1}{t}\right)_+ + a\epsilon \left(\frac{\ln t}{t}\right)_+ + \cdots \right]$$

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$$\int_0^1 dt \left(\frac{\ln^n t}{t}\right)_+ f(t) = \int_0^1 dt\, \frac{\ln^n t}{t} \left[f(t) - f(0)\right]$$

pySecDec

[Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke ’18]

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} = \frac{1}{(4\pi)^{3d/2}}$$

```
[ + ep^(-1)*((-1.2020569040793 7649) +
(6.74709950249940753e-9)*numerr )

 + ep^(0)*((-11.440962423725 6917) +
(4.99888756503079786e-8)*numerr ) ]
```

[RH, Nellopoulos ’22 (unpublished)]
earlier work: [RH, Neumann ’16]

Effective Field Theories

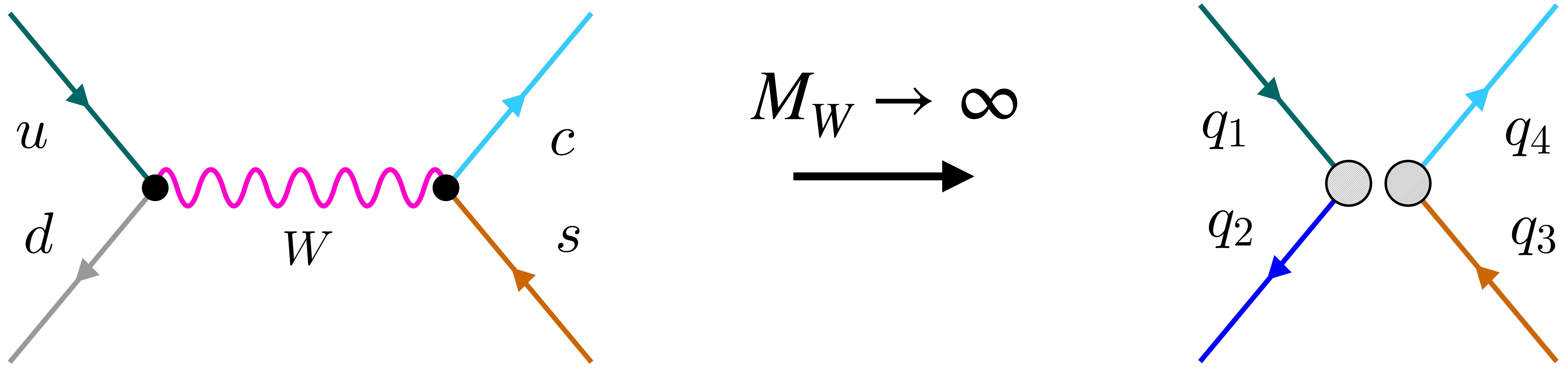
$$\mathscr{L} = \mathscr{L}^{\leq 4} + \sum_{d>4} \frac{1}{\Lambda^{d-4}} \sum_i C_i^{(d)} \mathcal{O}_i^{(d)}$$

some problems:

- many operators (SMEFT: 2499 @ dim 6)
- get a non-redundant basis (EoMs, IbP, Fierz, Schouten, …)
- determine $C_i^{(d)}$
- determine $\langle \mathcal{O}_i^{(d)} \rangle$ ←
- renormalization ←

Gradient Flow

Example



$$\mathcal{O}_1 = (\bar{q}_1 \gamma^L_\mu T q_2)(\bar{q}_3 \gamma^\mu_L T q_4)$$
$$\mathcal{O}_2 = (\bar{q}_1 \gamma^L_\mu q_2)(\bar{q}_3 \gamma^\mu_L q_4)$$

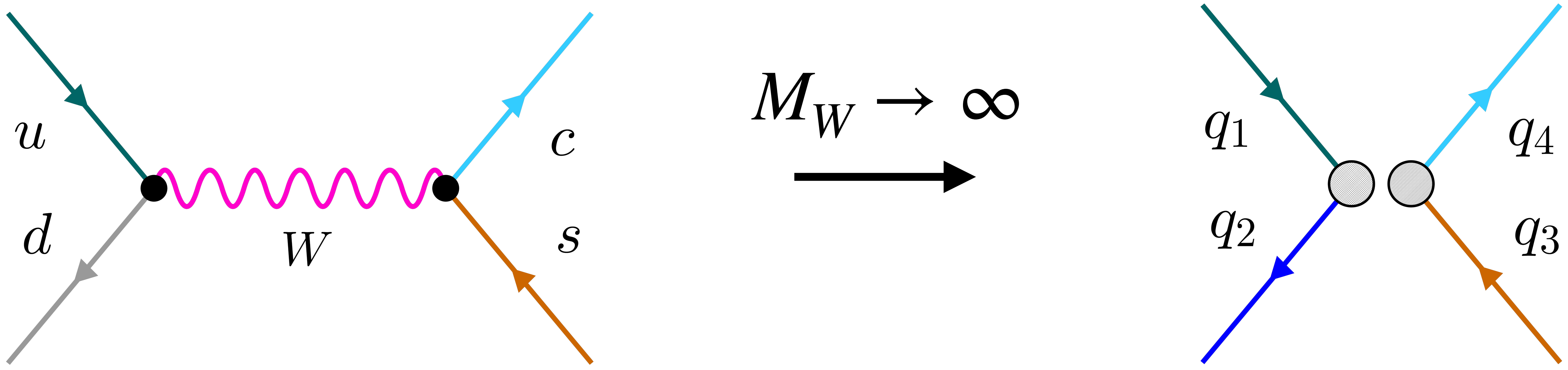
$$\mathcal{L}_{\text{eff}} \ni \sum_n C_n^B \, \mathcal{O}_n$$

$$\langle T \rangle = \sum_n C_n \langle \mathcal{O}_n^R \rangle$$

pert.th.

lattice

Example



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$$\mathcal{L}_{\text{eff}} \ni \sum_n C_n^B \, \mathcal{O}_n \equiv \sum_n \tilde{C}(t)_n \, \tilde{\mathcal{O}}(t)_n$$

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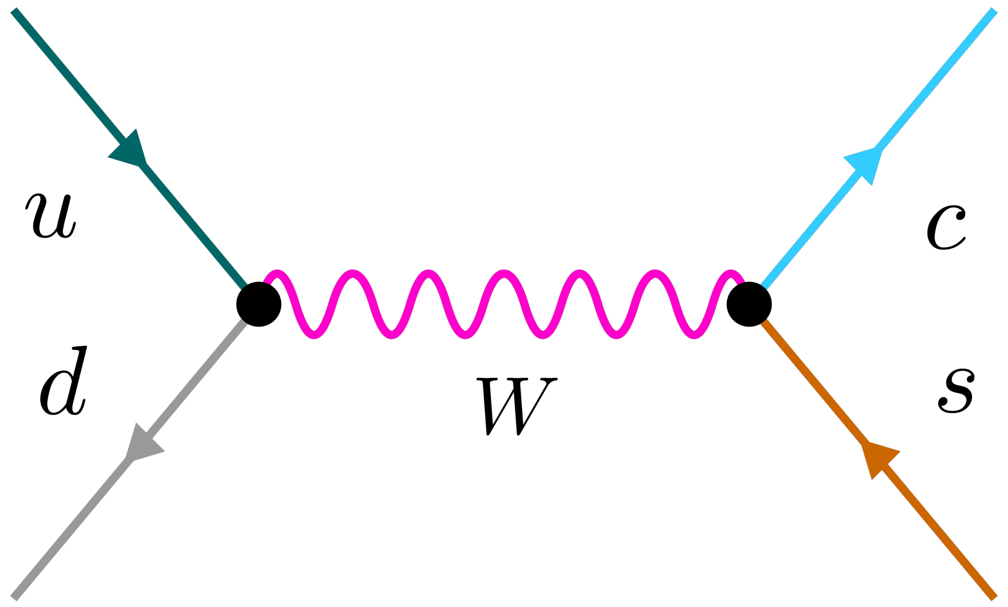
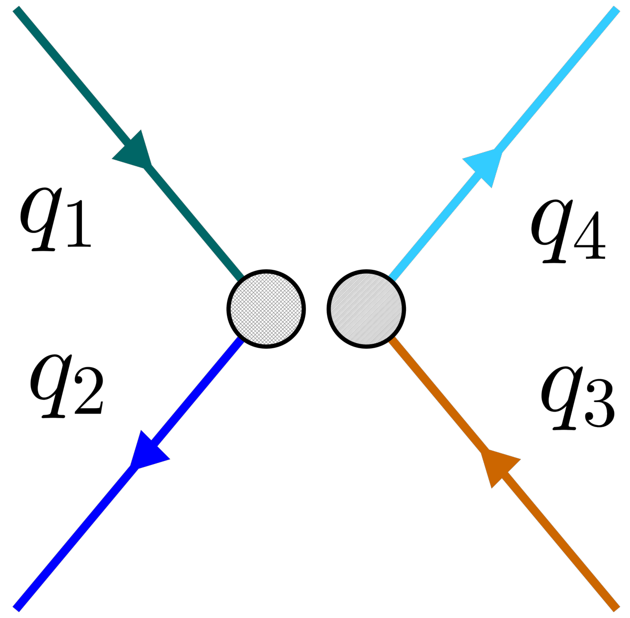
Example



$$\xrightarrow{M_W \to \infty}$$



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$$\Big\downarrow \quad q_i \to \chi_i(t)$$

$$\tilde{\mathcal{O}}_1(t), \tilde{\mathcal{O}}_2(t)$$

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$$\langle T \rangle = \sum_n \underset{\text{pert.th.}}{C_n} \underset{\text{lattice}}{\langle \mathcal{O}_n^R \rangle}$$

Small-flow-time expansion

Lüscher, Weisz 2011

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m$$

perturbative

Small-flow-time expansion

Lüscher, Weisz 2011

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad = \sum_{m,k} \zeta^B_{nm}(t) Z^{-1}_{mk} \mathcal{O}^R_k$$

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$$\tilde{C}_n(t) = C_m\, \zeta^{-1}_{mn}(t)$$

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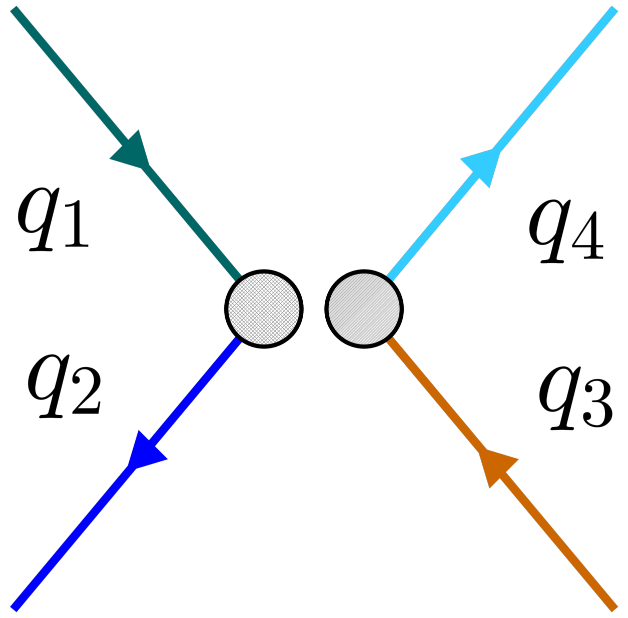
Need to compute suitable Green's functions of the operators…

Method of projectors

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \quad \longrightarrow \quad \langle k|\tilde{\mathcal{O}}_n(t)|0\rangle = \sum_m \zeta^B_{nm}(t)\,\langle k|\mathcal{O}_m|0\rangle$$

ideally: $\langle k|\mathcal{O}_m|0\rangle = \delta_{km} \quad \Rightarrow \quad \zeta^B_{nk}(t) = \langle k|\tilde{\mathcal{O}}_m(t)|0\rangle$



$\longrightarrow$

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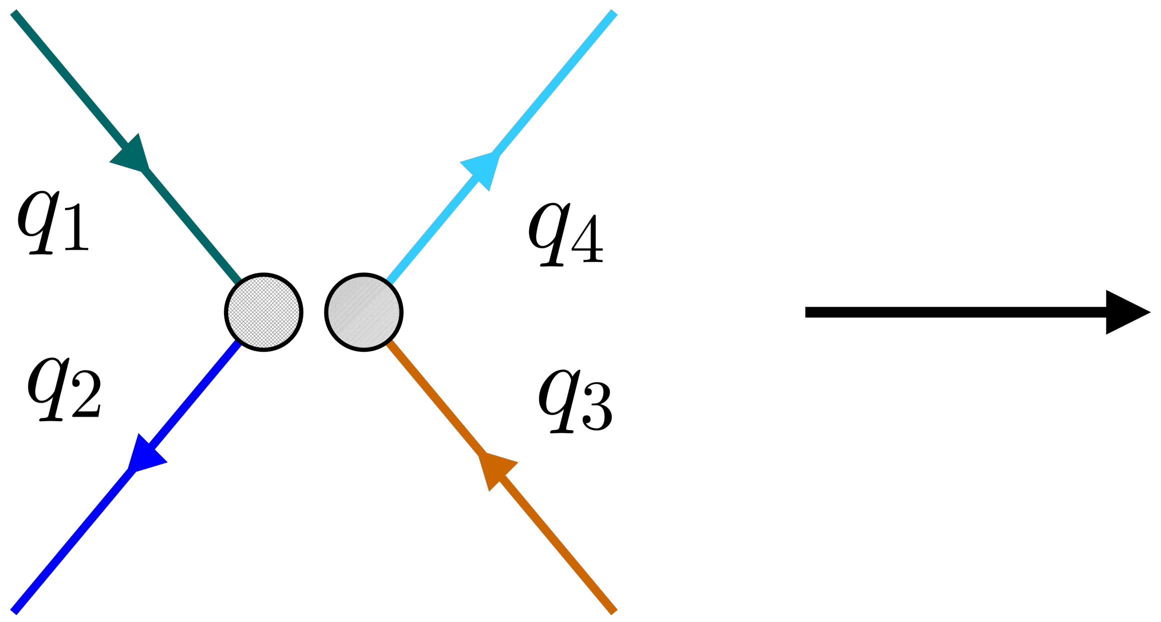

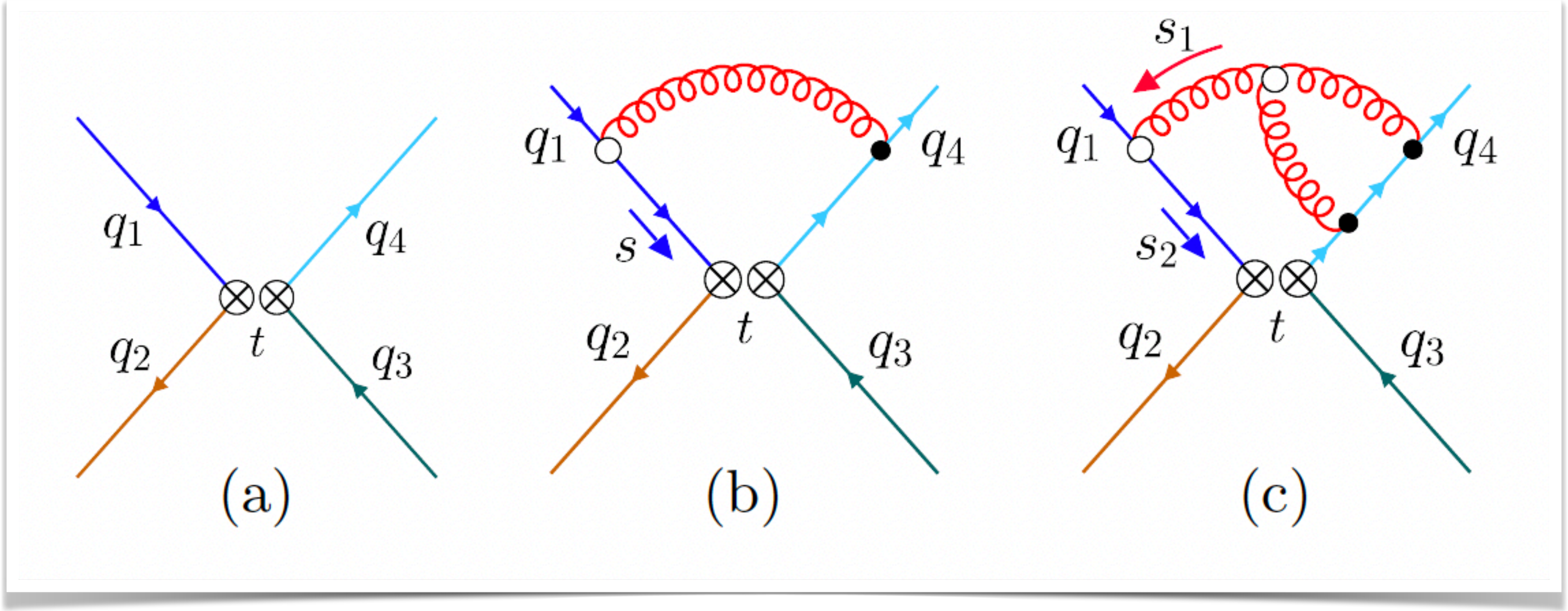

Energy-momentum tensor

$$T_{\mu\nu}(x) = \frac{2}{|g(x)|^{1/2}} \frac{\delta S}{\delta g^{\mu\nu}(x)}$$

here: $S = S_{\text{QCD}}$ and $g^{\mu\nu}$ = flat metric

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$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho} \qquad \mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma} \qquad \mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

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Noether current of space-time translations

Energy-momentum tensor

$$T_{\mu\nu}(x) = \frac{2}{|g(x)|^{1/2}} \frac{\delta S}{\delta g^{\mu\nu}(x)}$$

here: $S = S_{\mathrm{QCD}}$ and $g^{\mu\nu}$ = flat metric

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

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Noether current of space-time translations

→ ill-defined on the lattice!

Energy-momentum tensor

in QCD:

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idea and NLO result: H. Suzuki '14

Flowed operators

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m$$

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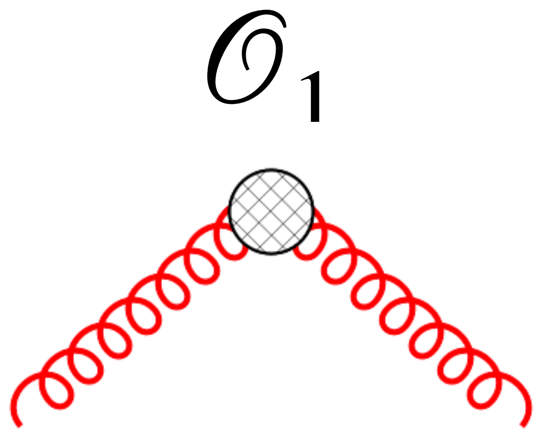
e.g.

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$



$$-g^2\delta^{ab}\left(\delta_{\mu\nu}p\cdot q - p_\mu q_\nu\right)$$

Flowed operators

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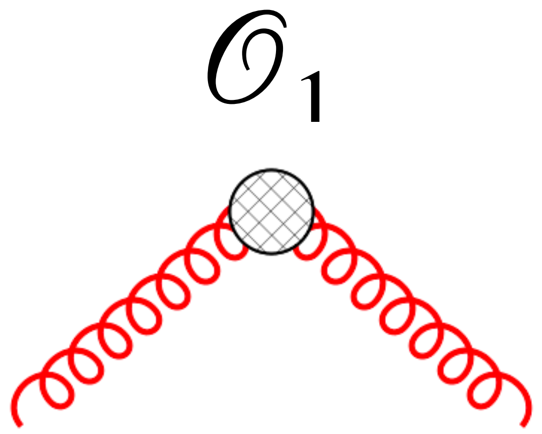
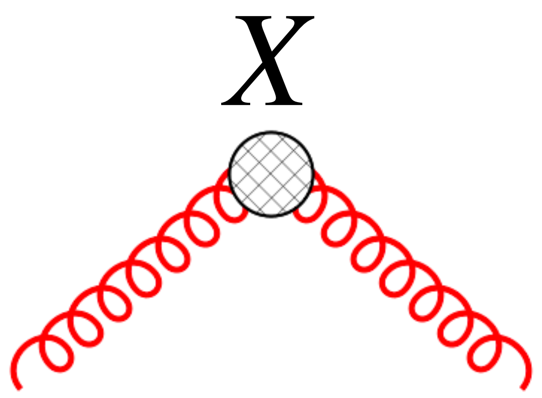


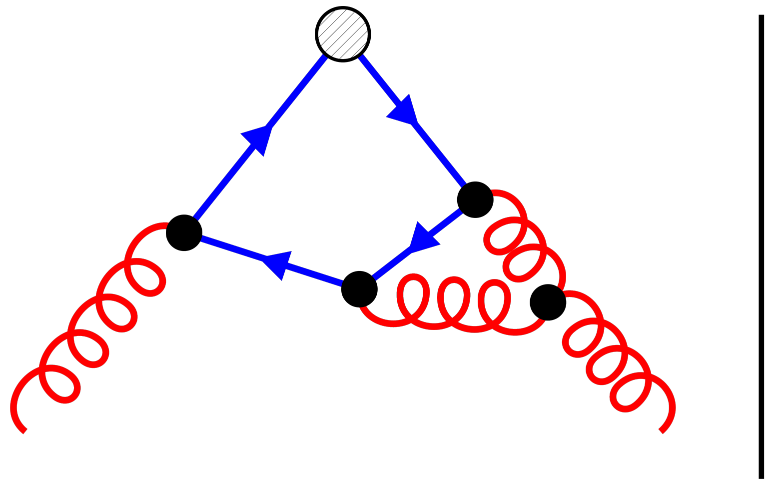
$$-g^2\delta^{ab}\left(\delta_{\mu\nu}p\cdot q - p_\mu q_\nu\right)$$

projector: $P^{ab}_{1,\mu\nu}[X] \sim \delta^{ab}\dfrac{\partial}{\partial p_\mu}\dfrac{\partial}{\partial q_\nu}$



$$\Big|_{p=q=0}$$

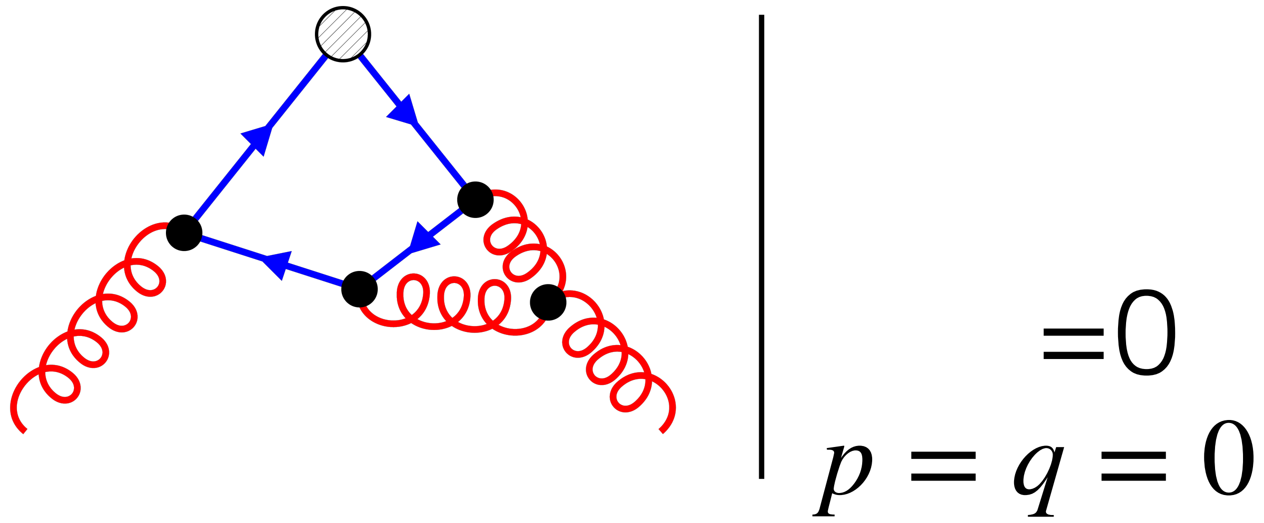
Method of projectors



$$=0 \Big|_{p = q = 0}$$

Method of projectors

If X is a regular QCD operator, all higher orders = 0.



$$\left. \quad \right|_{p=q=0} = 0$$

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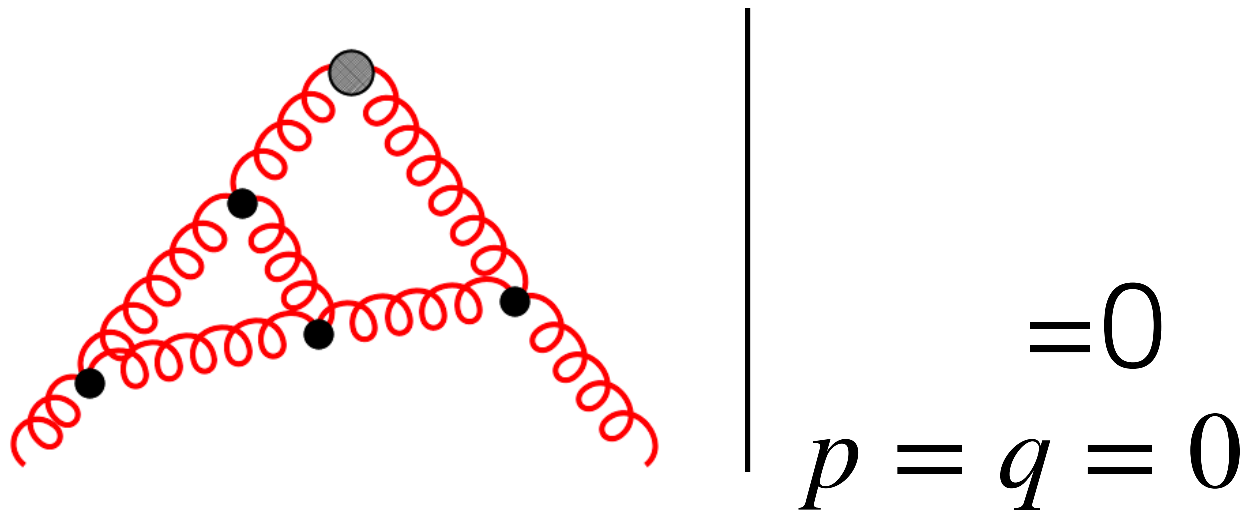
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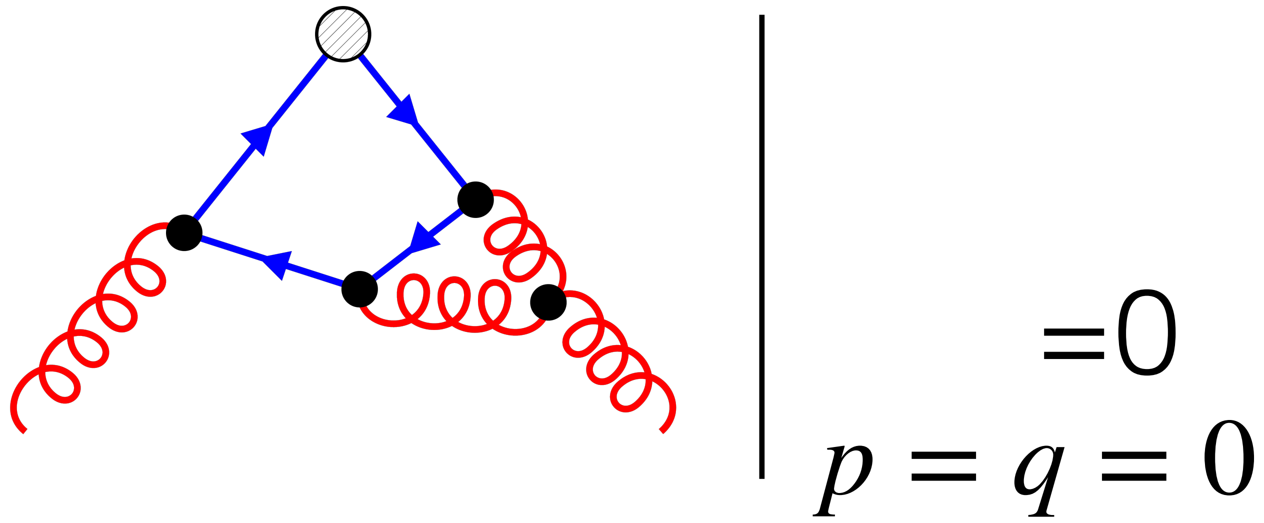
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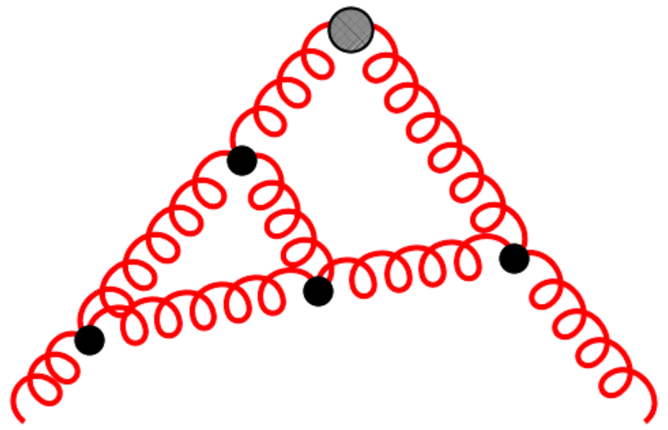

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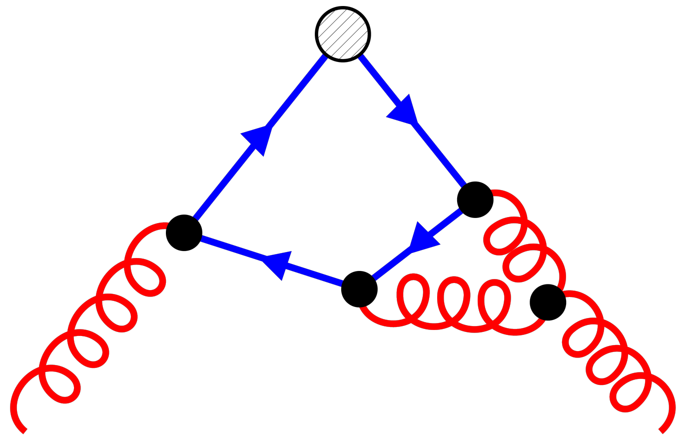

$$\Big|_{p=q=0} = 0$$

$$\Rightarrow \qquad P_1[\tilde{\mathcal{O}}_n(t)] = \sum_m \zeta_{nm}(t) P_1[\mathcal{O}_m] = \zeta_{n1}(t)$$

to all orders

Gorishny, Larin, Tkachov ’83

Energy-momentum tensor

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idea and NLO result: H. Suzuki ‘14

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$\mathcal{O}_{3,\mu\nu} = \bar\psi\Big($

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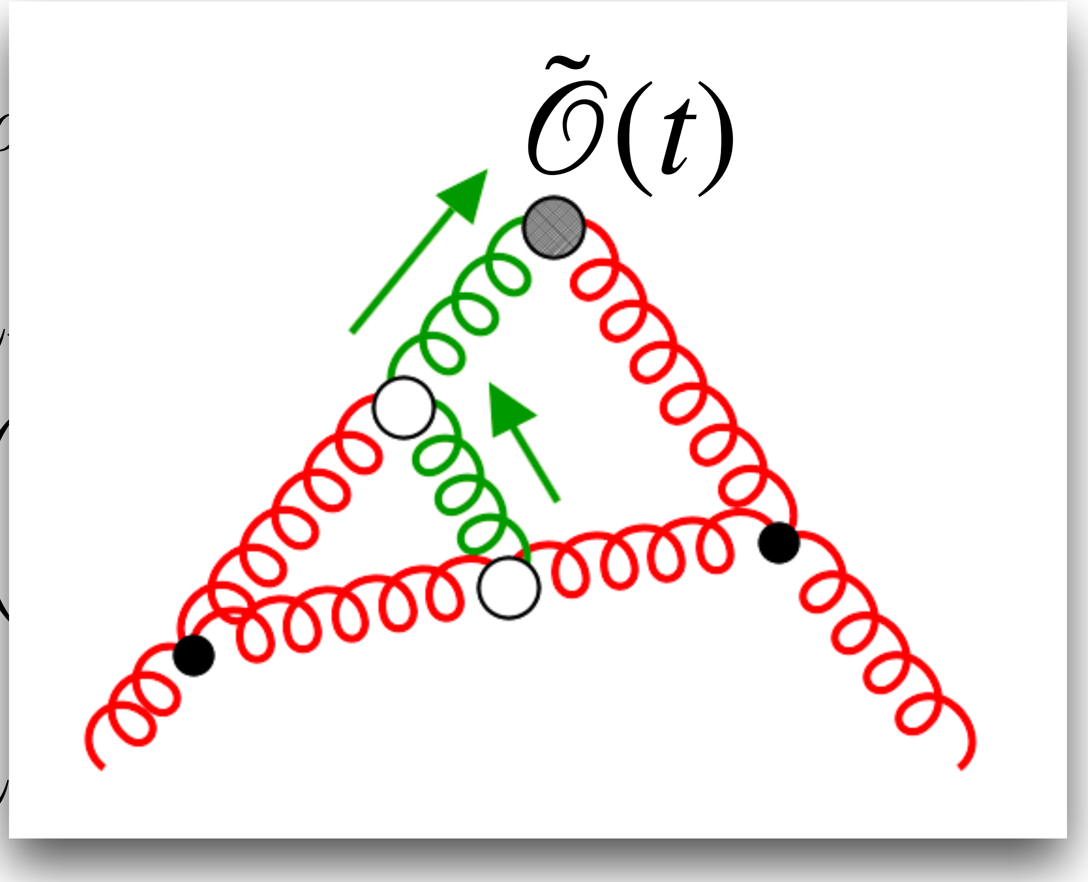

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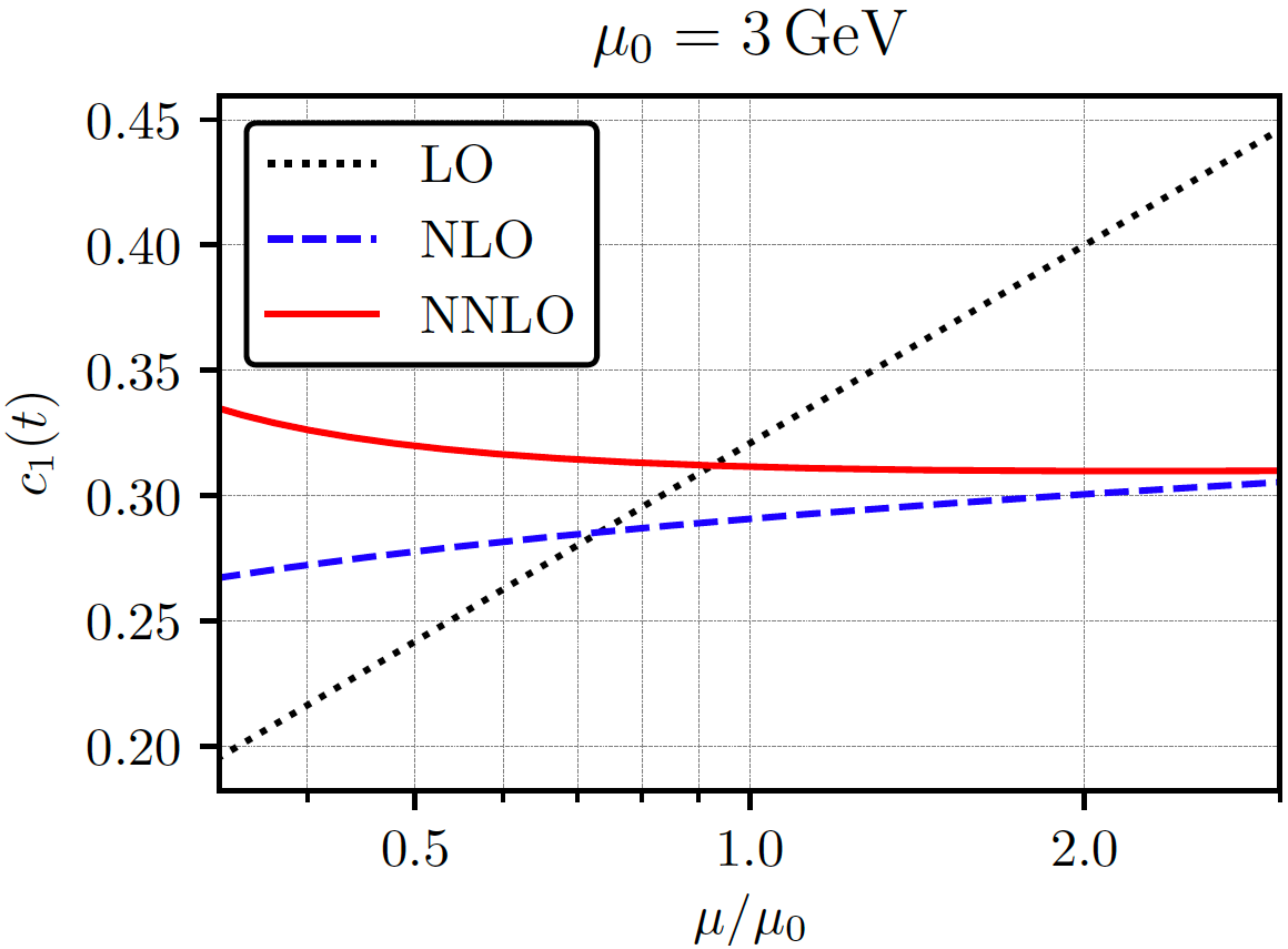
idea and NLO result: H. Suzuki '14

NNLO result

$$
\begin{aligned}
c_1(t) = \frac{1}{g^2}\Bigg\{ 1 &+ \frac{g^2}{(4\pi)^2}\left[-\frac{7}{3}C_A + \frac{3}{2}T_F - \beta_0\, L(\mu,t)\right] \\
&+ \frac{g^4}{(4\pi)^4}\Bigg[-\beta_1\, L(\mu,t) + C_A^2\left(-\frac{14482}{405} - \frac{16546}{135}\ln 2 + \frac{1187}{10}\ln 3\right) \\
&\quad + C_A T_F\left(\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{10873}{810} + \frac{73}{54}\pi^2 - \frac{2773}{135}\ln 2 + \frac{302}{45}\ln 3\right) \\
&\quad + C_F T_F\left(-\frac{256}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{2587}{108} - \frac{7}{9}\pi^2 - \frac{106}{9}\ln 2 - \frac{161}{18}\ln 3\right)\Bigg] \\
&+ \mathcal{O}(g^6)\Bigg\}\,, \qquad L(\mu,t) \equiv \ln\left(2\mu^2 t\right) + \gamma_{\mathrm{E}}
\end{aligned}
$$

etc.

RH, Kluth, Lange ‘18

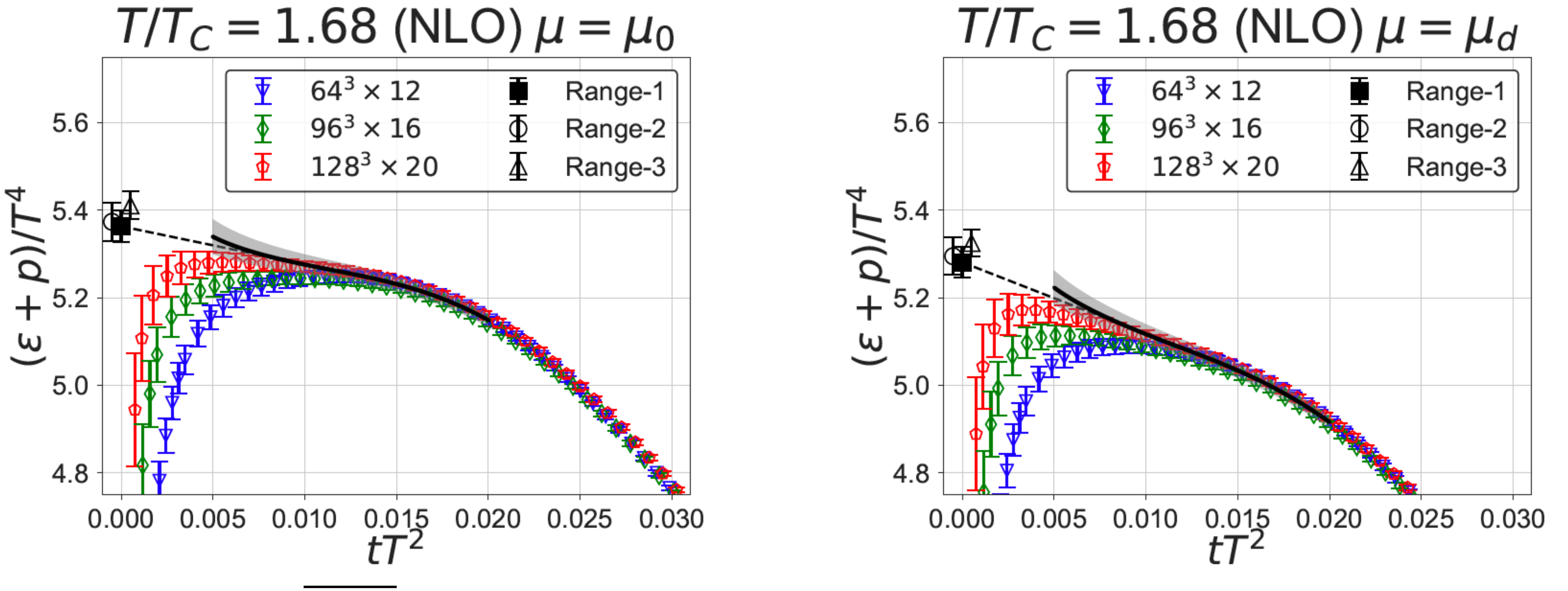

$\mu_0 = 3\,\mathrm{GeV}$
LO
NLO
NNLO
0.45
0.40
0.35
0.30
0.25
0.20
$c_1(t)$
0.5
1.0
2.0
μ/μ_0

Application

Entropy density: $$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

$$\mu_0 = \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$$

$$\mu_d = \frac{1}{\sqrt{8t}}$$



Iritani, Kitazawa, Suzuki, Takaura 2019

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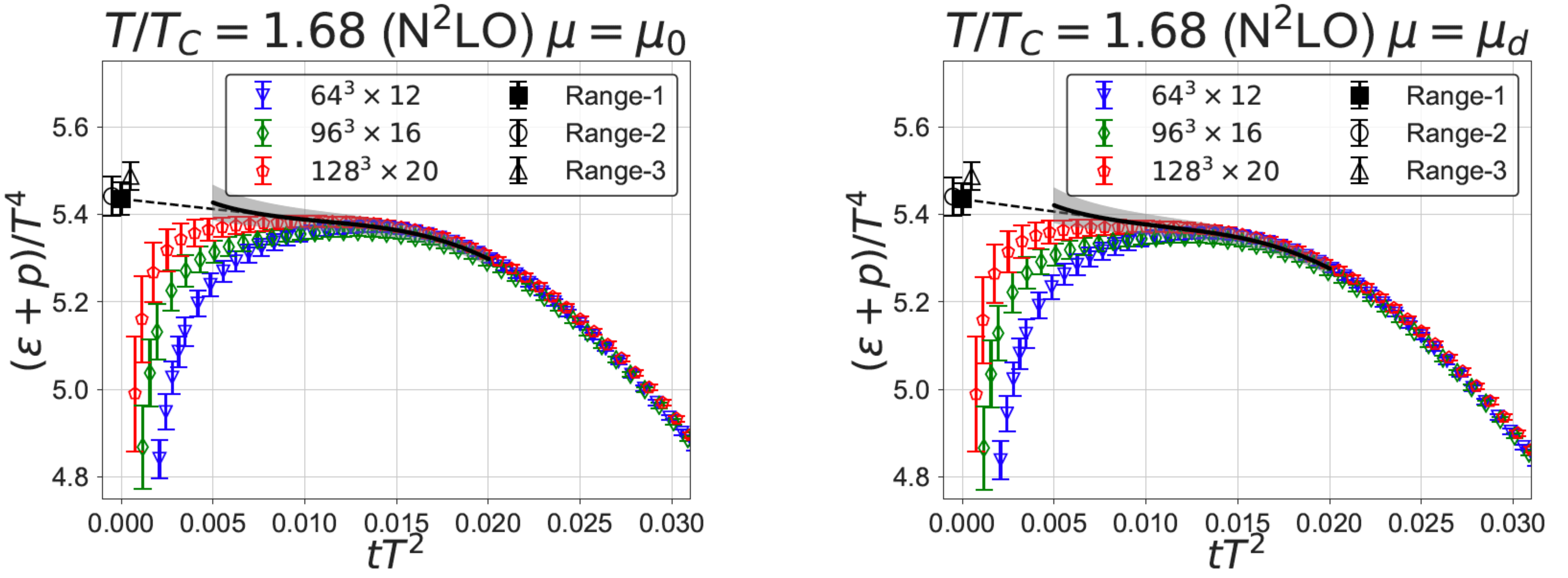
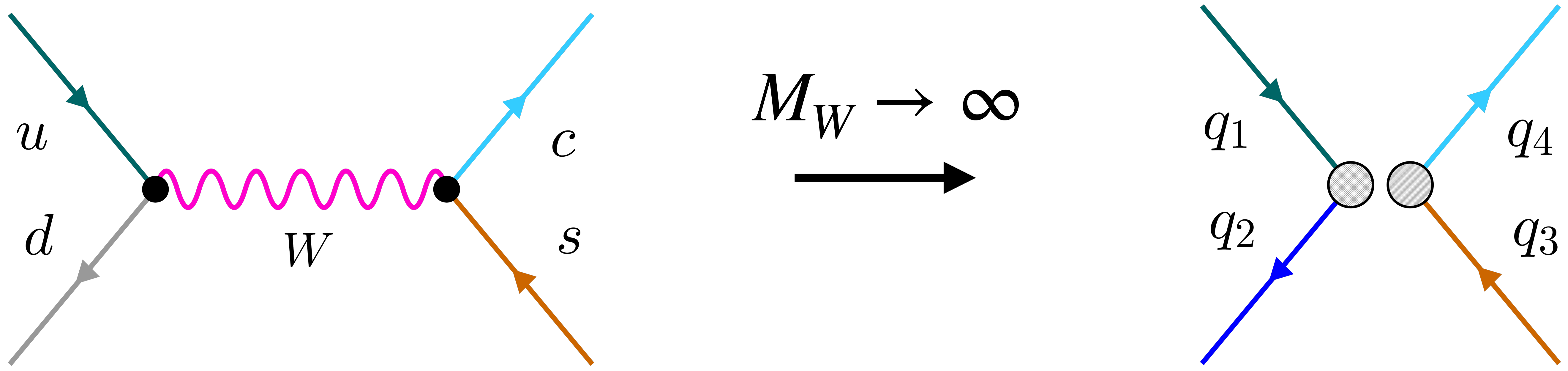


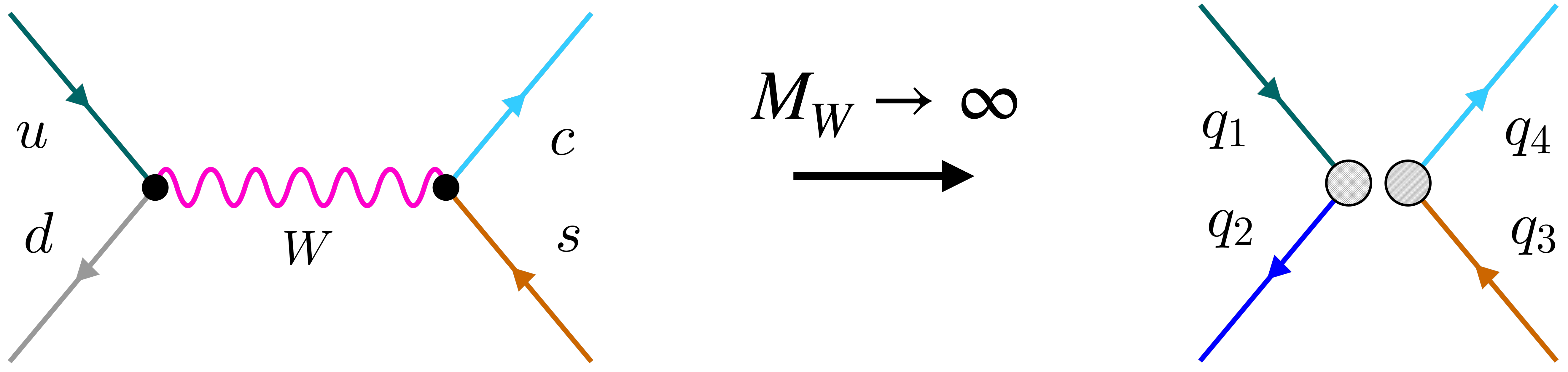
Iritani, Kitazawa, Suzuki, Takaura 2019

Application to EFT



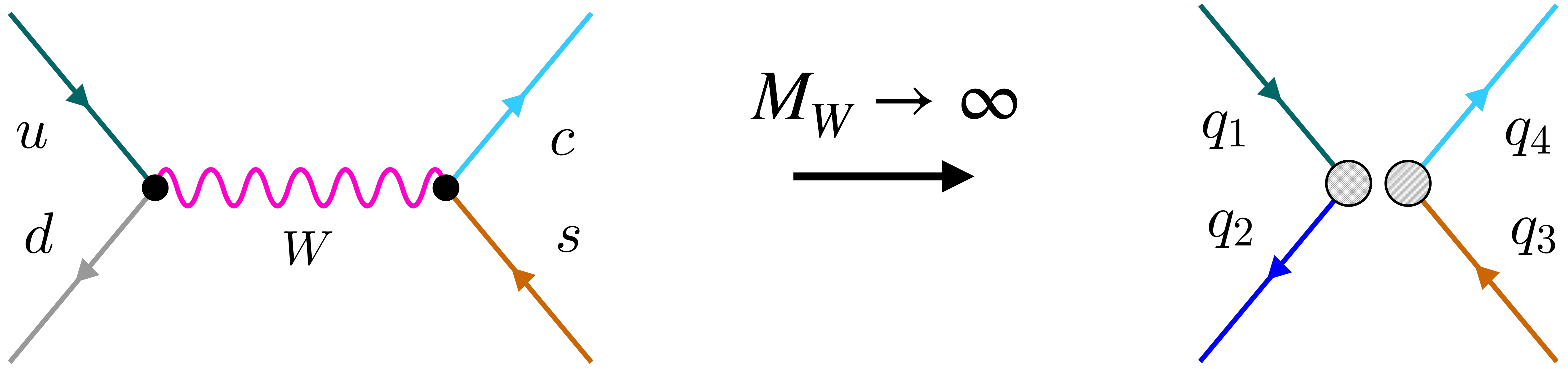
$$\sum_n C_n^B \mathcal{O}_n$$

Application to EFT



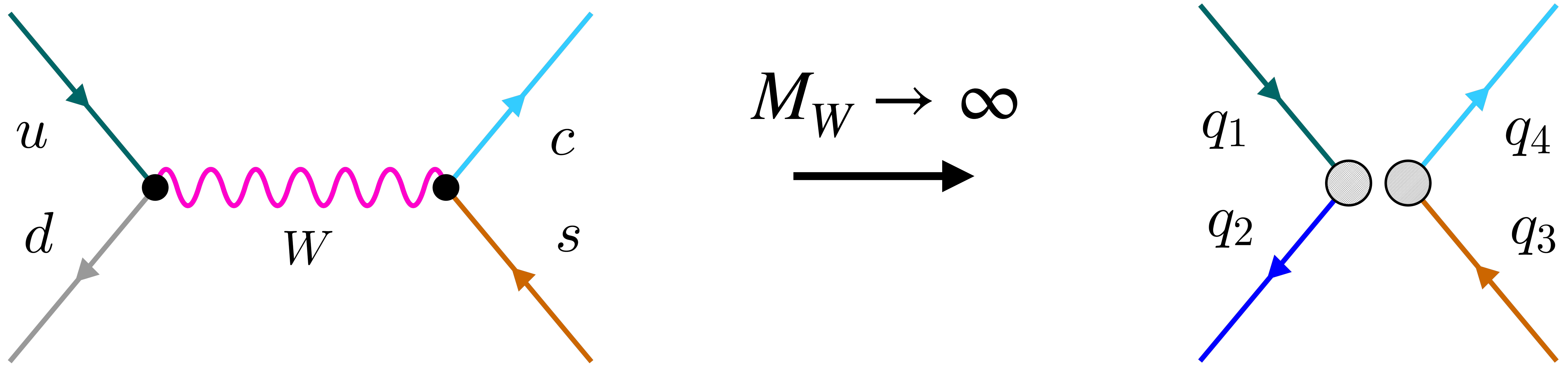
$$\sum_n C_n^B \mathcal{O}_n \equiv \sum_n \tilde{C}(t)_n \tilde{\mathcal{O}}(t)_n$$

Application to EFT



$$\sum_n C_n^B \mathcal{O}_n \equiv \sum_n \tilde{C}(t)_n \tilde{\mathcal{O}}(t)_n = \sum_n (C\zeta^{-1}(t))_n (\zeta(t)\mathcal{O}^R)_n$$

Application to EFT



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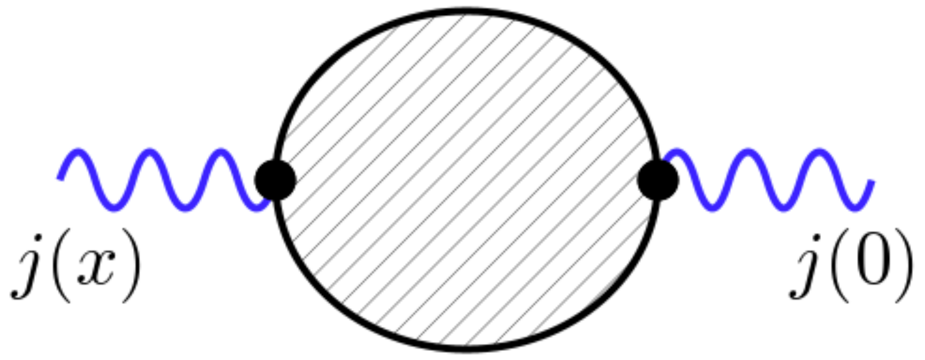
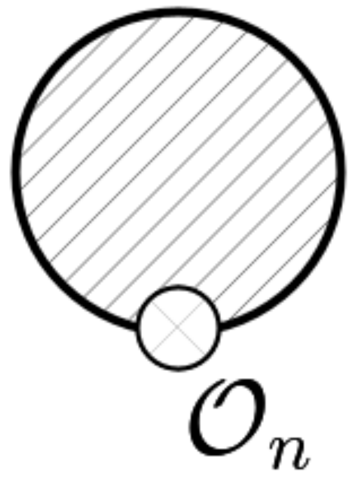
$\overline{\text{MS}}$ coefficient

Hadronic vacuum polarization

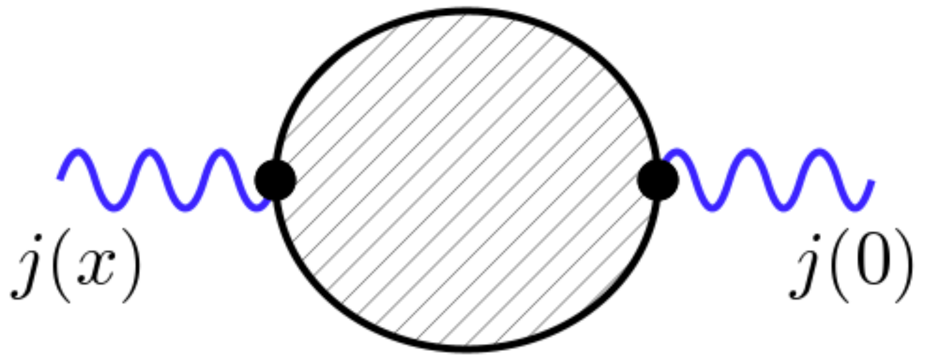
$$\int d^4x\, e^{iQx} \langle T\, j(x)\, j(0) \rangle \rightarrow \sum_n C_n(Q) \langle \mathcal{O}_n(x=0) \rangle$$

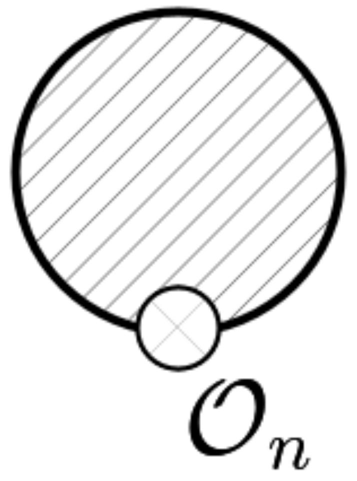




Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T\, j(x)\, j(0)\rangle \to \sum_n C_n(Q)\langle \mathcal{O}_n(x=0)\rangle = \sum_n \tilde{C}_n(Q, t)\langle \tilde{\mathcal{O}}_n(t)\rangle$$





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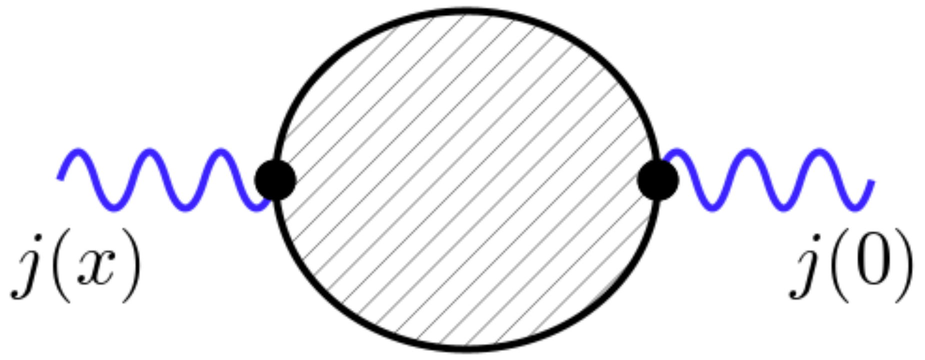
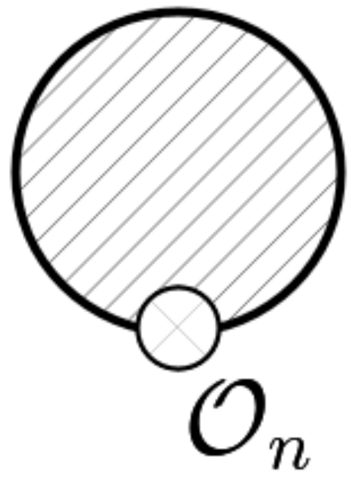
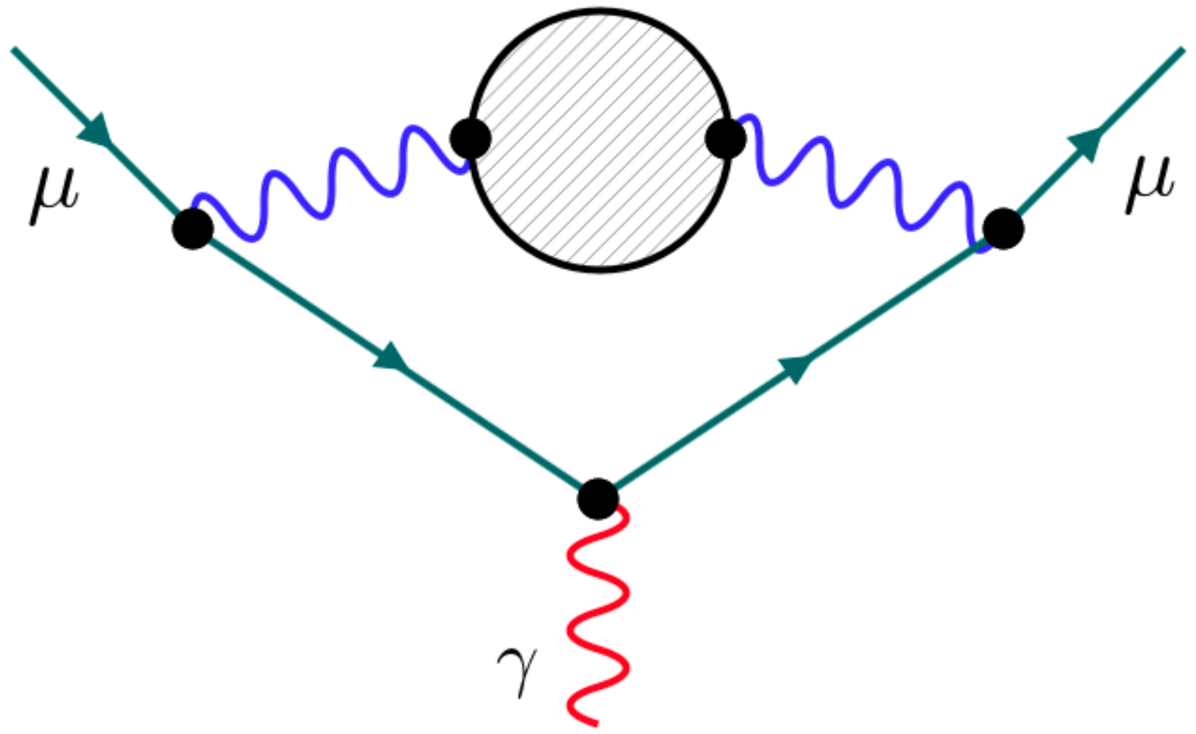


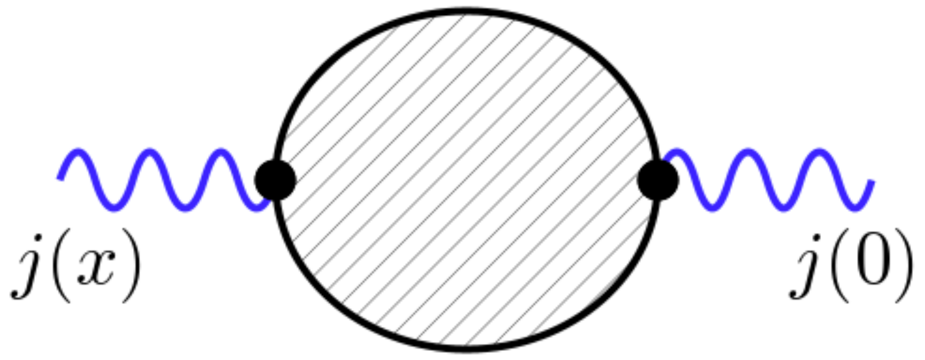


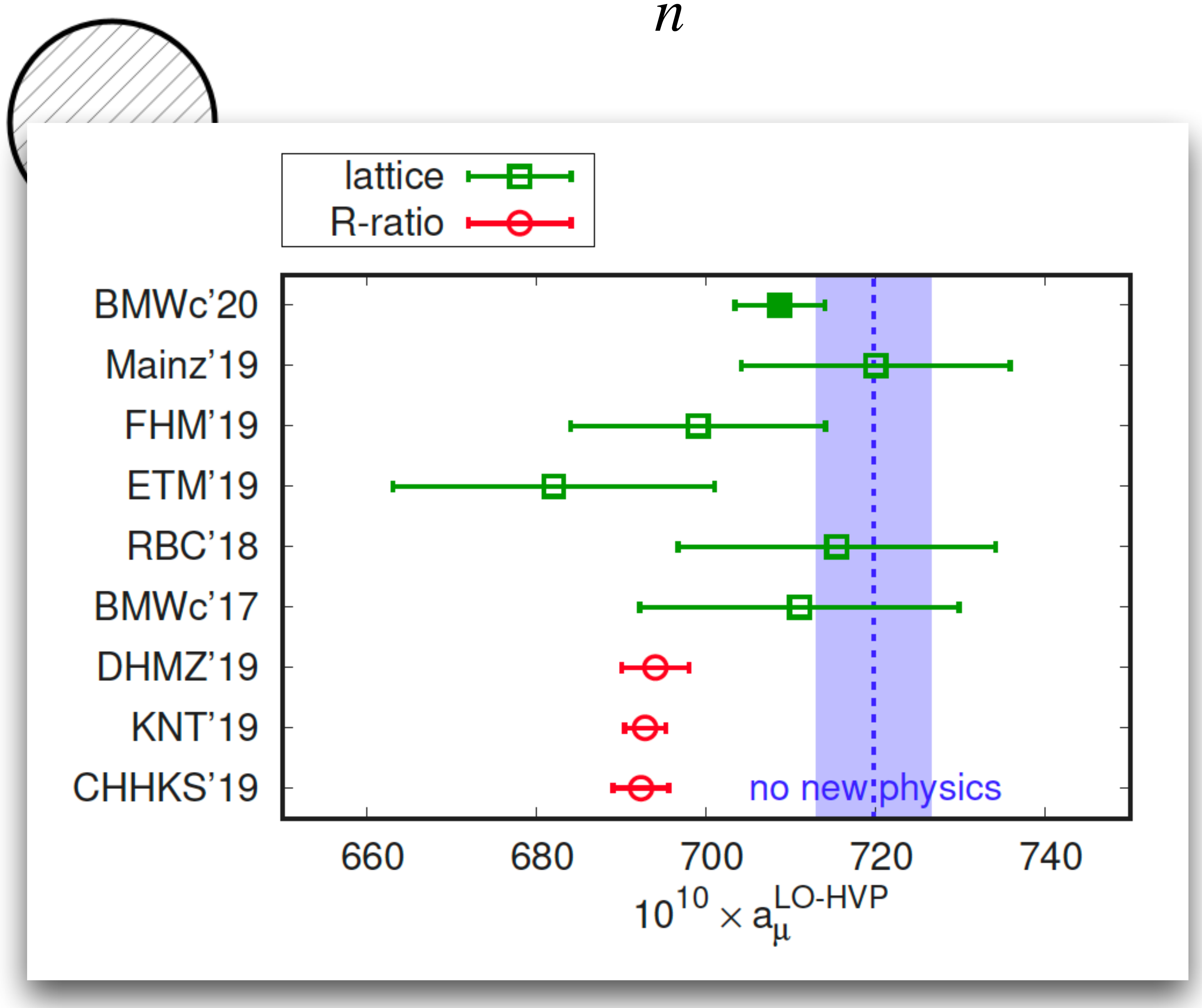
contribution to (g-2)mu



Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T j(x) j(0)\rangle \to \sum_n C_n(Q)\langle \mathcal{O}_n(x=0)\rangle = \sum_n \tilde{C}_n(Q,t)\langle \tilde{\mathcal{O}}_n(t)\rangle$$





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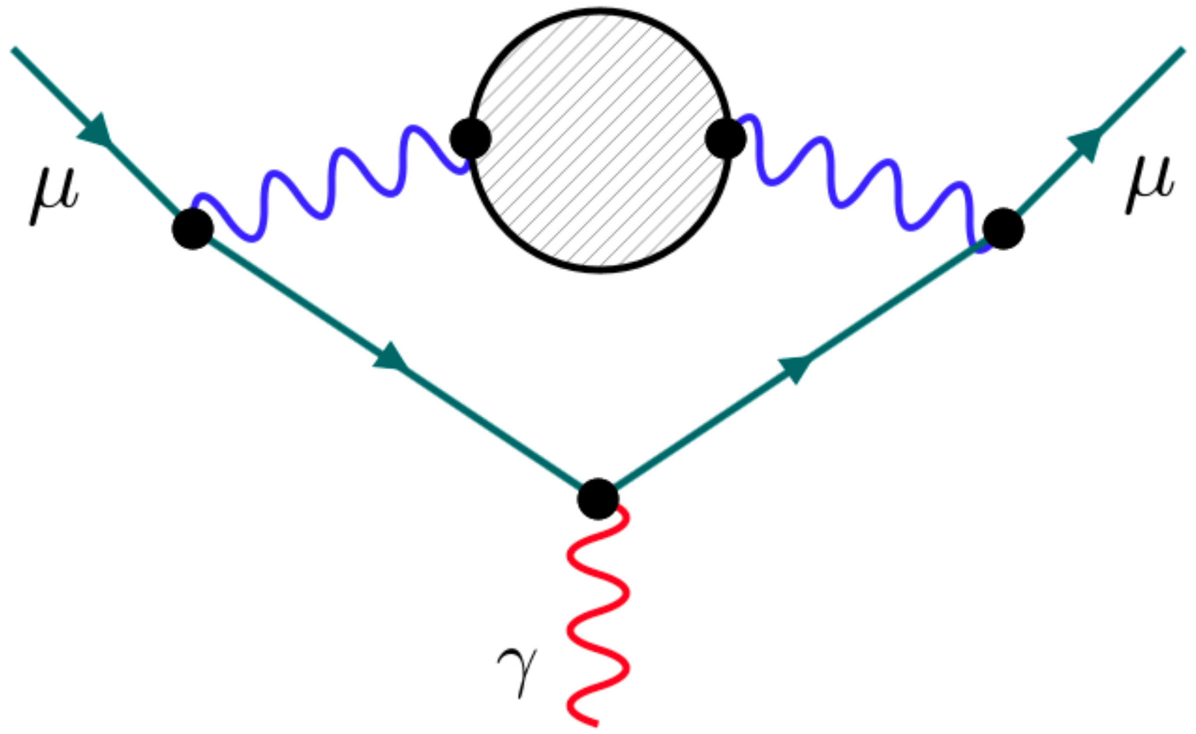

Renormalization

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m$$

Renormalization

$$\underset{\text{finite}}{\tilde{\mathcal{O}}_n(t)} = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m$$

Renormalization

$$\tilde{\mathcal{O}}_n({\color{red}t}) = \sum_m \zeta^B_{nm}({\color{red}t})\mathcal{O}_m$$

finite divergent

Renormalization

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)\mathcal{O}$$

finite divergent

Renormalization

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finite divergent

$$\mathcal{O}^R = Z\,\mathcal{O}$$

Renormalization

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finite divergent

$$\mathcal{O}^R = Z\,\mathcal{O} \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)Z^{-1}Z\mathcal{O} = \zeta^B(t)Z^{-1}\,\mathcal{O}^R = \zeta(t)\mathcal{O}^R$$

Renormalization

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$$P_k[\tilde{\mathcal{O}}_n(t)] = \sum_m \zeta^B_{nm}(t)P_k[\mathcal{O}_m] = \zeta^B_{nk}(t)$$

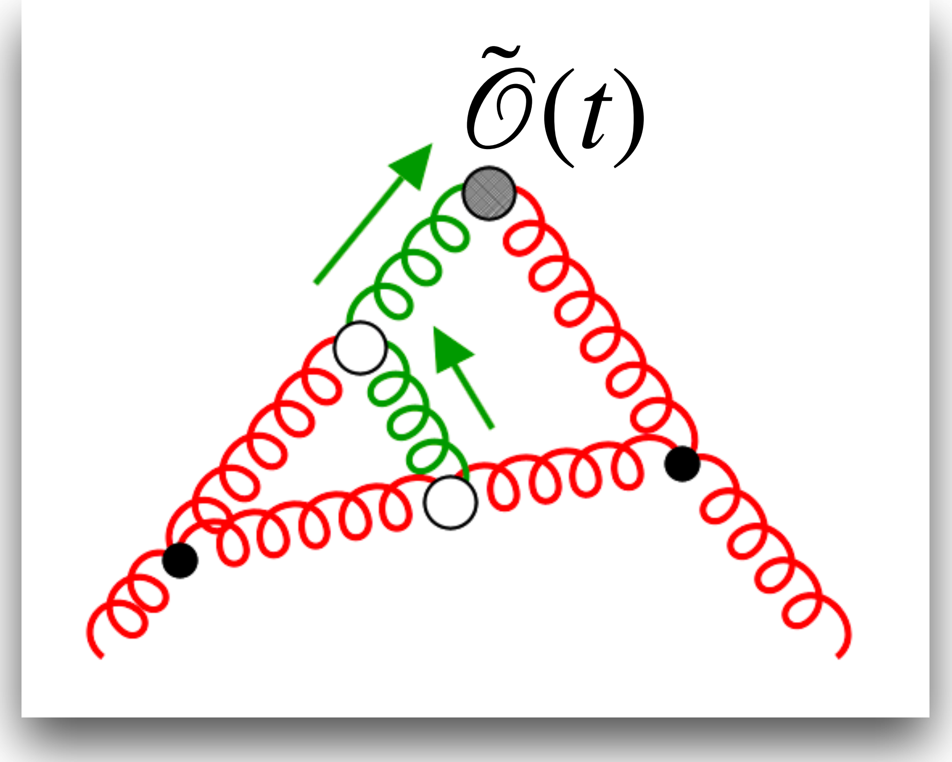
Renormalization

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)\mathcal{O}$$

finite divergent

$$\mathcal{O}^R = Z\,\mathcal{O} \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)Z^{-1}Z\mathcal{O} = \zeta^B(t)Z^{-1}\,\mathcal{O}^R = \zeta(t)\mathcal{O}^R$$

$$P_k[\tilde{\mathcal{O}}_n(t)] = \sum_m \zeta^B_{nm}(t)P_k[\mathcal{O}_m] = \zeta^B_{nk}(t)$$



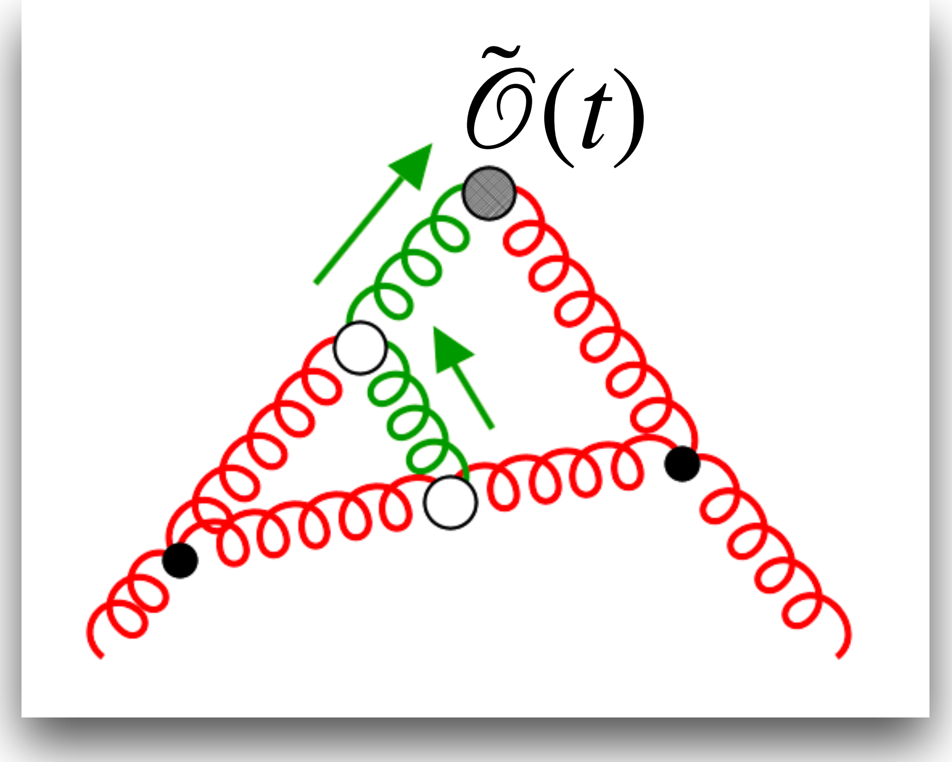
Renormalization

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)\mathcal{O}$$

finite divergent

$$\mathcal{O}^R = Z\,\mathcal{O} \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)Z^{-1}Z\mathcal{O} = \zeta^B(t)Z^{-1}\,\mathcal{O}^R = \zeta(t)\mathcal{O}^R$$

$$P_k[\tilde{\mathcal{O}}_n(t)] = \sum_m \zeta^B_{nm}(t)P_k[\mathcal{O}_m] = \zeta^B_{nk}(t)$$



UV = 0

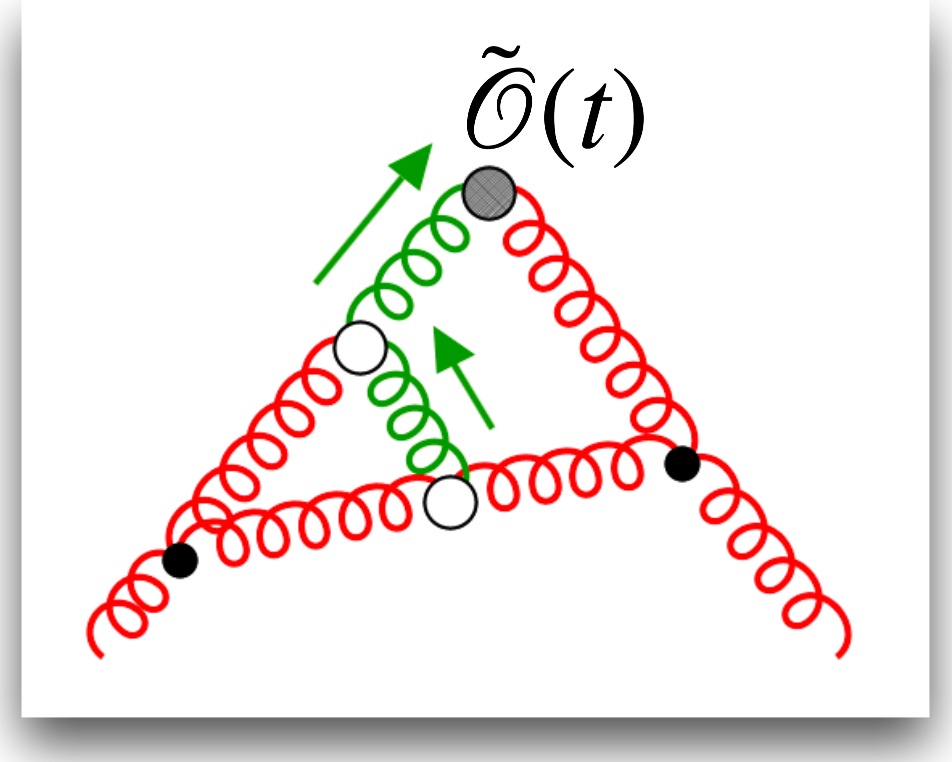
Renormalization

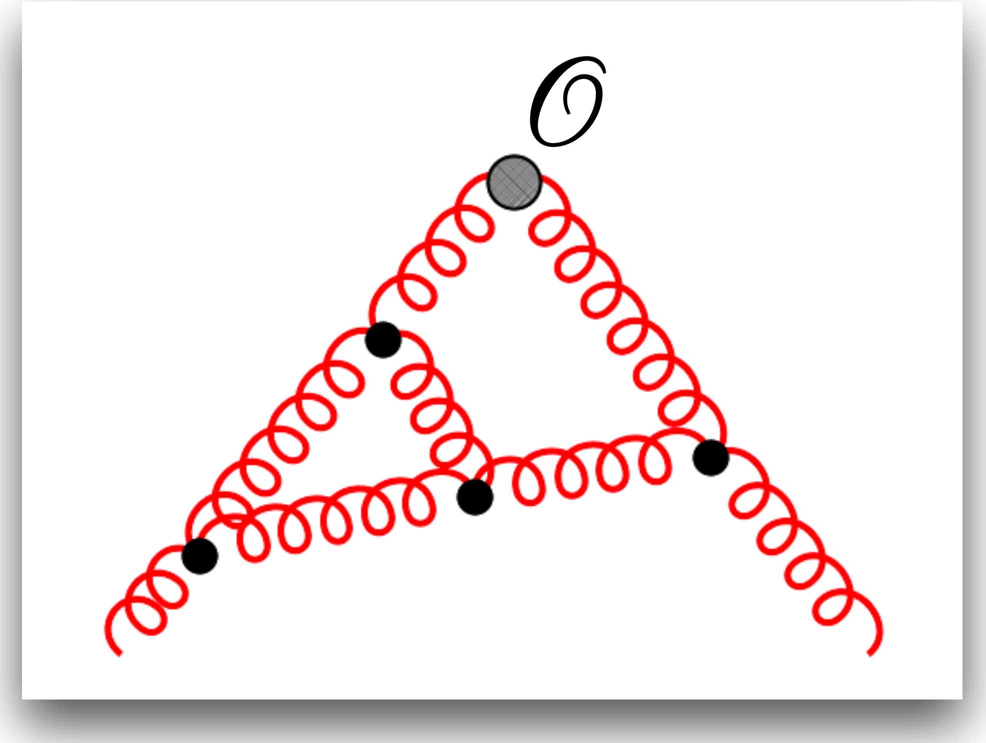
$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)\mathcal{O}$$

finite divergent

$$\mathcal{O}^R = Z\,\mathcal{O} \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)Z^{-1}Z\mathcal{O} = \zeta^B(t)Z^{-1}\,\mathcal{O}^R = \zeta(t)\mathcal{O}^R$$

$$P_k[\tilde{\mathcal{O}}_n(t)] = \sum_m \zeta^B_{nm}(t)P_k[\mathcal{O}_m] = \zeta^B_{nk}(t)$$





UV = 0

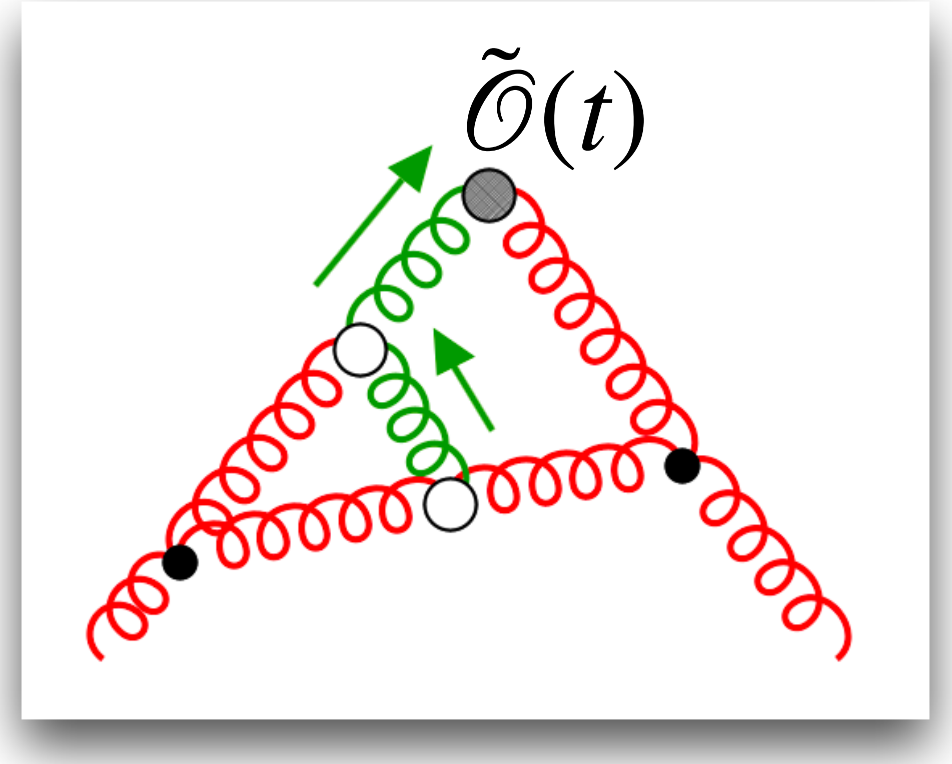
Renormalization

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)\mathcal{O}$$

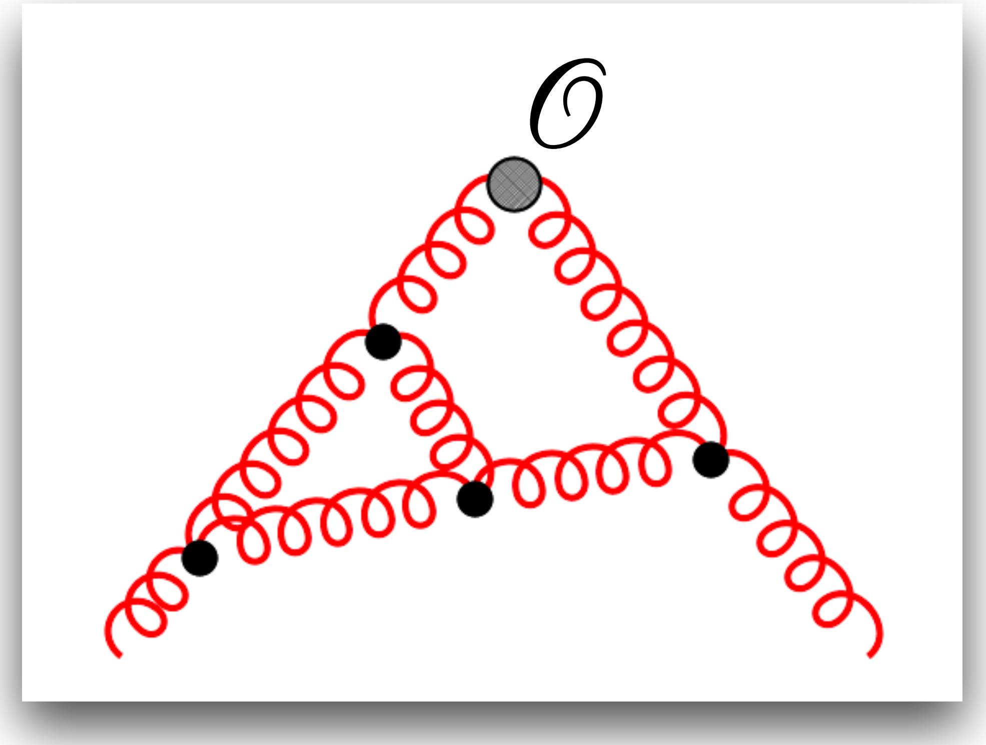
finite divergent

$$\mathcal{O}^R = Z\,\mathcal{O} \qquad \tilde{\mathcal{O}}(t) = \zeta^B(t)Z^{-1}Z\mathcal{O} = \zeta^B(t)Z^{-1}\,\mathcal{O}^R = \zeta(t)\mathcal{O}^R$$

$$P_k[\tilde{\mathcal{O}}_n(t)] = \sum_m \zeta^B_{nm}(t)P_k[\mathcal{O}_m] = \zeta^B_{nk}(t)$$



UV = 0



UV + IR = 0

Summary

- Gradient Flow is a (relatively) new tool
- Extremely successful in lattice QCD
- Perturbative approach not yet fully explored