The perturbative gradient flow

Robert Harlander
RWTH Aachen University

What is the gradient flow?
Perturbative solution
Effective Field Theories
Calculational techniques

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY
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Effective Field Theories

$$\mathscr{L} = \mathscr{L}^{\leq 4} + \sum_{d>4} \frac{1}{\Lambda^{d-4}} \sum_i C_i^{(d)} \mathscr{O}_i^{(d)}$$

some problems:

- many operators (SMEFT: 2499 @ dim 6)
- get a non-redundant basis (EoMs, IbP, Fierz, Schouten, …)
- determine $C_i^{(d)}$
- determine $\langle \mathscr{O}_i^{(d)} \rangle$
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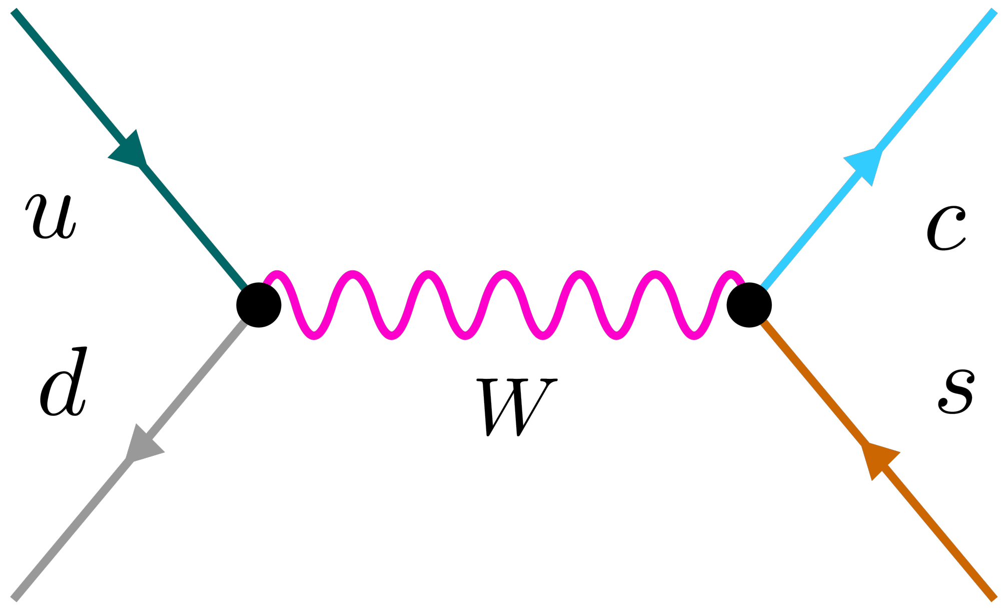
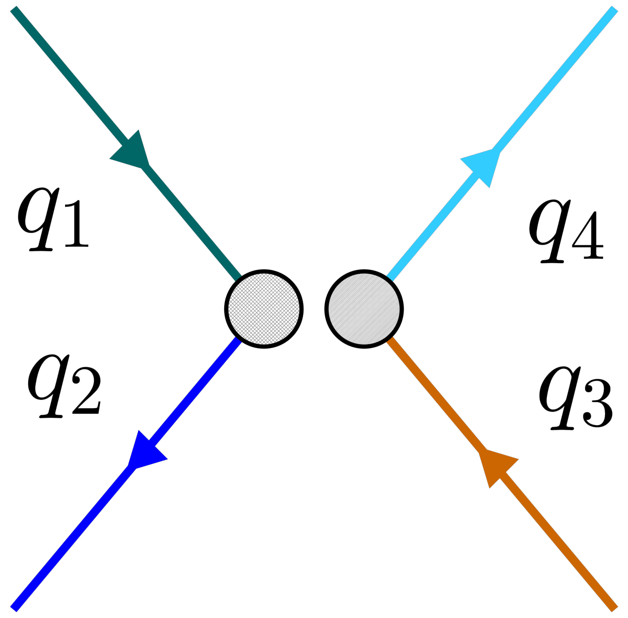
Gradient Flow

Example



$\xrightarrow{M_W \to \infty}$



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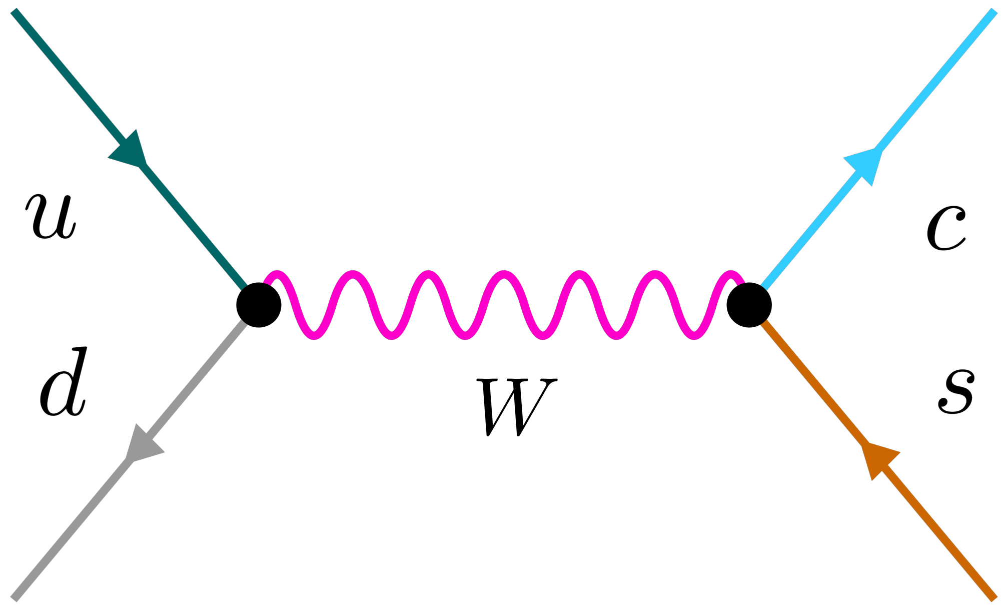

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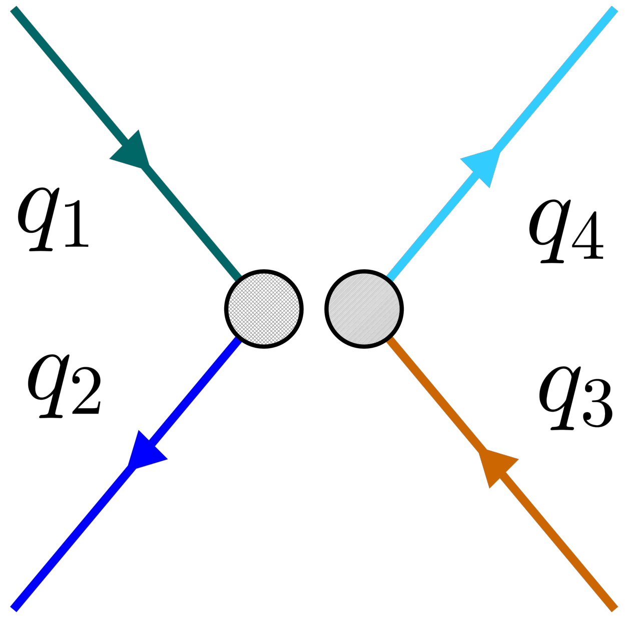

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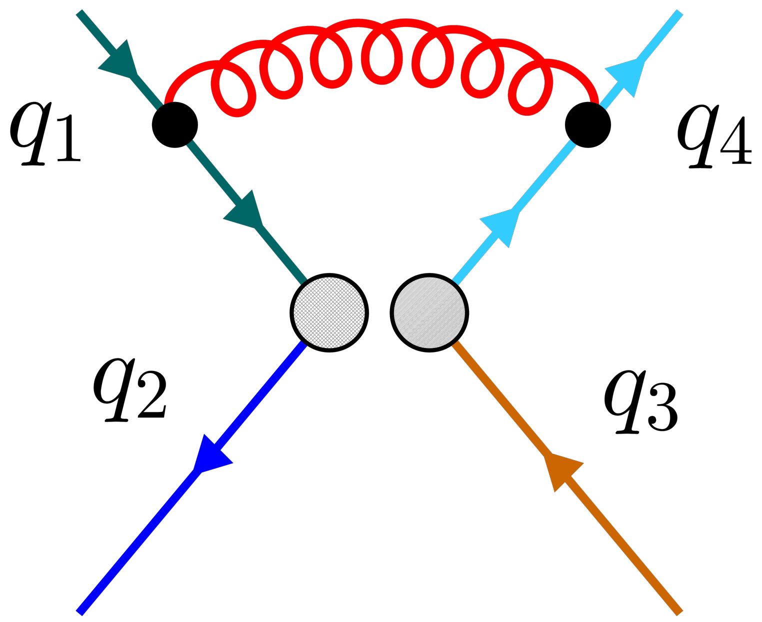

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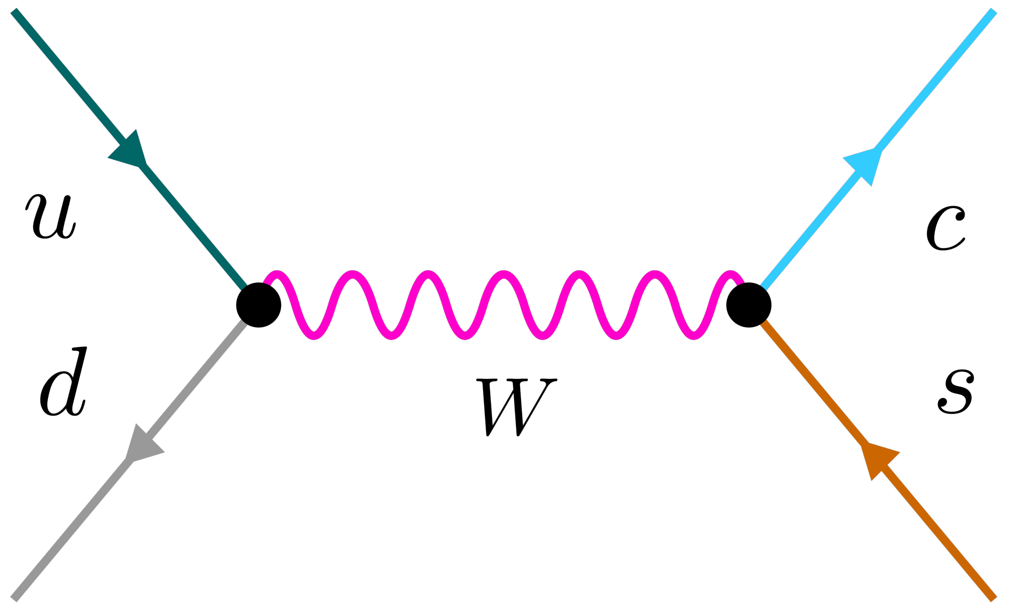

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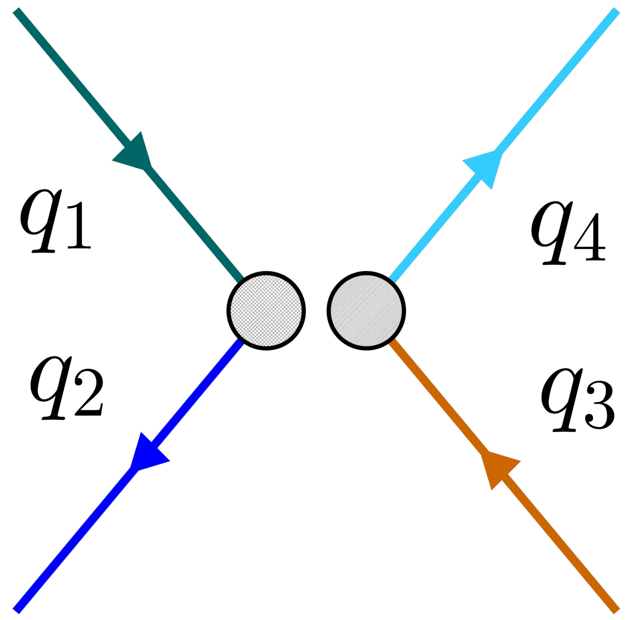

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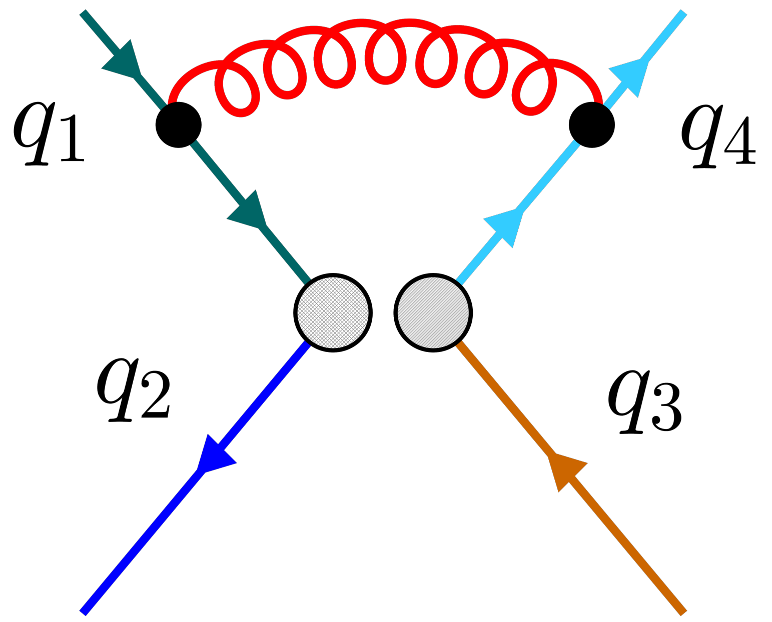

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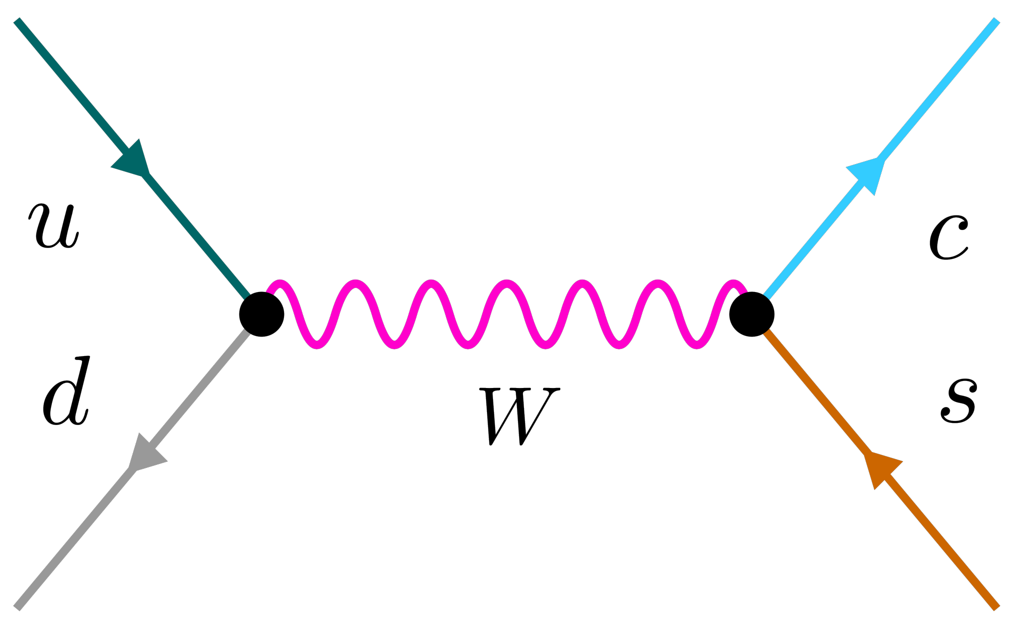

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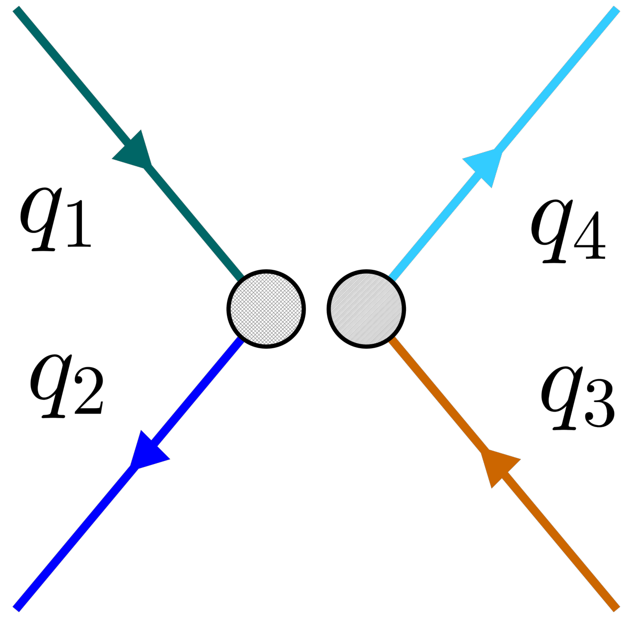

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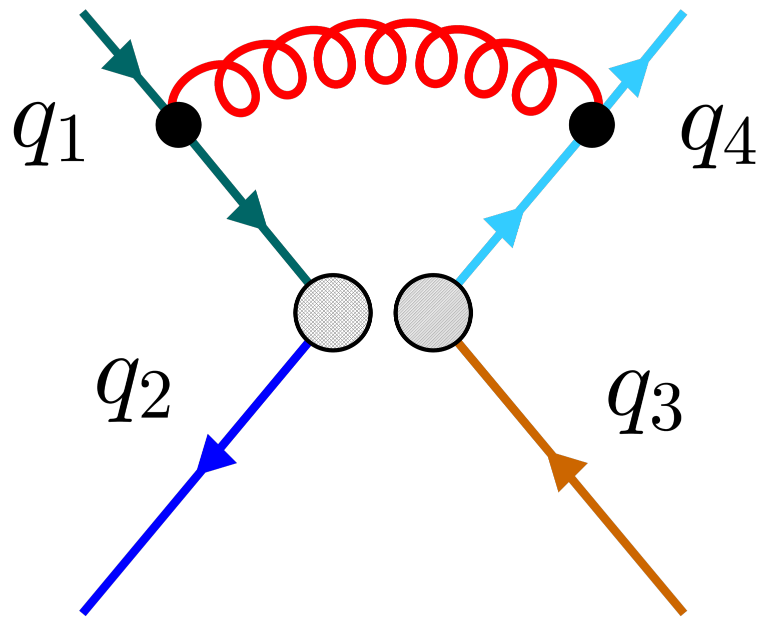
$$\mathscr{L}_{\text{eff}} \ni \sum_n C_n^B \mathcal{O}_n = \sum_n (CZ^{-1})_n \mathcal{O}_n = \sum_n (CZ^{-1})_n (Z\mathcal{O}^R)_n$$



Perturbative Renormalization

$$\mathcal{O}_n = \sum_m Z_{nm} \mathcal{O}_m^R \qquad \mathcal{O}_n^R = \sum_m (Z^{-1})_{nm} \mathcal{O}_m$$

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Determine Z: evaluate suitable Green's functions

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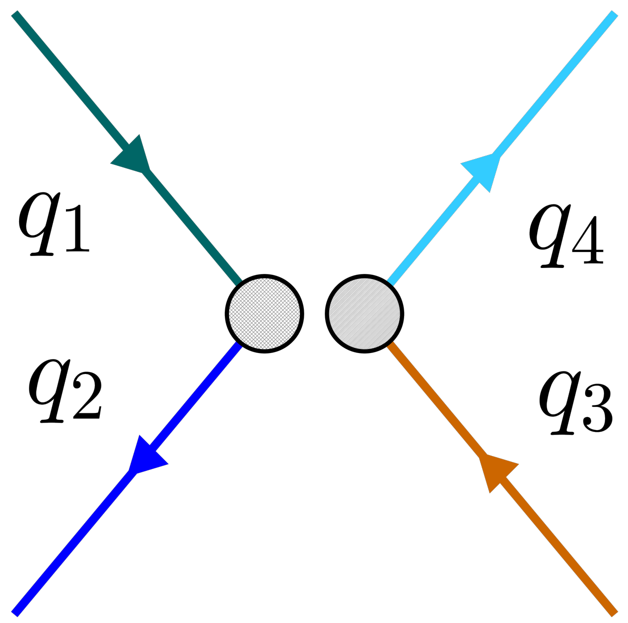

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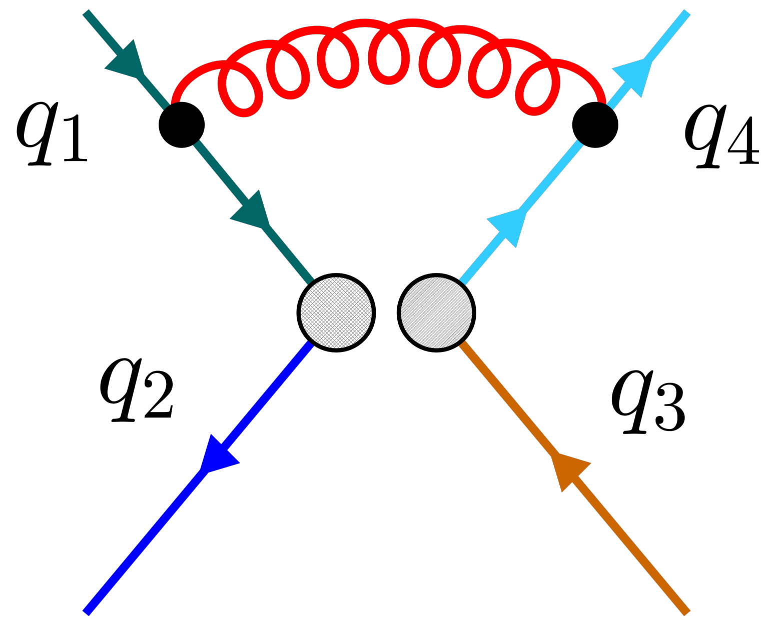

\+ …) = finite

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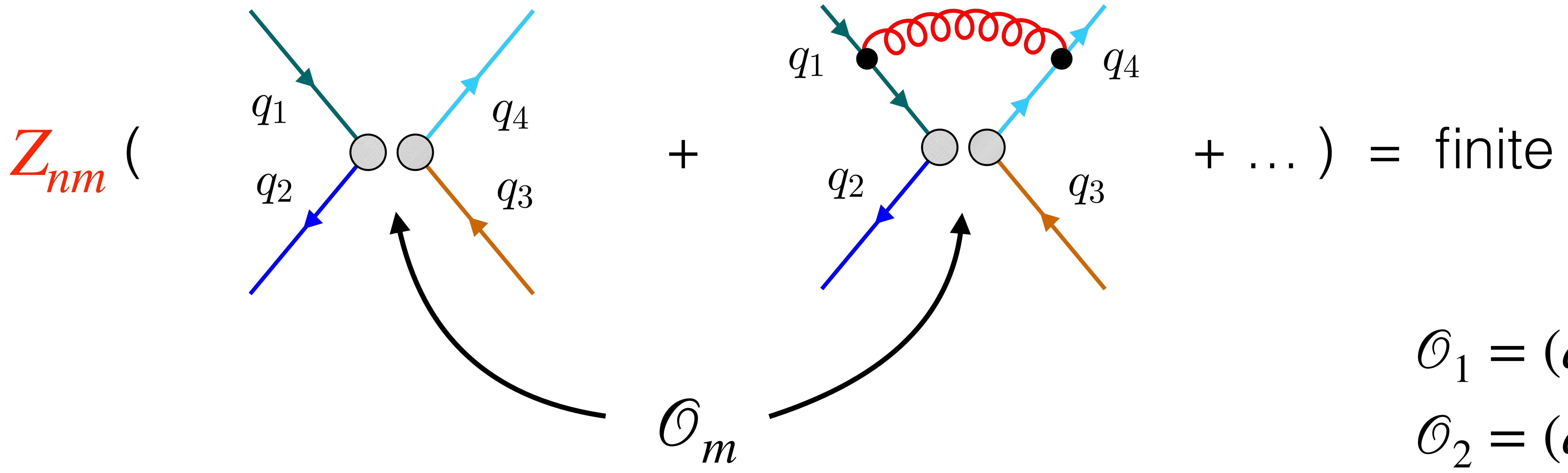

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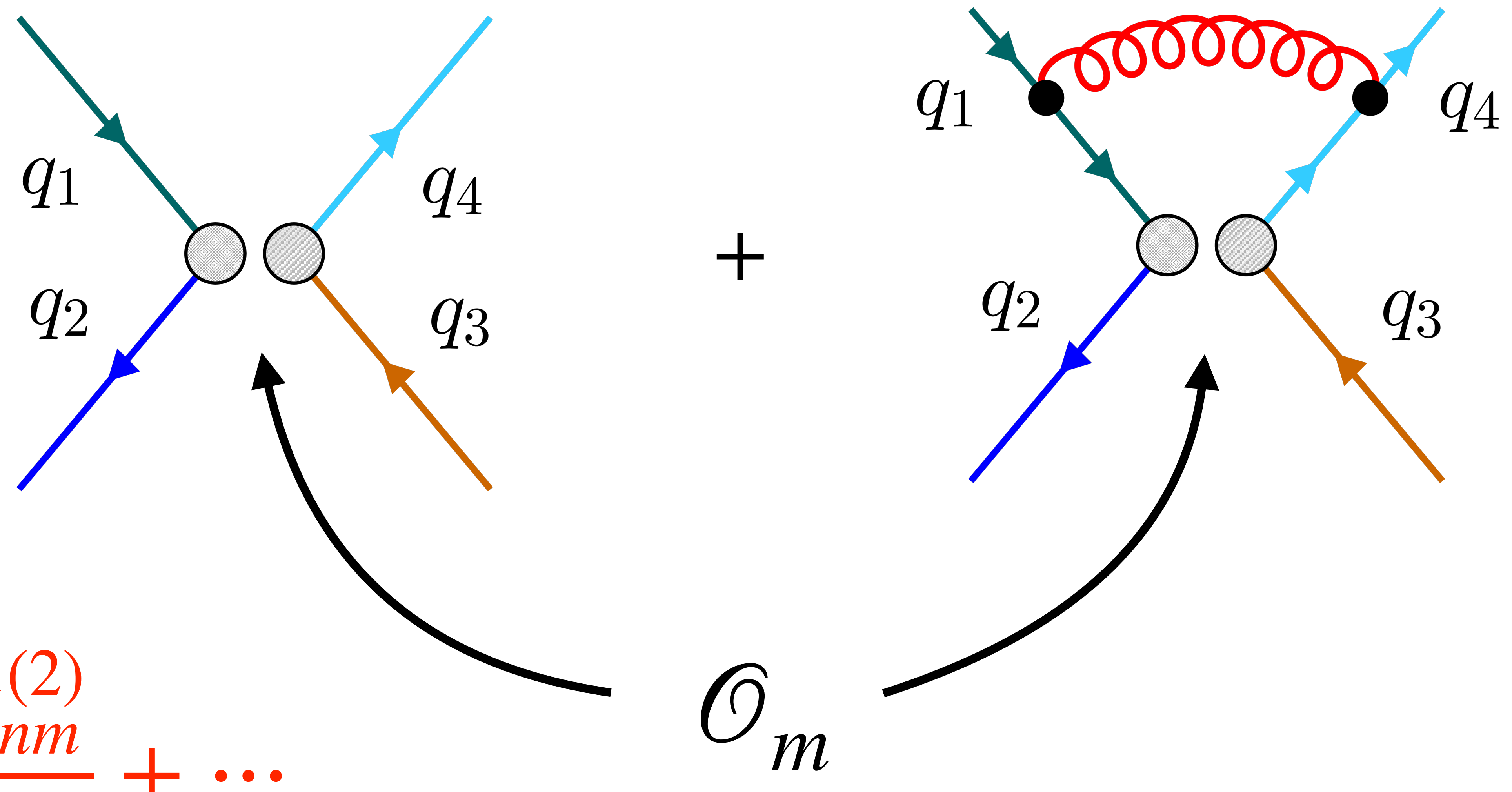

$\overline{\text{MS}}$

$$= \delta_{nm} + \frac{z_{nm}^{(1)}}{\epsilon} + \frac{z_{nm}^{(2)}}{\epsilon^2} + \cdots$$

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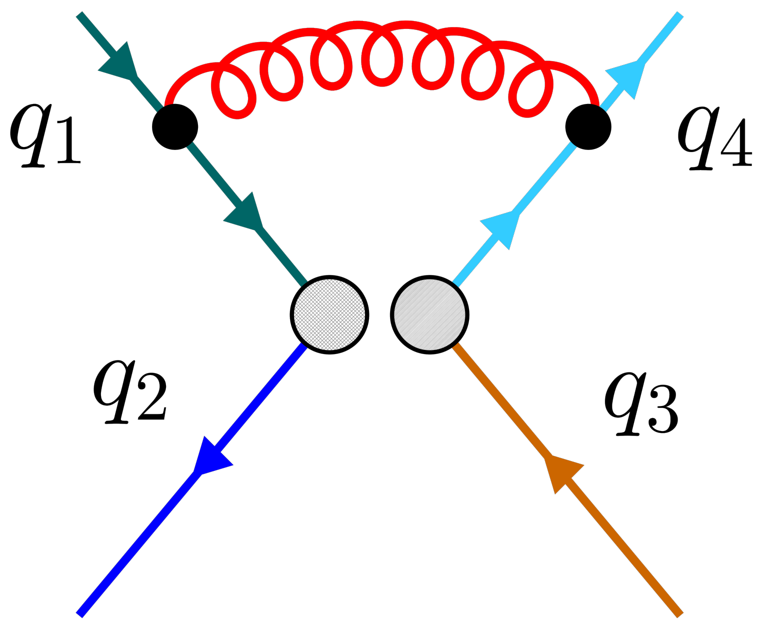

$$Z_{nm} = \delta_{nm} + \frac{z^{(1)}_{nm}}{\epsilon} + \frac{z^{(2)}_{nm}}{\epsilon^2} + \cdots$$

independent of masses and momenta

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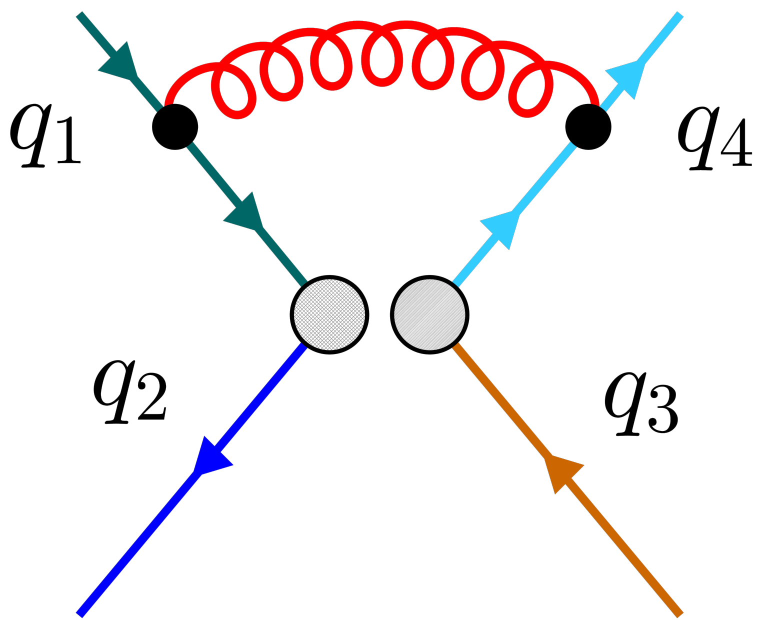

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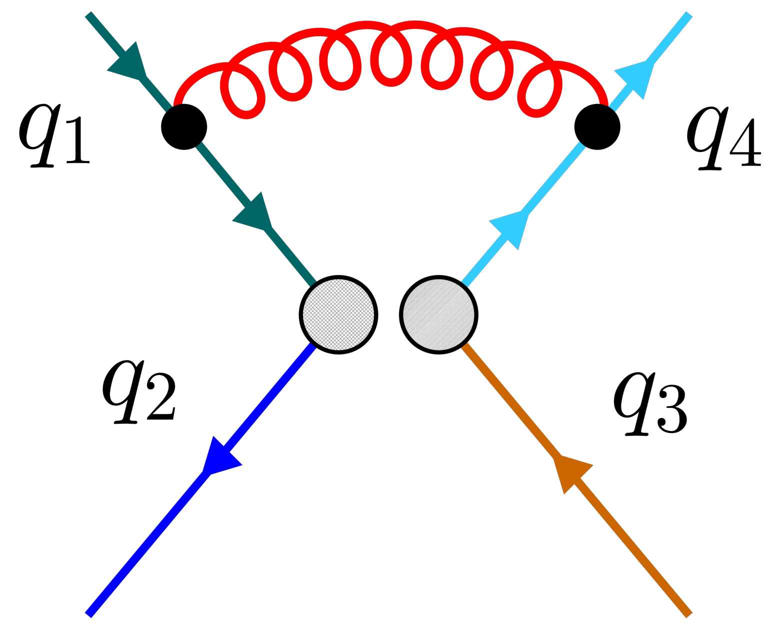

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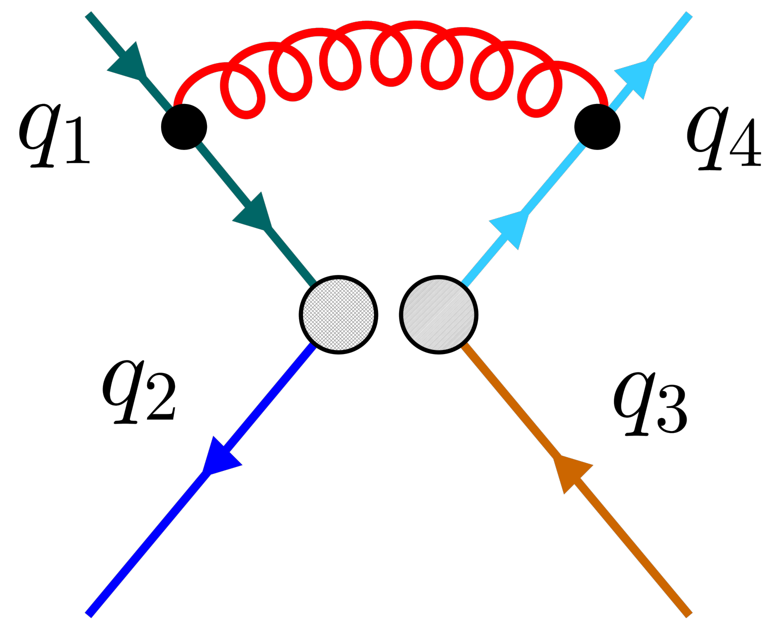

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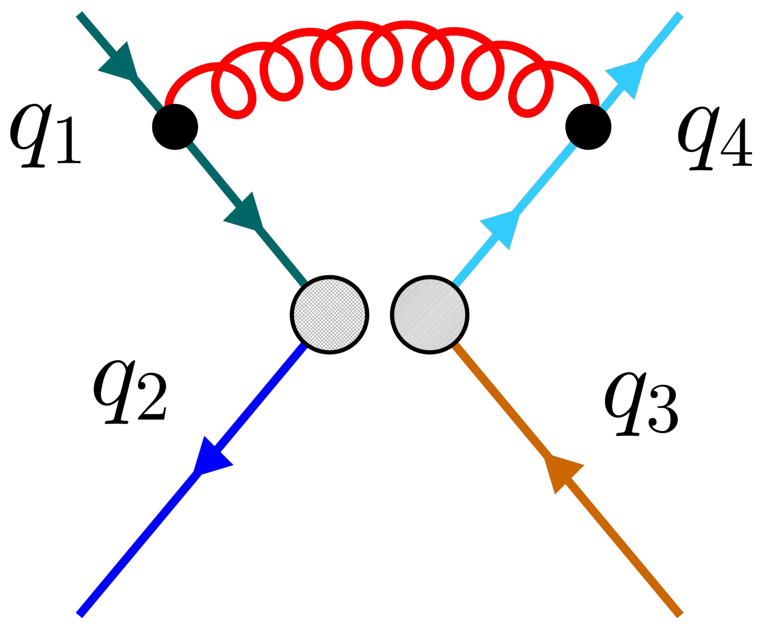

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$$\int d^D p \frac{1}{(p^2 + m^2)^a} \sim m^{D-2a}$$

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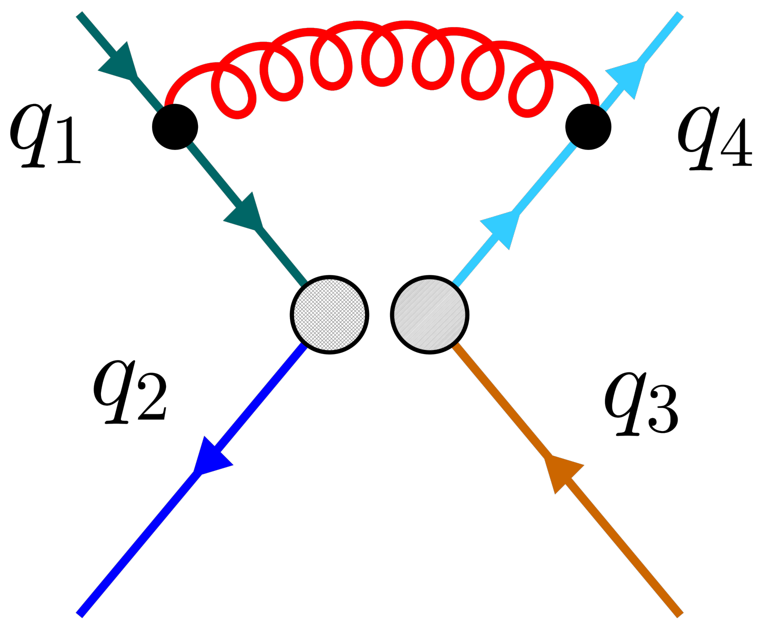

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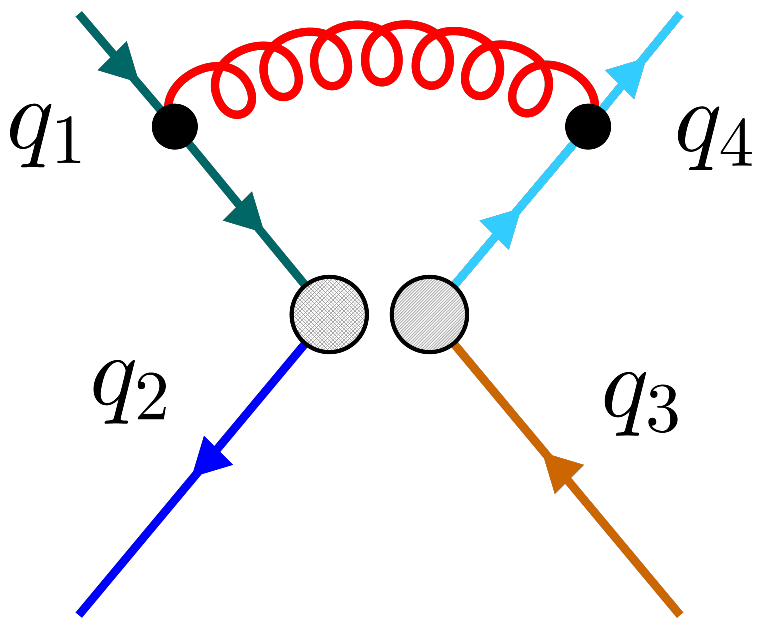

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↓ meromorphic in D

$= 0$ for $m = 0$ $\forall D$

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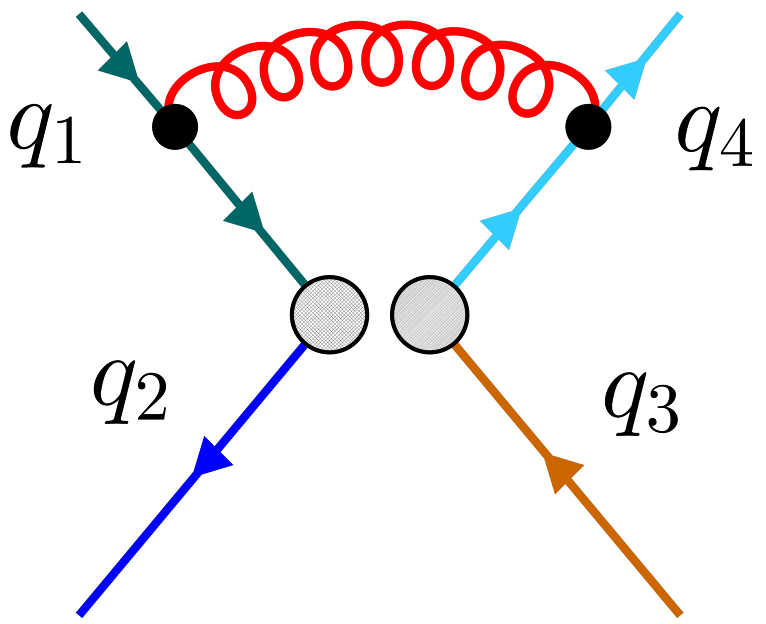

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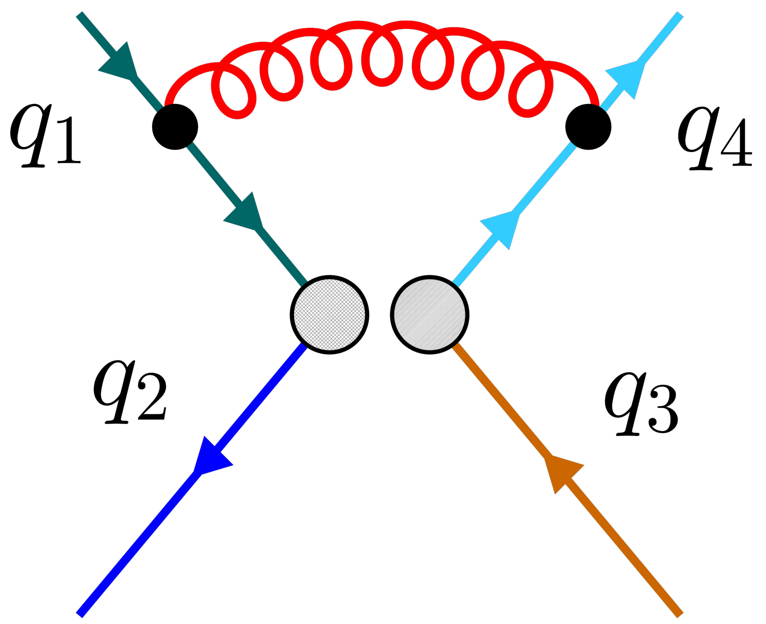

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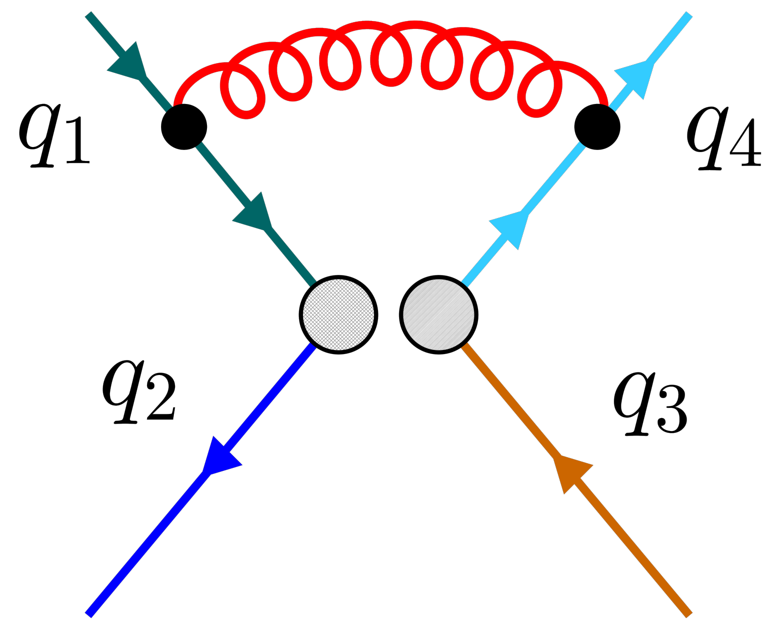

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Gradient flow is a great tool to achieve this.

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Gradient Flow

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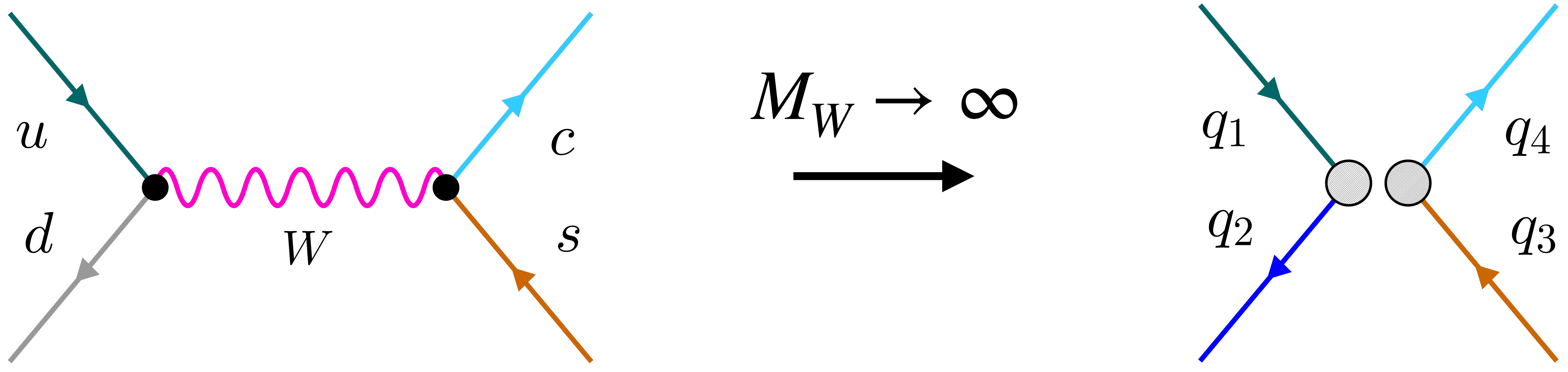

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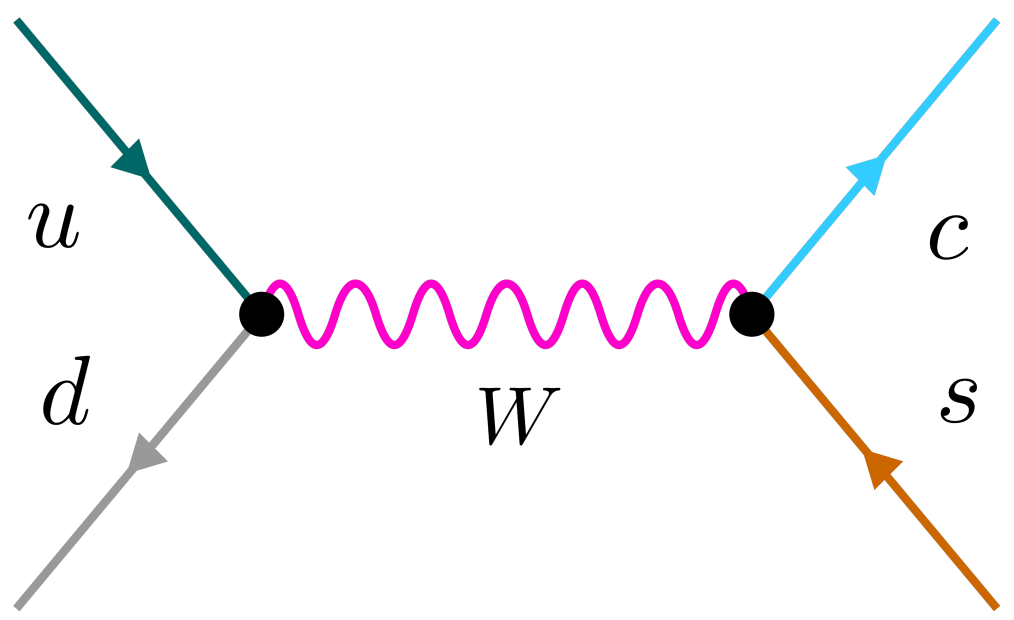

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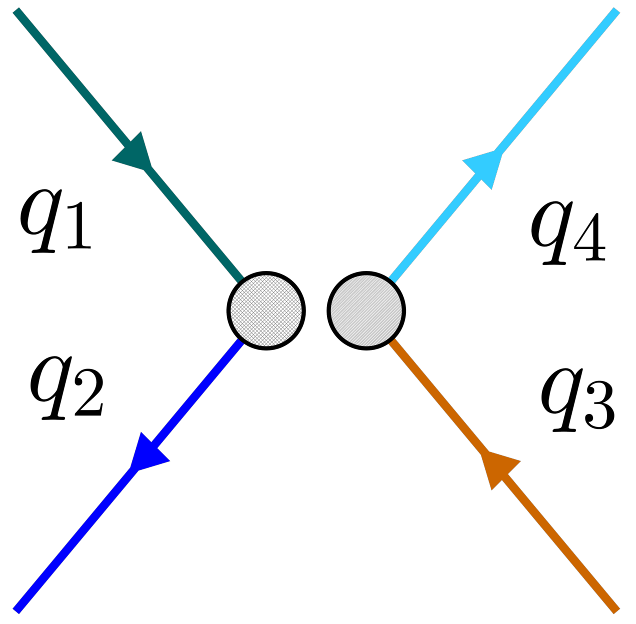

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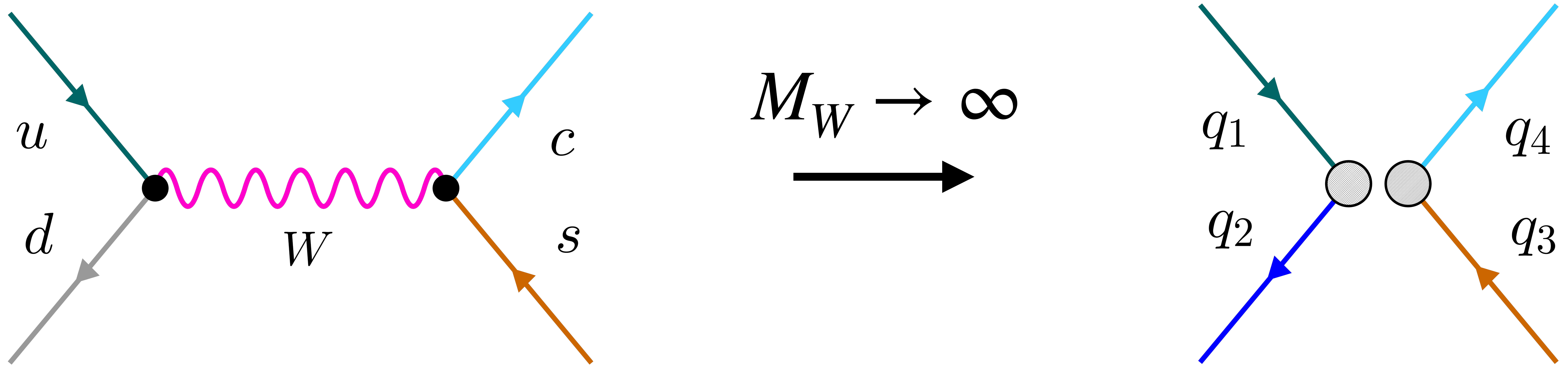

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pert.th. ↑

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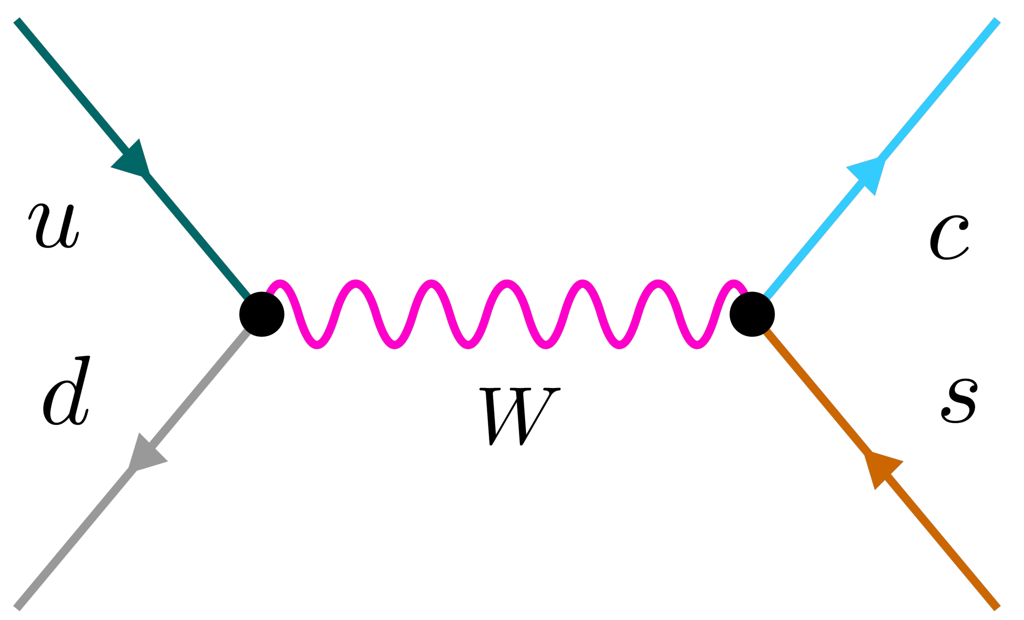

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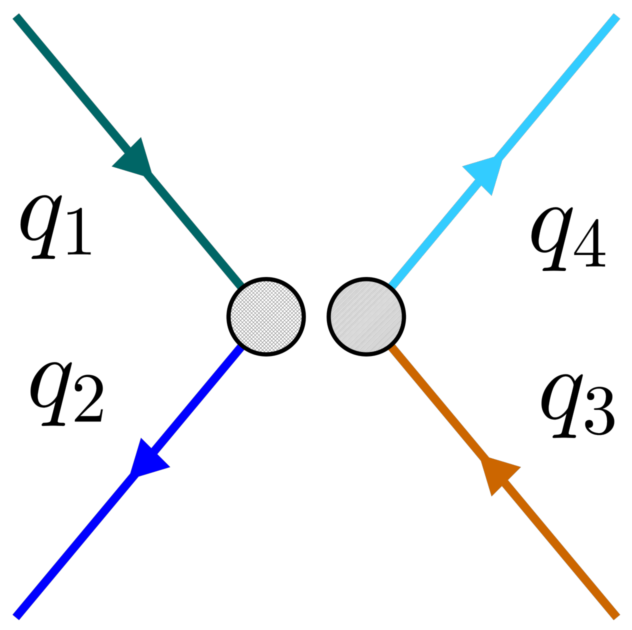

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pert.th. (C_n)

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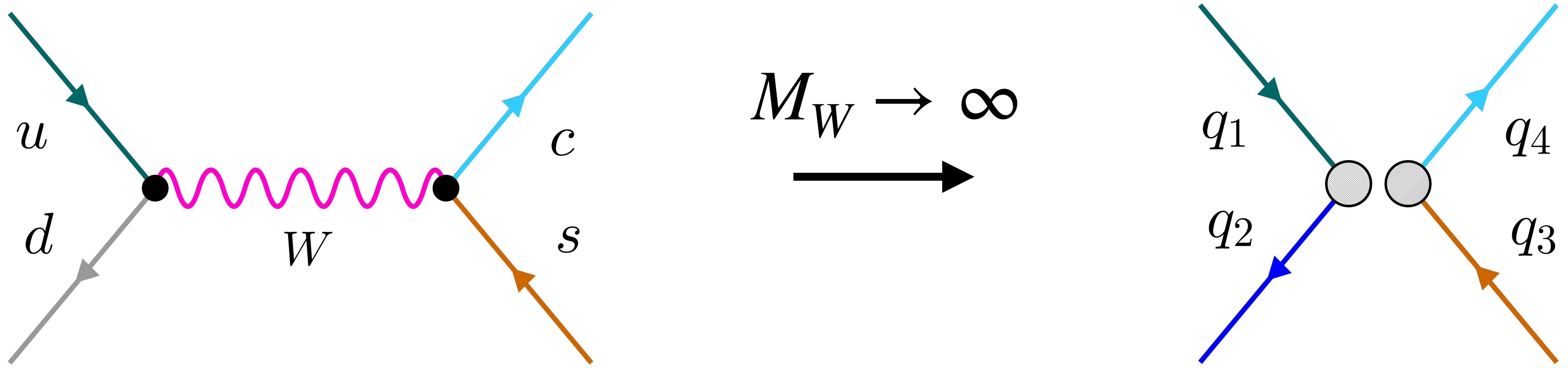

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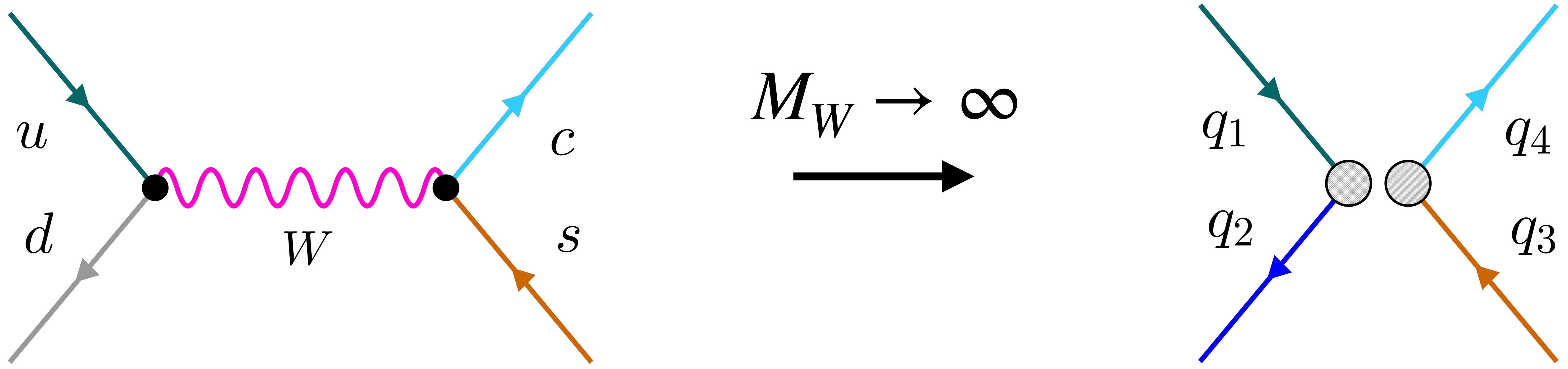

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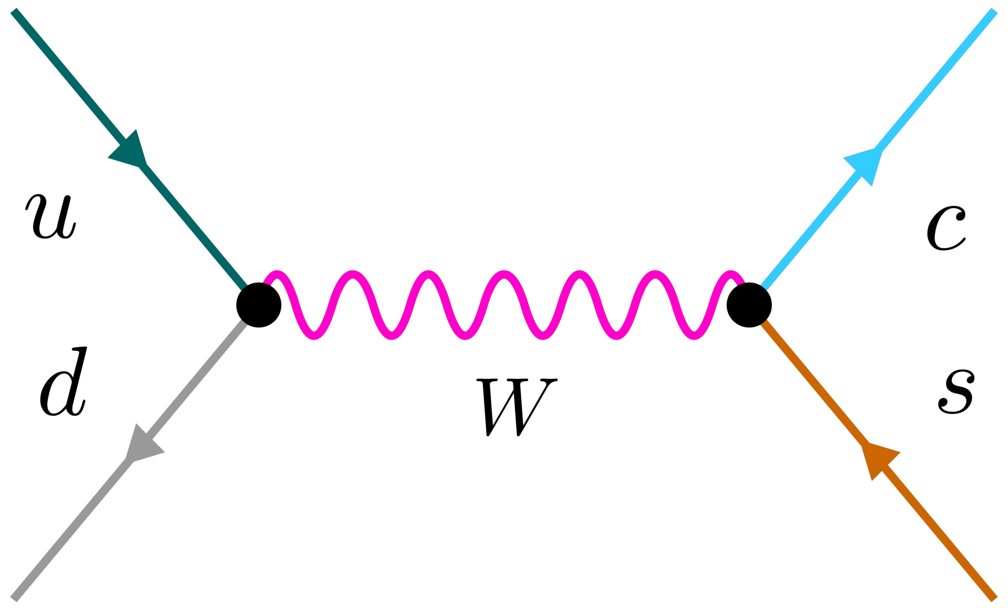

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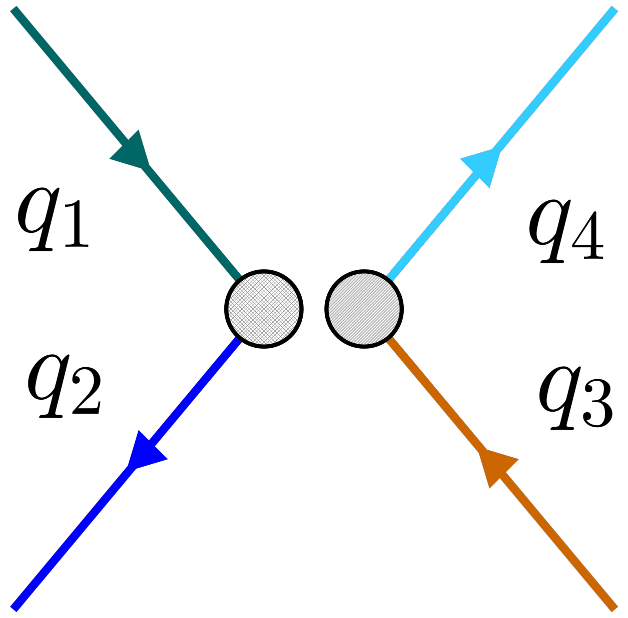

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$$q_i \to \chi_i(t)$$

$$\tilde{\mathcal{O}}_1(t), \tilde{\mathcal{O}}_2(t)$$

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pert.th.

lattice

The QCD gradient flow

Lüscher 2010, 2013

$$\frac{\partial}{\partial t}\chi(t) = \mathcal{D}^2(t)\,\chi(t) \qquad \chi(t=0) = q$$

$$\mathcal{D}_\mu(t) = \partial_\mu - ig_0 T^a B_\mu^a(t)$$

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$$\frac{\partial}{\partial t}B_\mu(t) = \mathscr{D}_\nu(t)\,G_{\nu\mu}(t) \qquad B_\mu(t=0) = A_\mu$$

$$G_{\mu\nu}(t) = \frac{i}{g_0}[\mathscr{D}_\mu(t), \mathscr{D}_\nu(t)]$$

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Schematically…

$$\frac{\partial}{\partial t} B_\mu(t) = \mathscr{D}_\nu G_{\nu\mu}(t)$$

$$\mathscr{D}_\mu = \partial_\mu - iT^a g_0 B^a_\mu(t)$$

Schematically…

$$\frac{\partial}{\partial t} B \quad = \mathscr{D} \; G$$

$$\mathscr{D} \quad = \partial \; - \quad g_0 B$$

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$$\mathscr{D} = \partial - g_0 B$$

$$G \sim \frac{1}{g_0}[\mathscr{D}, \mathscr{D}] \sim \partial B + g_0 B^2$$

Schematically…

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flow equation: $\partial_t B \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$

Perturbative solution

flow equation: $\partial_t B \sim \partial^2 B + \partial B^2 + B^3$

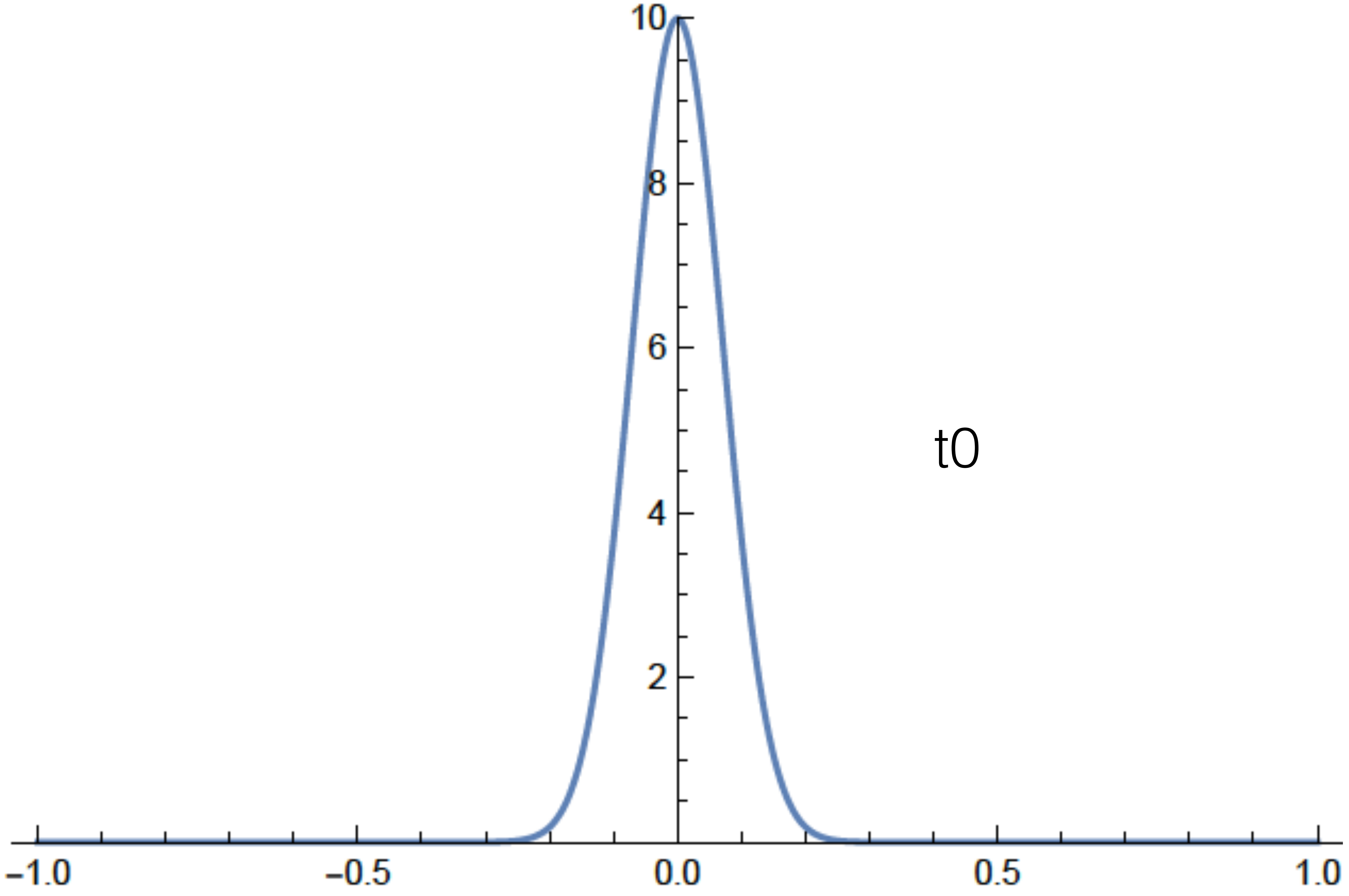
perturbative ansatz: $B = gB_1 + g^2 B_2 + \ldots$

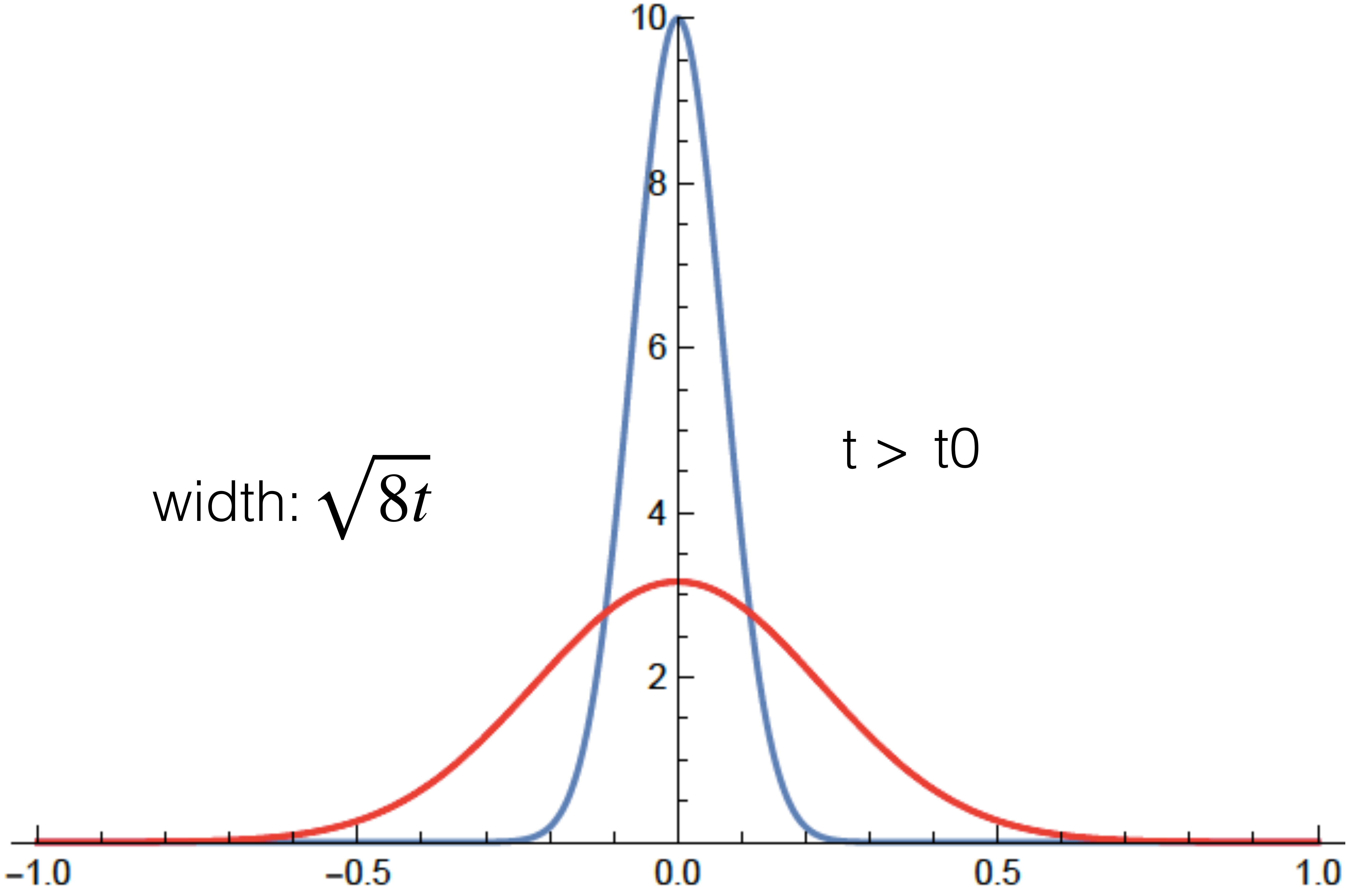
$\mathcal{O}(g)$: $\partial_t B_1 \sim \partial^2 B_1$ heat equation!

momentum space: $\tilde{B}_1(p) = e^{-tp^2} \tilde{A}(p)$

Example: $B_1(t=0,x) = c\,\delta(x)$

$$B_1(t,x) = \frac{c}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$ + higher orders


10
8
6
4
2
−1.0
−0.5
0.0
0.5
1.0
t0

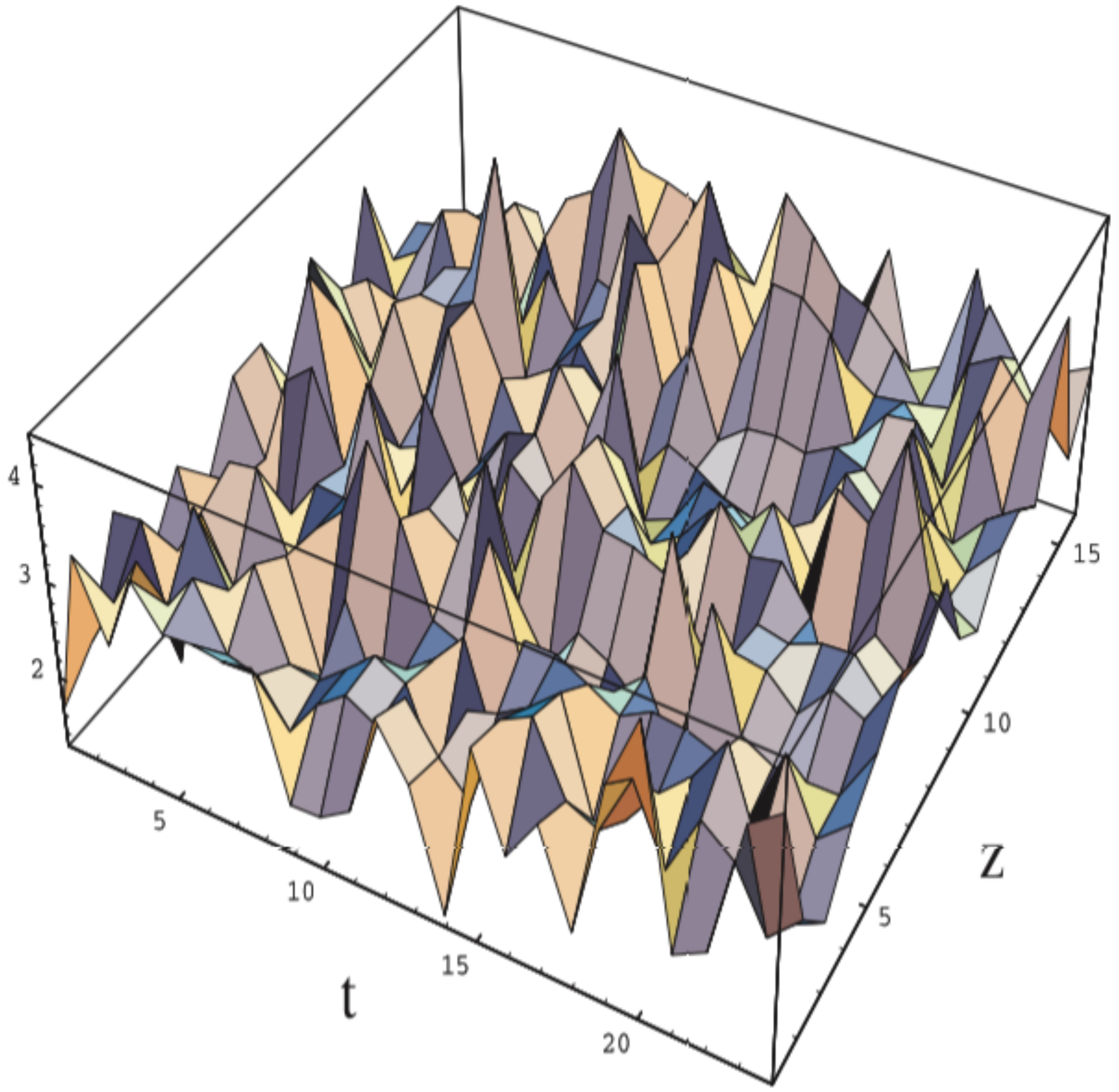


width: $\sqrt{8t}$

t > t0

Lattice QCD

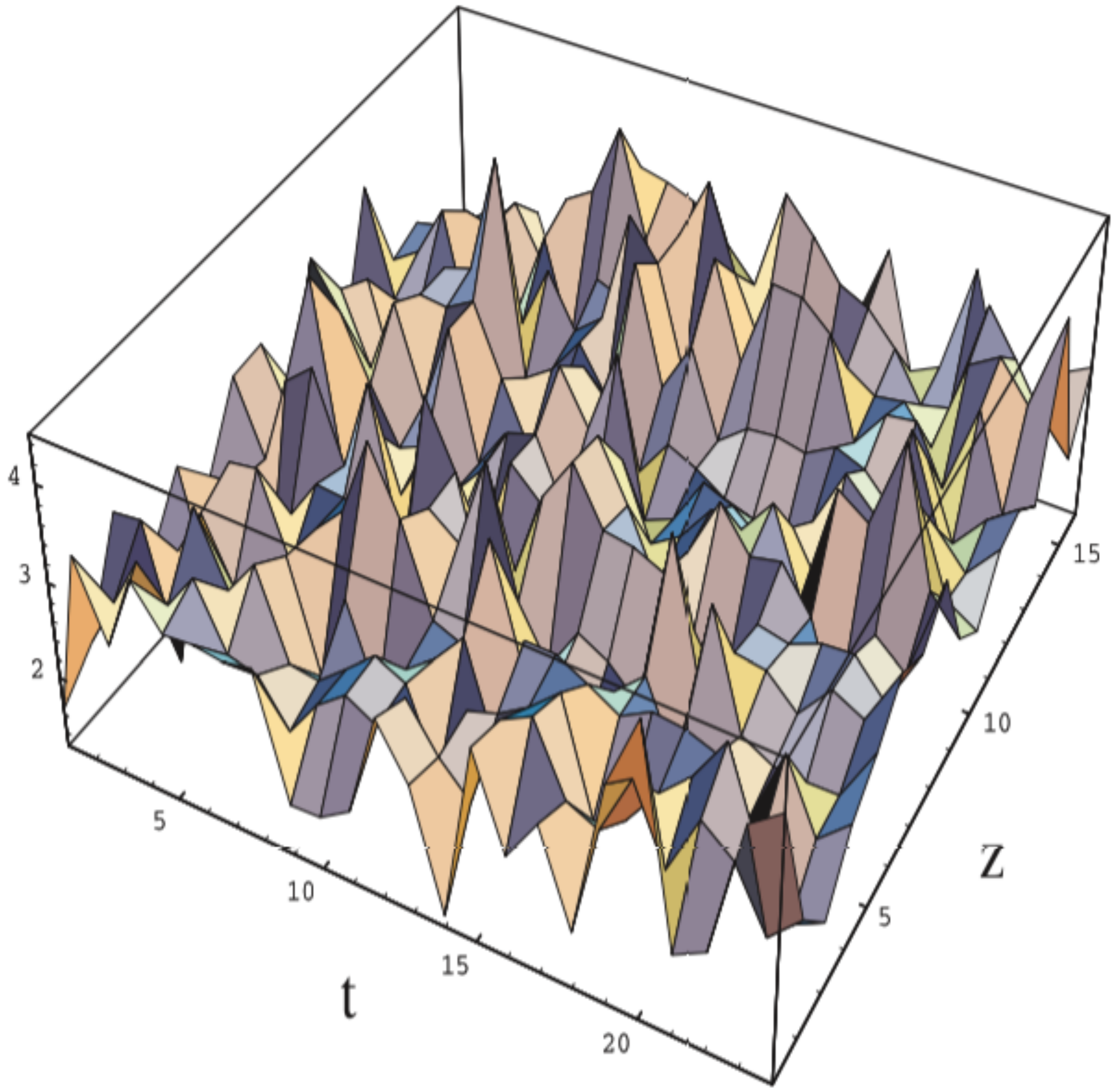
quantum fluctuations:



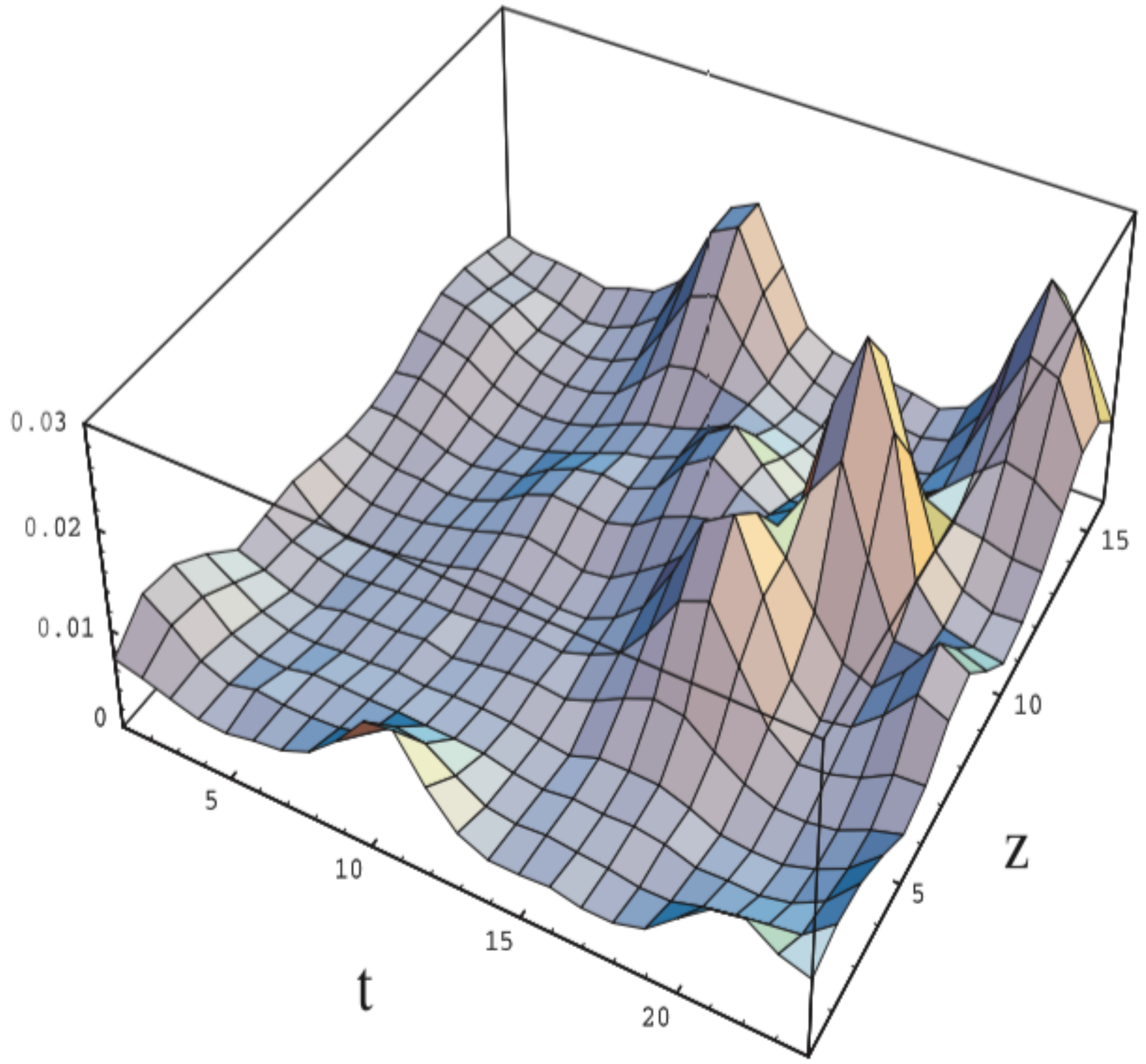
Engel 2009

Lattice QCD

quantum fluctuations:



"smearing":



Engel 2009

Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher (CERN and Geneva U.)
Jun 23, 2010

21 pages
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Report number: CERN-PH-TH-2010-143
View in: AMS MathSciNet, ADS Abstract Service, CERN Document Server

pdf cite 702 citations

Higher orders

flow equation: $\partial_t B \sim \partial^2 B + \partial B^2 + B^3$

perturbative ansatz: $B = gB_1 + g^2 B_2 + \ldots$

momentum space: $\tilde{B}_1(p) = e^{-tp^2} \tilde{A}(p)$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, \tilde{A}(p)\, \tilde{A}(p-q)$$

$$K(t,s,p,q) \sim \exp\left[-tp^2 - 2sq(q-p) \right]$$

etc.

Higher orders

flow equation: $\partial_t B \sim \partial^2 B + \partial B^2 + B^3$

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etc.

Exponential damping in momentum integrals!

Feynman rules

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

L_μ Lagrange multiplier field

Feynman rules

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$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

L_μ Lagrange multiplier field

μ, a, t $\qquad$ ν, b, s

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

Feynman rules

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B$$

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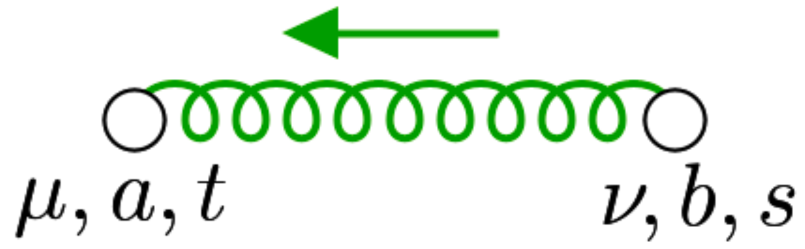

$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

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$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

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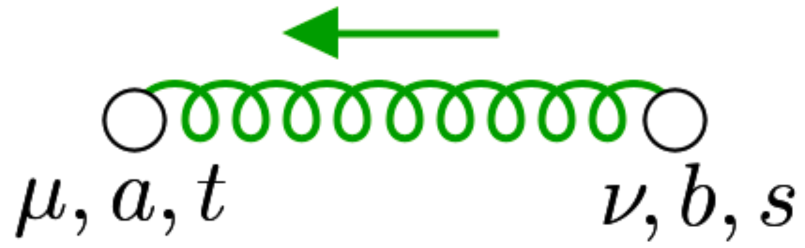


$$\delta_{ab} \delta_{\mu\nu}\, \theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

analogously for quarks:

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \Delta \right) \chi + \text{h.c.}$$

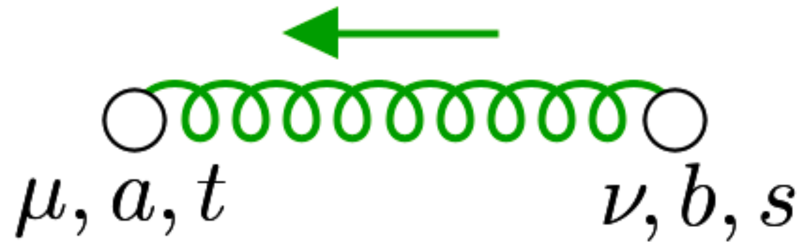
Feynman rules

$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

L_μ Lagrange multiplier field

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$



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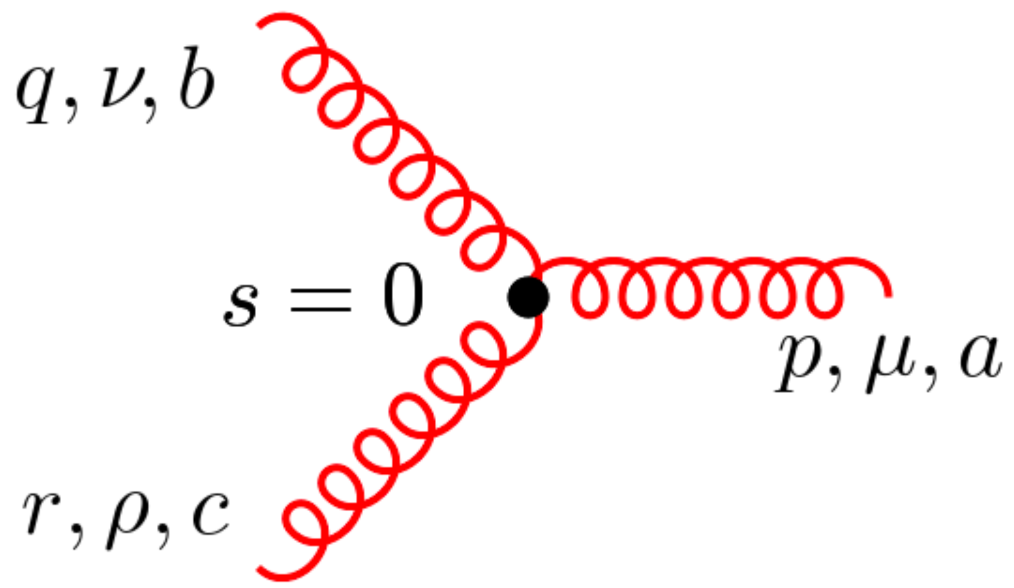
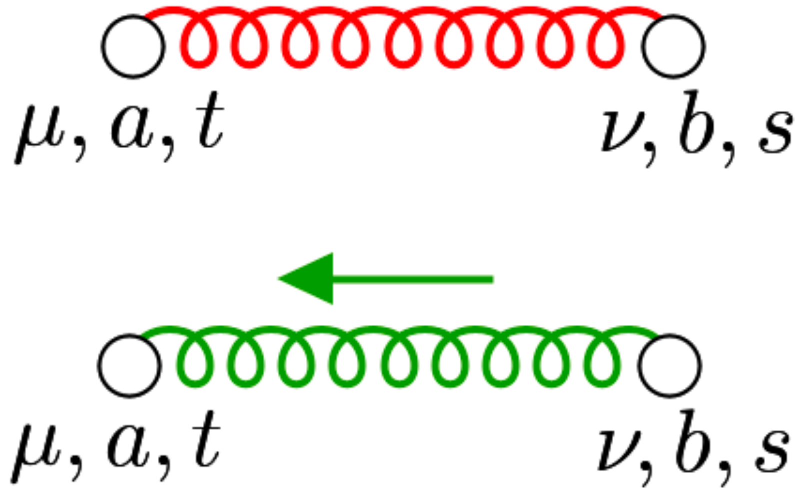
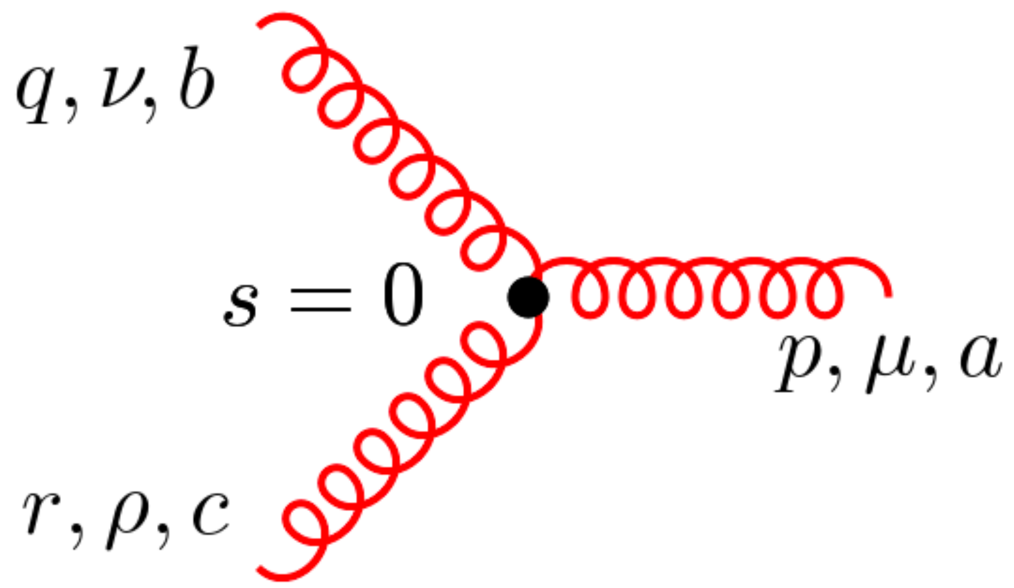
Vertices



regular 3-gluon vertex



Vertices



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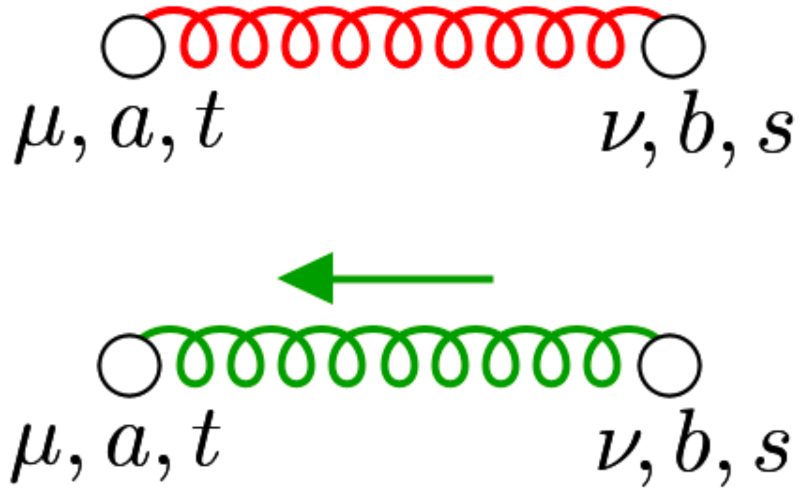
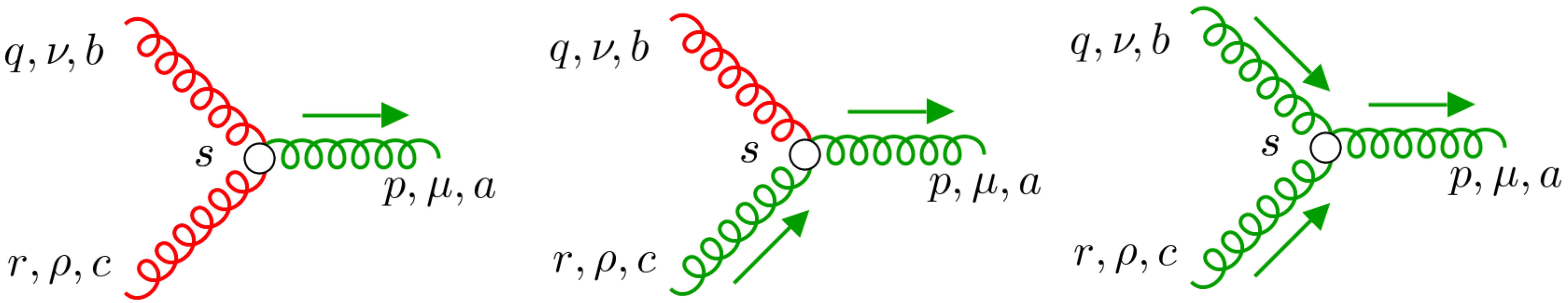
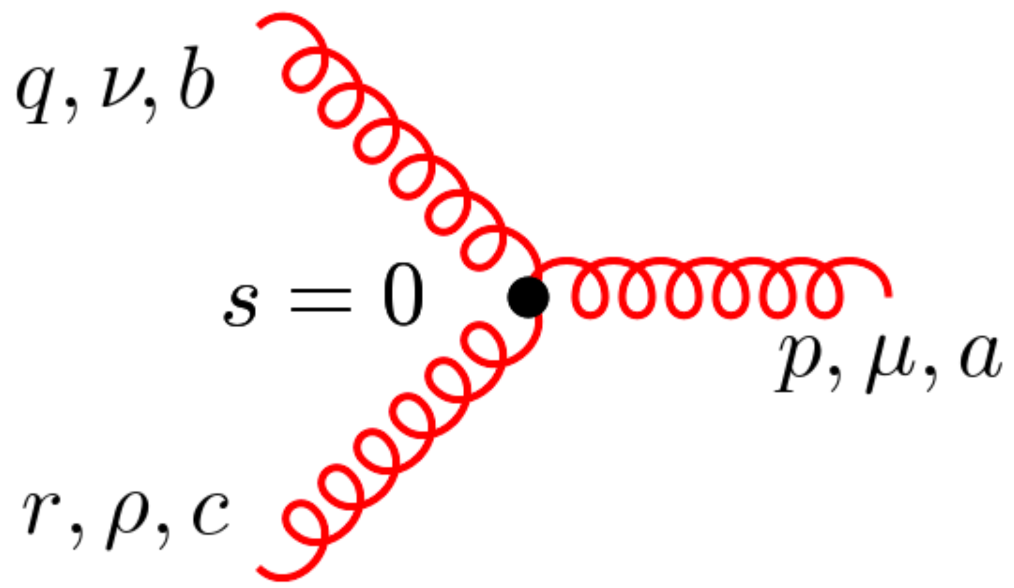




$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu$$
$$+(\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

Vertices



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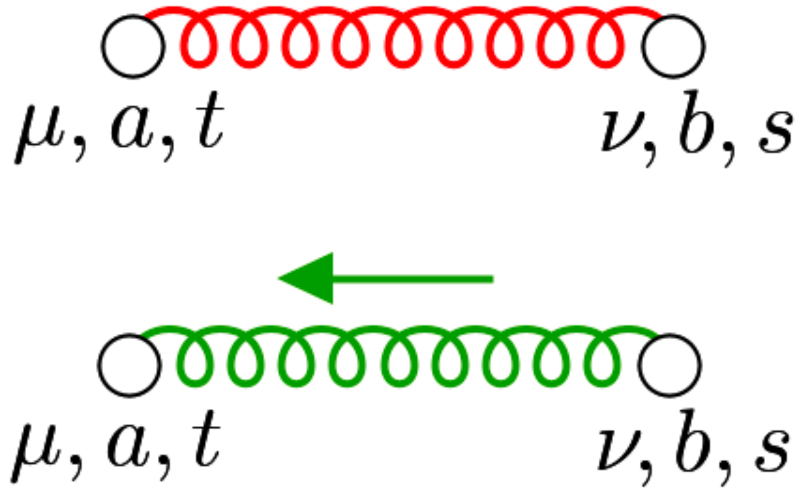
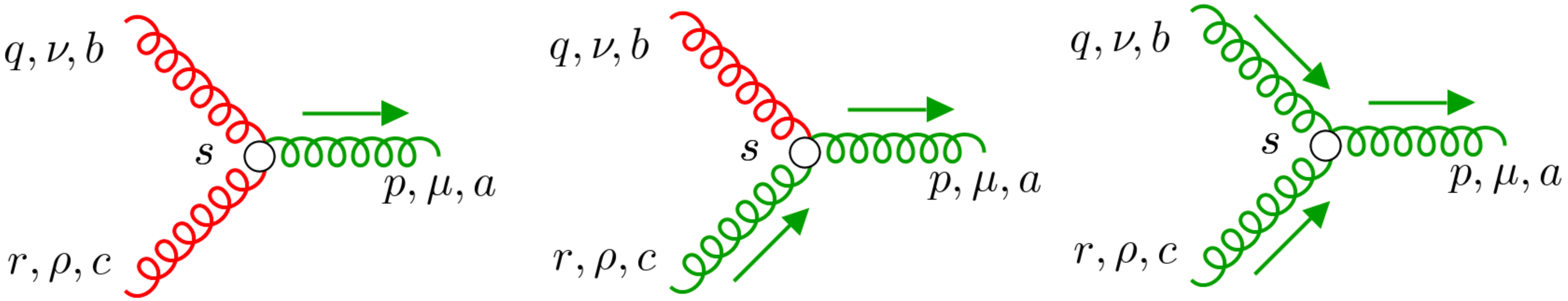
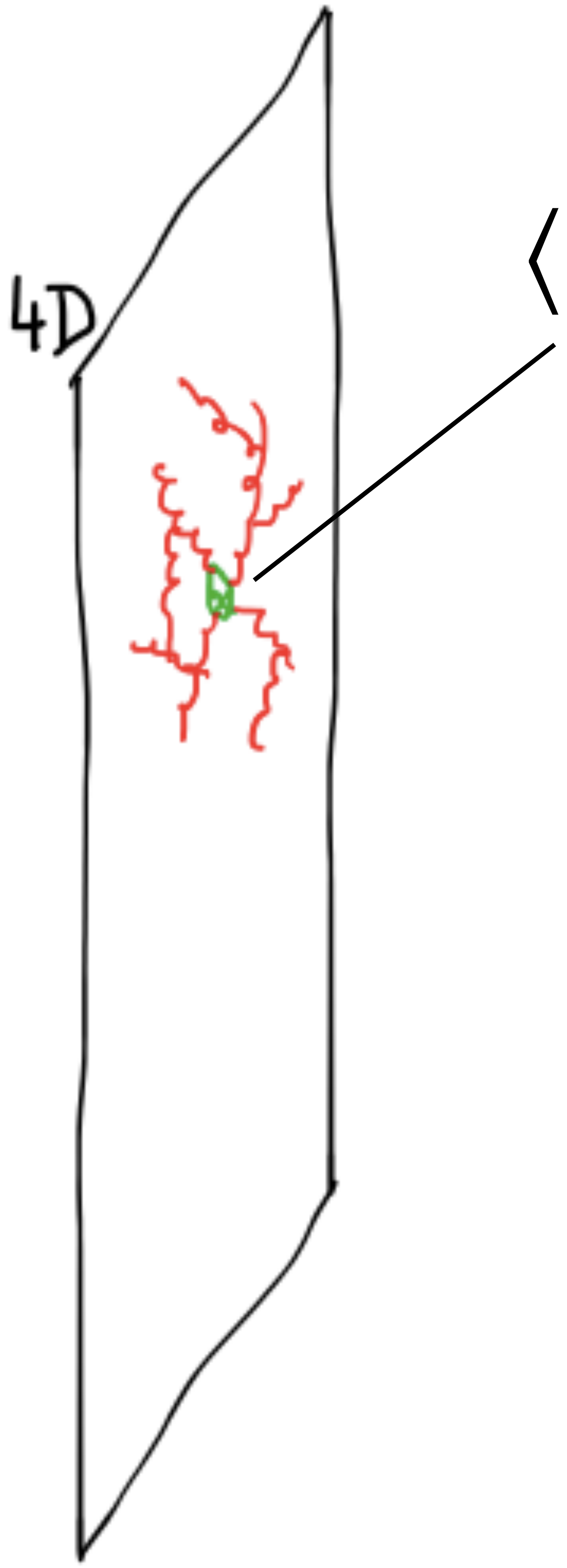




$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

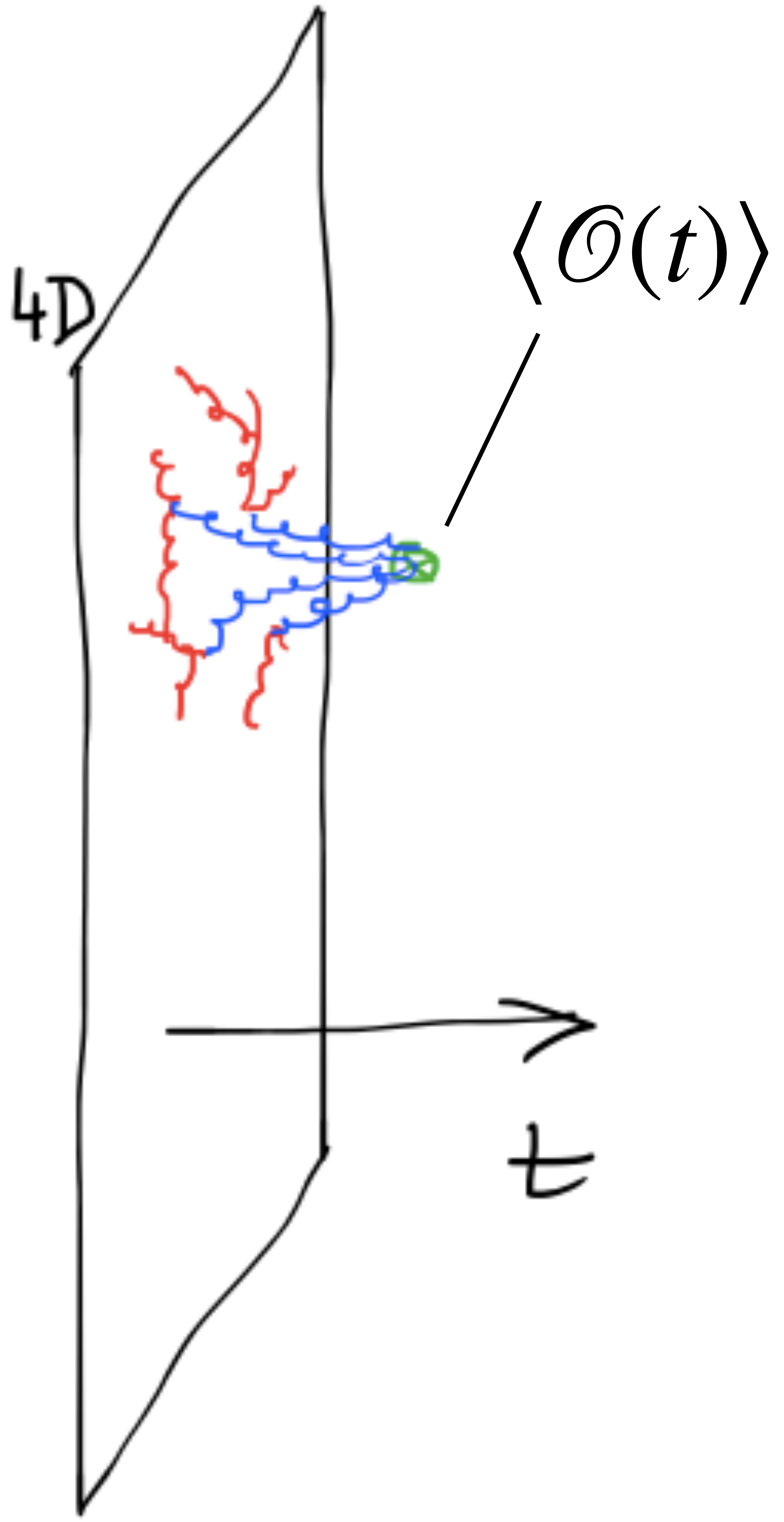
analogously for 4-gluon vertex and quarks

$$\mathscr{L} = \mathscr{L}_{\text{QCD}}$$



$\langle \mathcal{O} \rangle$

$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B$$



Let’s calculate

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

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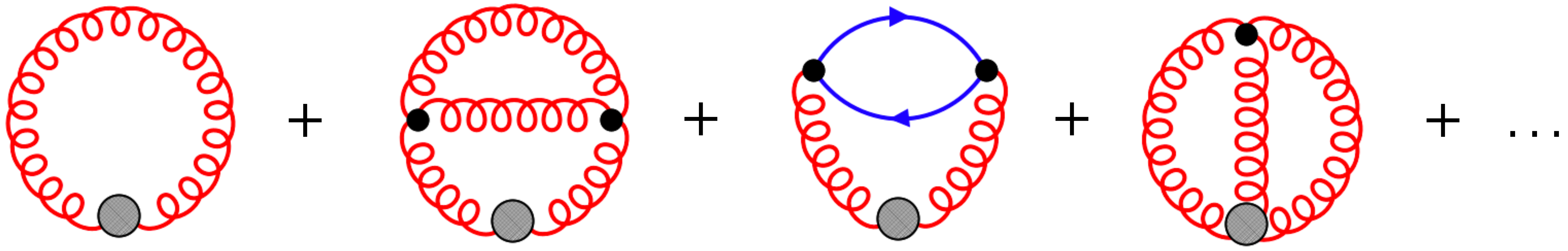
Let’s calculate

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

at t=0 (i.e. fundamental QCD):



$$\sim \int d^D p\, p^n = 0$$

Let's calculate

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

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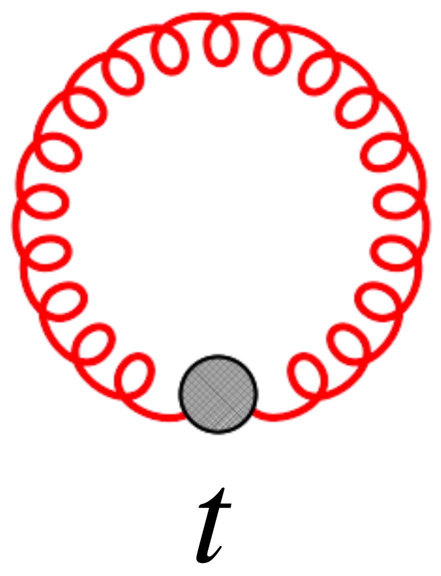
LO: 

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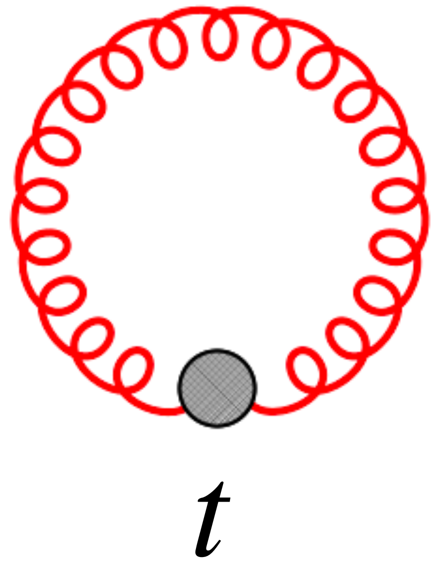
LO:



$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

Let's calculate

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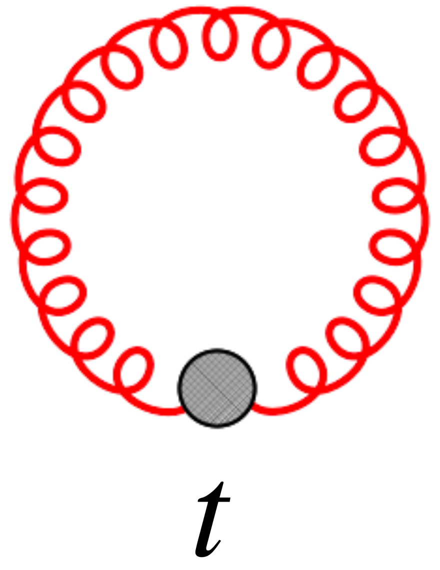
LO: 

$$\sim \int d^Dp\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

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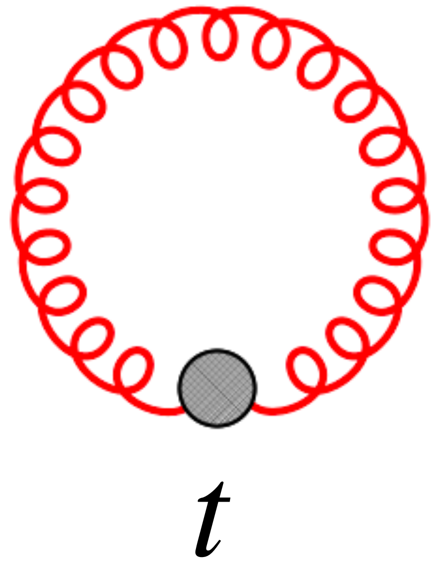
LO: 

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explicitly: $$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

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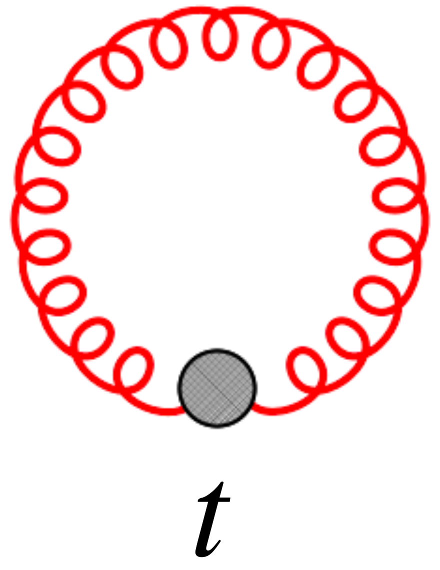
LO: 

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→ measure α_s on the lattice?

Let's calculate

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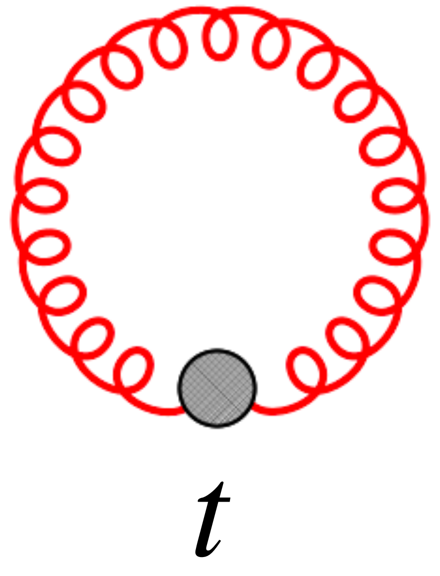
LO: 

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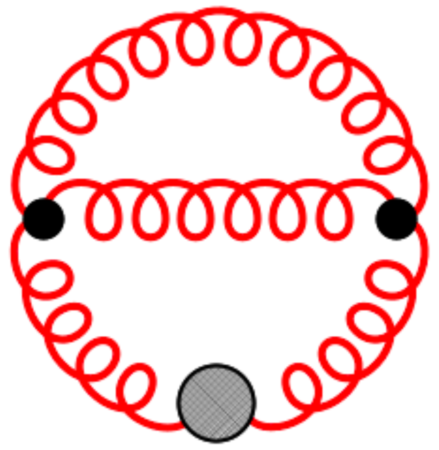
explicitly: $$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

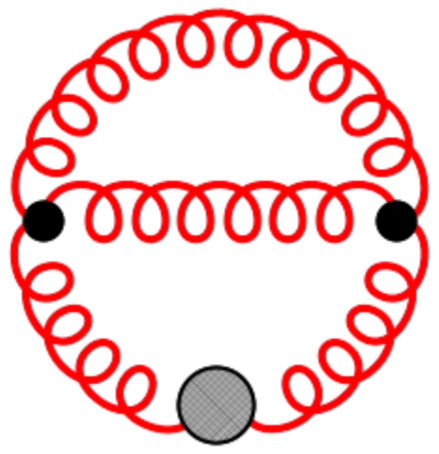
→ measure α_s on the lattice? $\alpha_s = \alpha_s(\mu)$

Higher orders



$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

Higher orders



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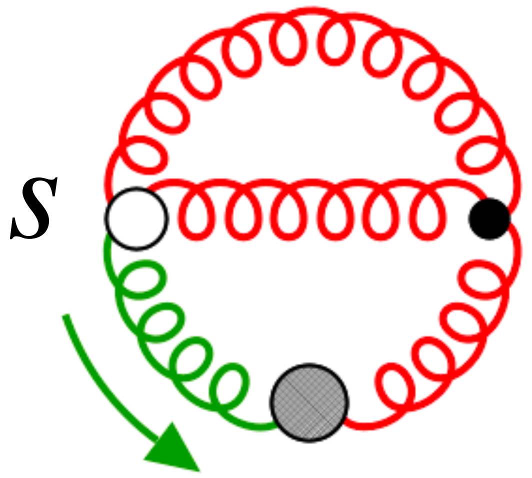
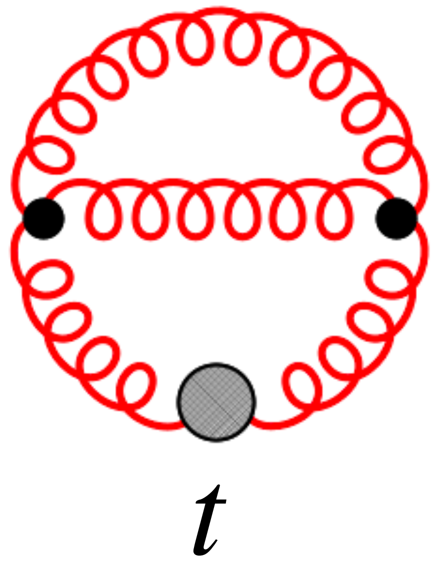
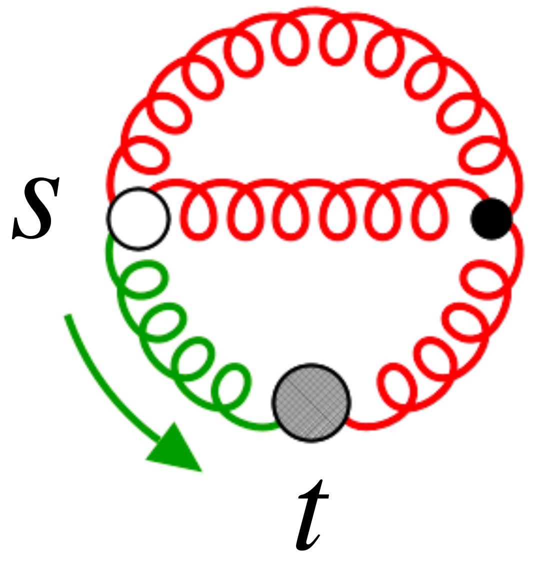


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

Higher orders





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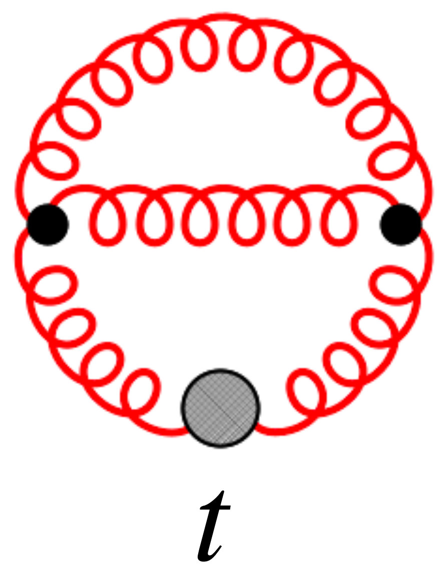
$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\, k^2\, (p-k)^2}$$

- more loop integrals

Higher orders



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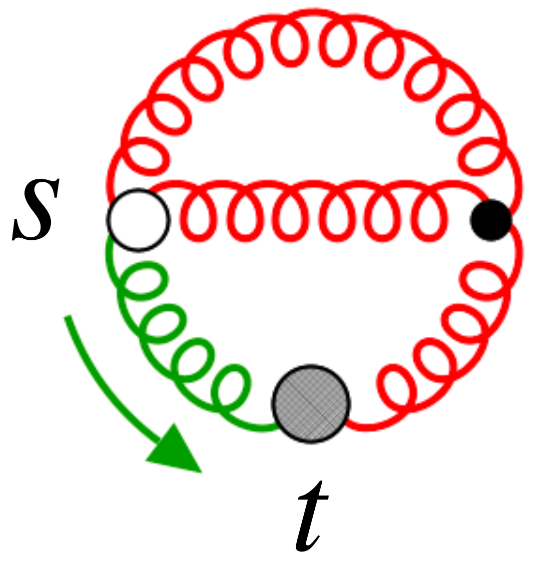
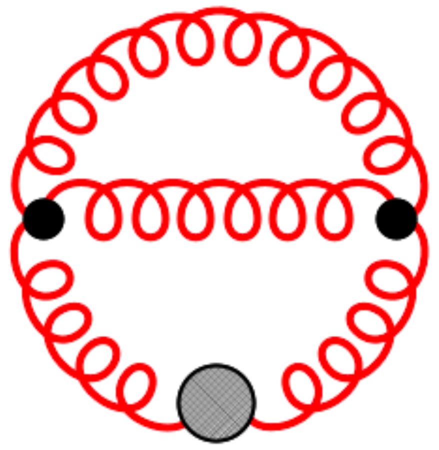
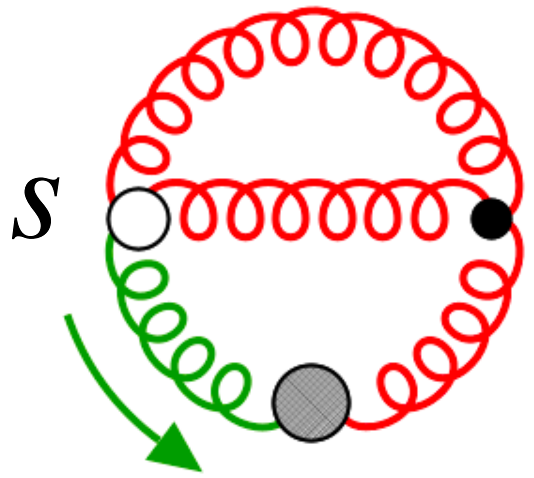


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

- more loop integrals
- integration over flow-time parameters

Higher orders





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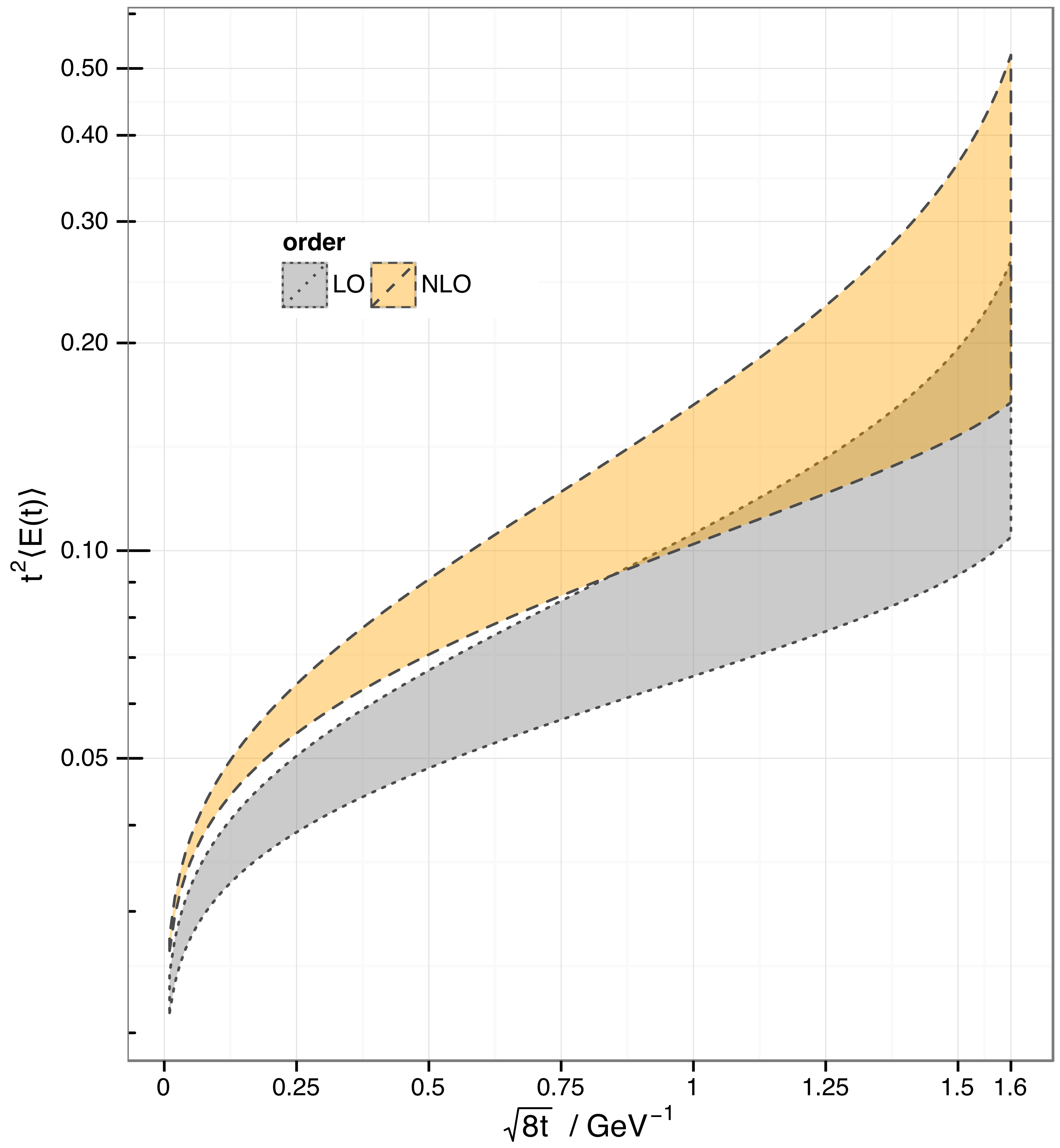
- more loop integrals
- integration over flow-time parameters
- renormalization: same as fundamental QCD!

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher '10



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3 - \frac{11}{3}L_{t\mu}\right) C_A - \frac{8}{9} n_f T_R$$

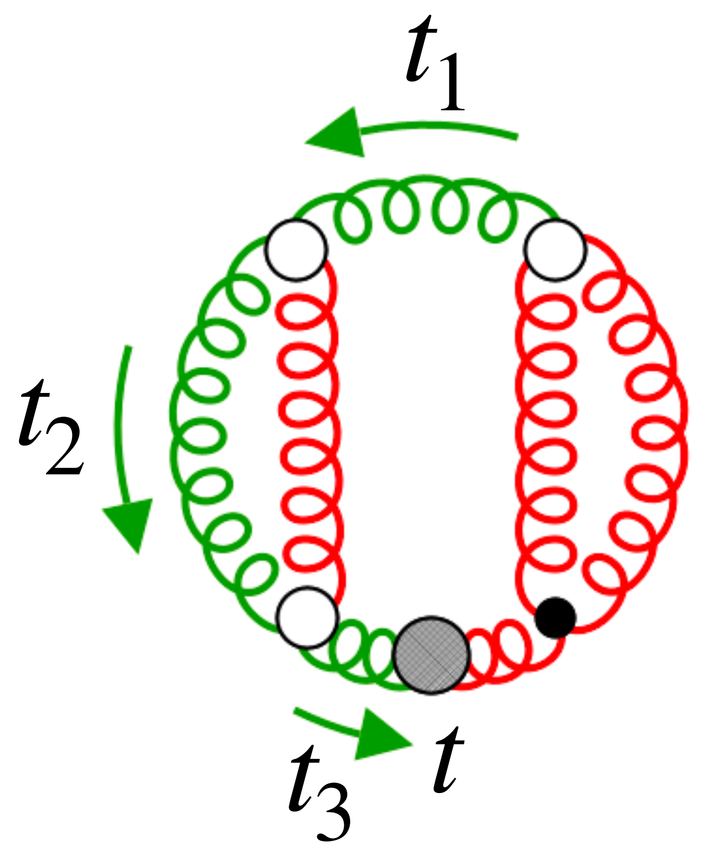
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative accuracy on αs: ± 3-5%

PDG: ± 1%

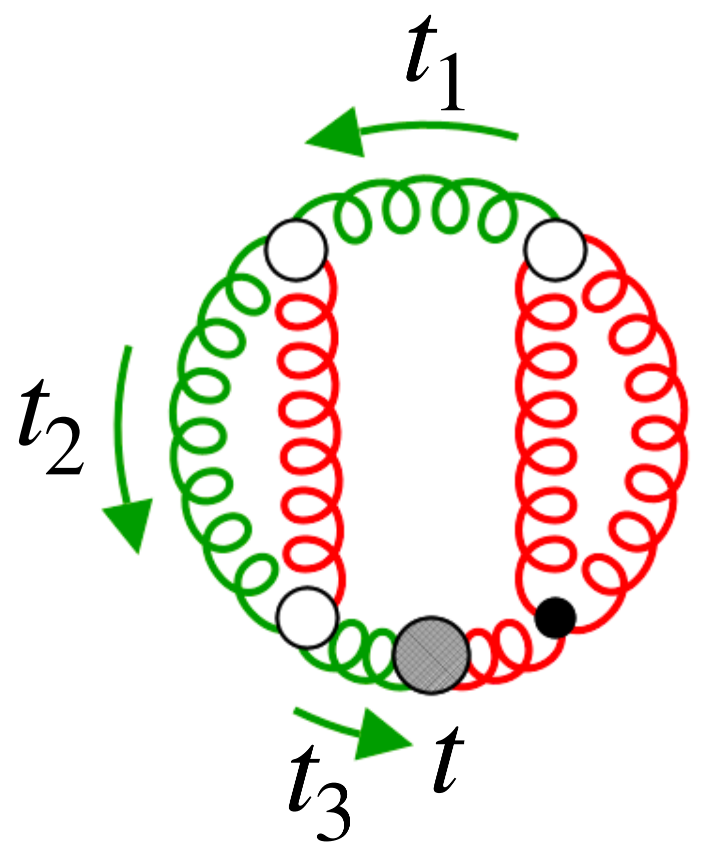
Three-loop calculation



Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



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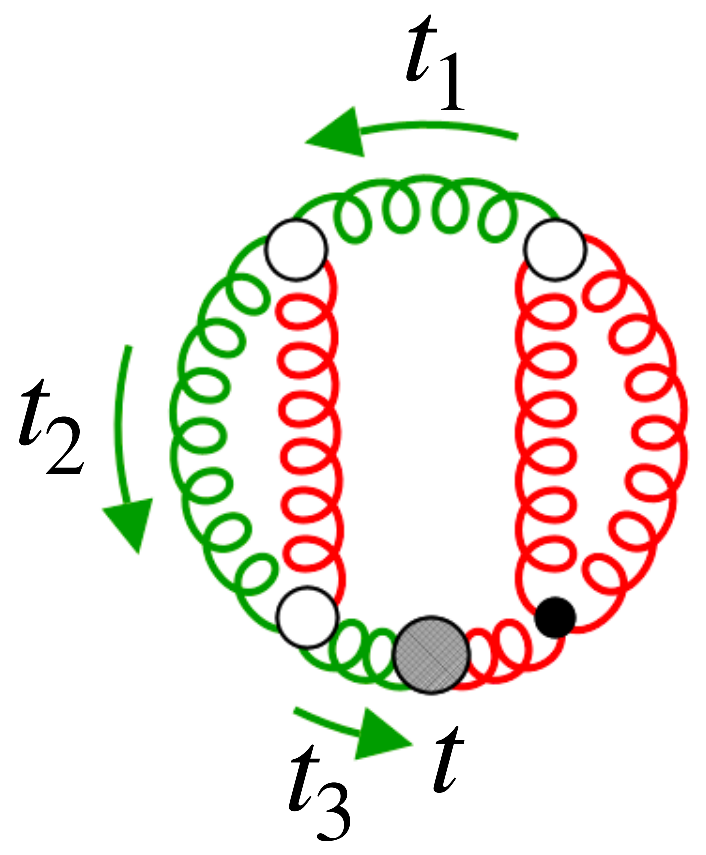
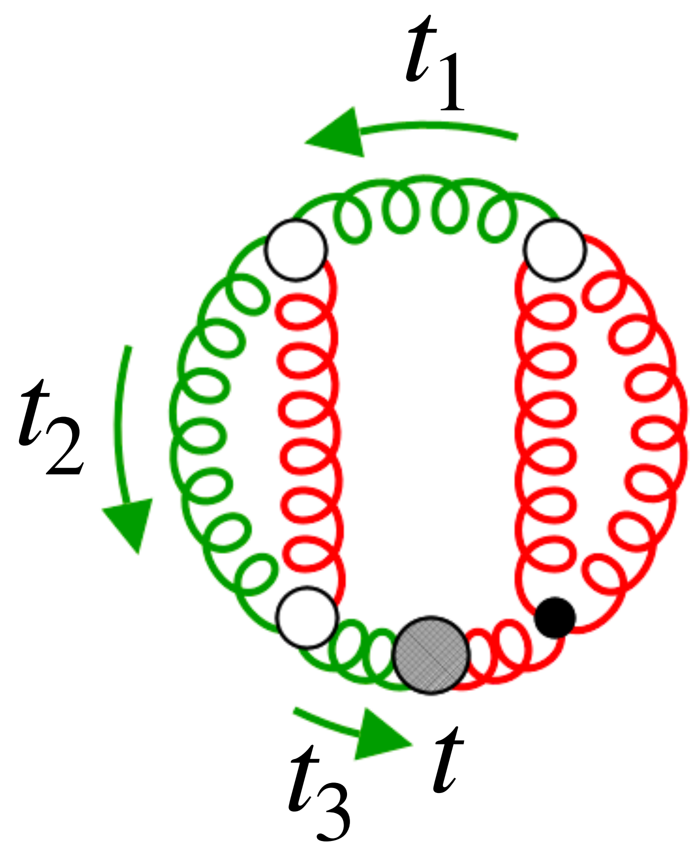


$$I(t, \mathbf{n}, \mathbf{a}, D) = \left(\prod_{f=1}^{N} \int_0^{t_f^{\rm up}} {\rm d}t_f \right) \int_{p_1,p_2,p_3} \frac{\exp[\sum_{k,i,j} a_{kij} t_k p_i p_j]}{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} p_4^{2n_4} p_5^{2n_5} p_6^{2n_6}}$$

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



$$I(t, \mathbf{n}, \mathbf{a}, D) = \left(\prod_{f=1}^{N} \int_0^{t_f^{\rm up}} \mathrm{d}t_f \right) \int_{p_1,p_2,p_3} \frac{\exp[\sum_{k,i,j} a_{kij} t_k p_i p_j]}{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} p_4^{2n_4} p_5^{2n_5} p_6^{2n_6}}$$

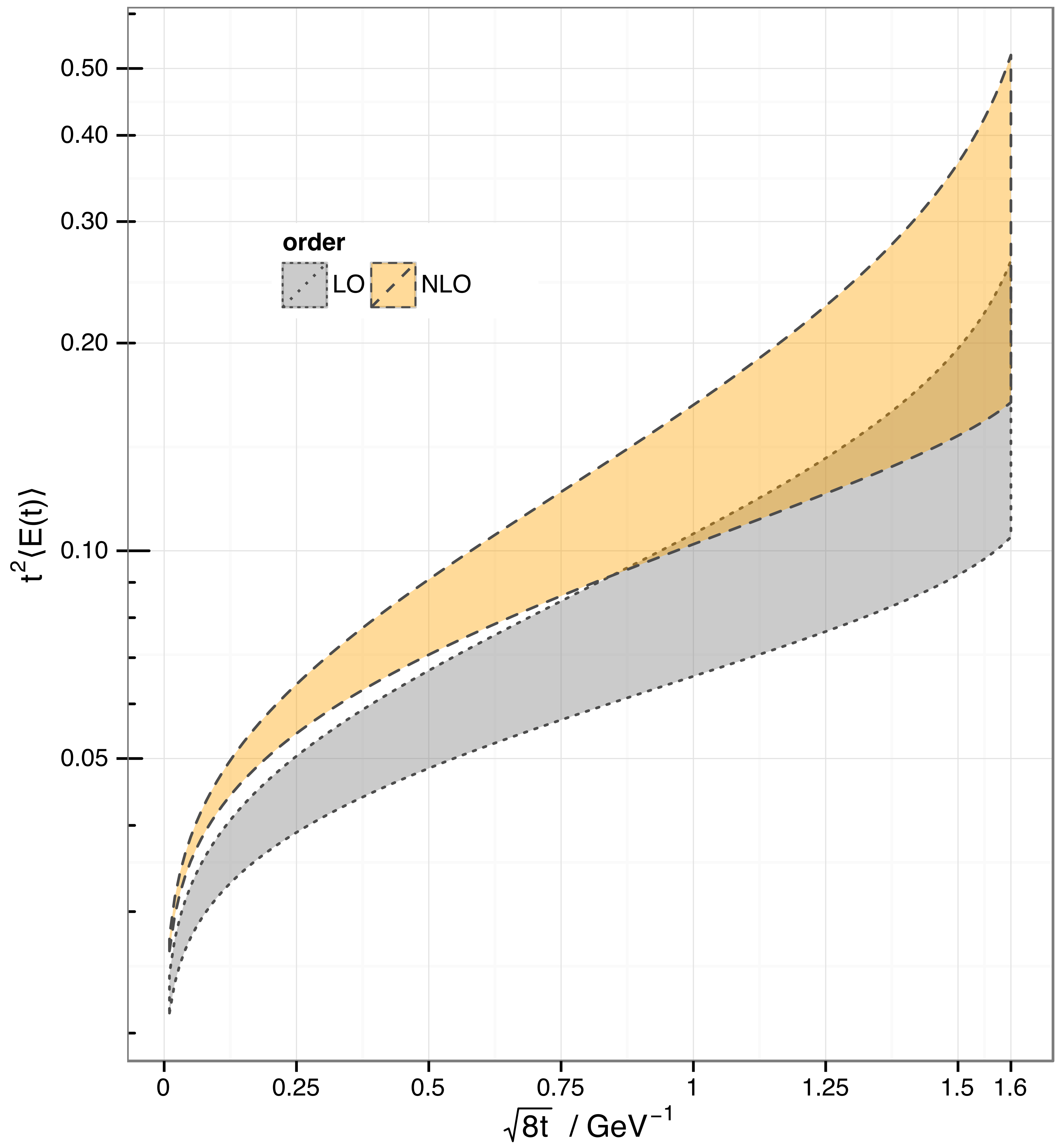
The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals

Artz, RH, Lange, Neumann, Prausa '19

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher '10



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3 - \frac{11}{3}L_{t\mu}\right) C_A - \frac{8}{9} n_f T_R$$

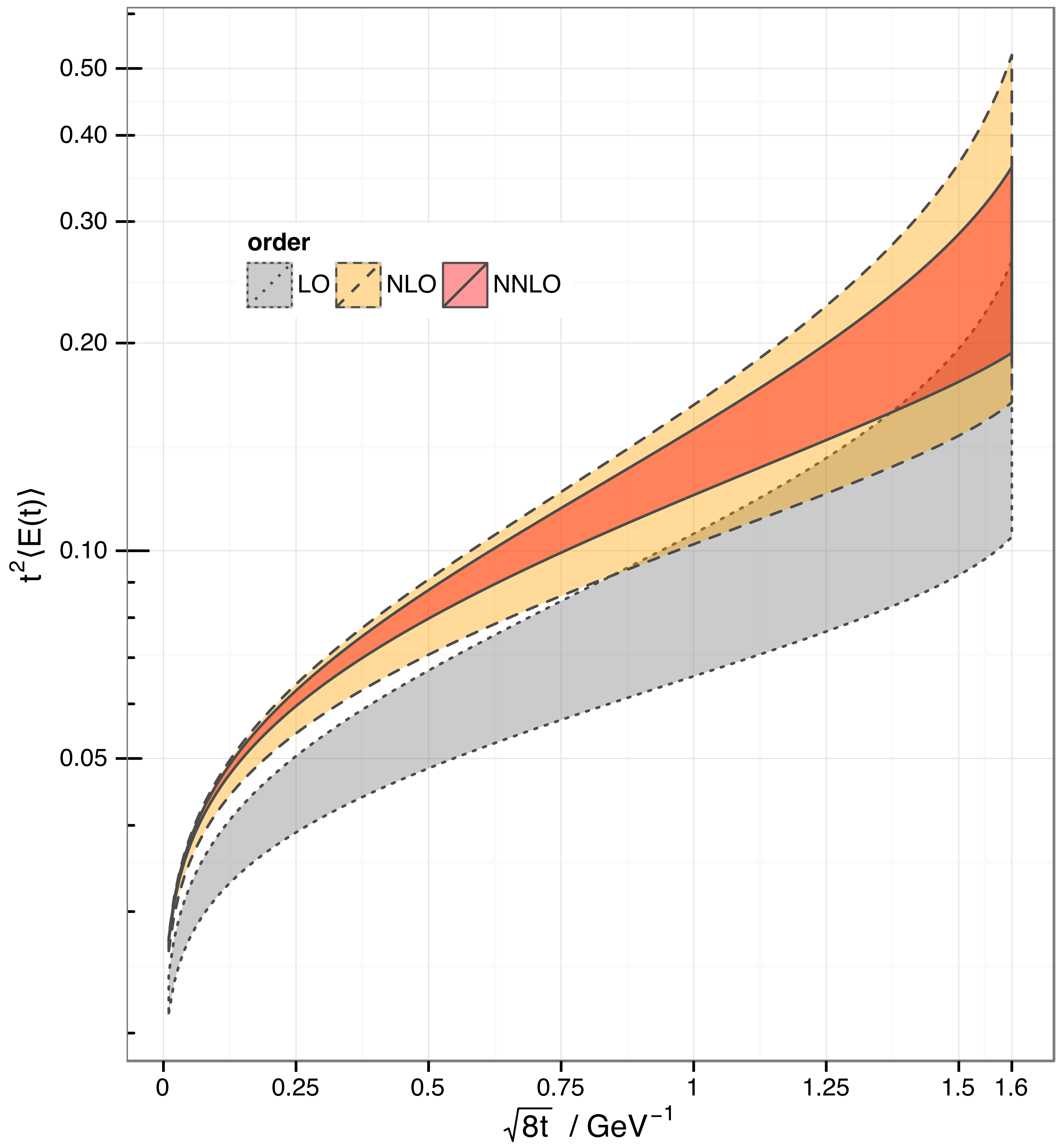
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

resulting perturbative accuracy on αs: ± 3-5%

PDG: ± 1%

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann ‘16



resulting perturbative accuracy on αs: O(1%)

PDG: ± 1%

Derive $\alpha_s(M_Z)$

	$t^2\langle E(t)\rangle \cdot 10^4$							
q_8	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

Gradient-flow coupling

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

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$$= -\,\hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$

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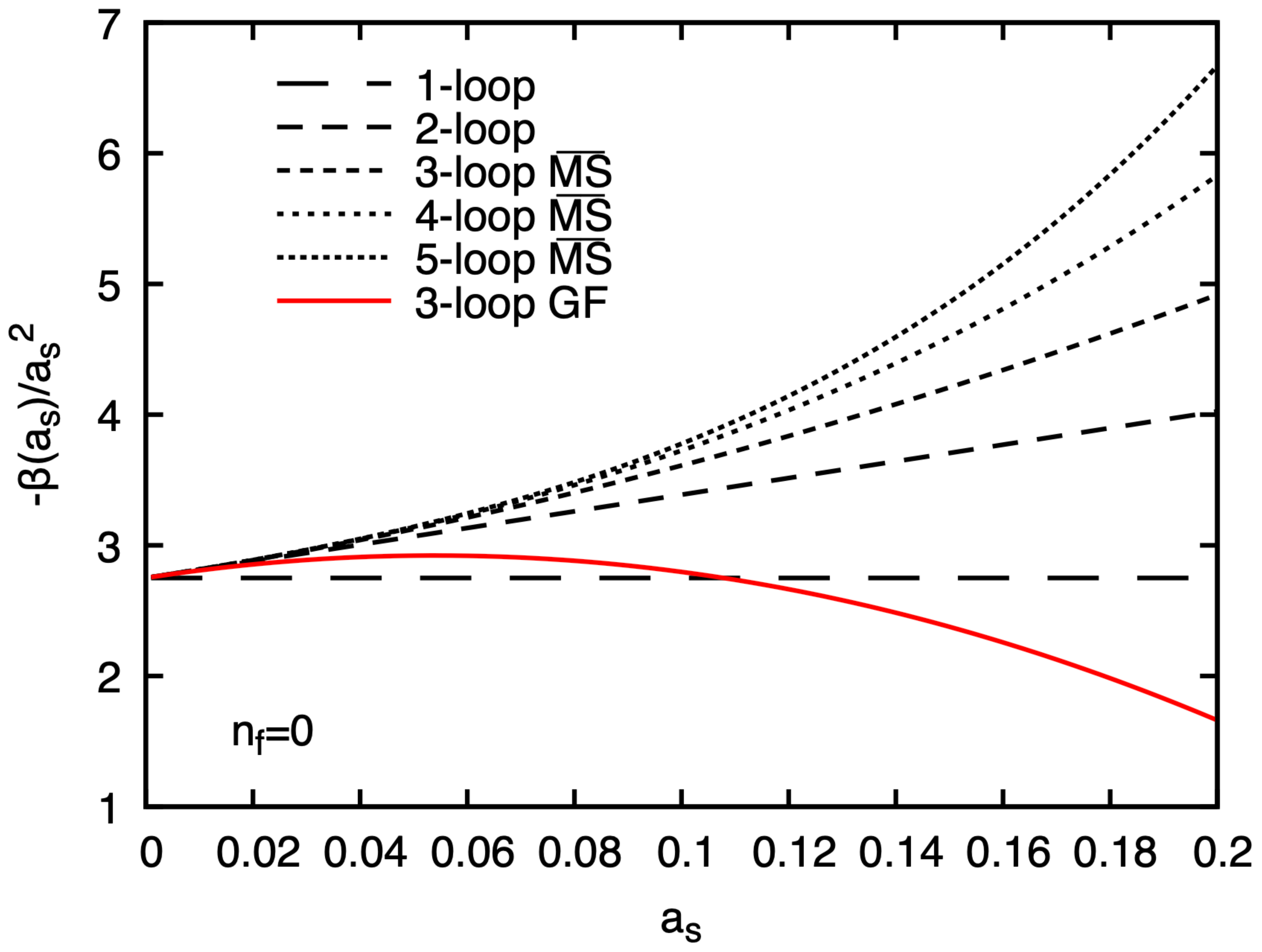

Integration-by-parts relations

Karlsruhe Institute of Technology

- After tensor reduction, we end up with many scalar integrals of the form

$$I(\{t_f^{\mathrm{up}}\},\{T_i\},\{a_i\}) = \left(\prod_{f=1}^{F}\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right)\int_{k_1,\ldots,k_L}\frac{\exp[-(T_1 q_1^2+\cdots+T_N q_N^2)]}{q_1^{2a_1}\cdots q_N^{2a_N}}$$

with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$
- Chetyrkin and Tkachov observed [Tkachov 1981; Chetyrkin, Tkachov 1981]

$$\int_{k_1,\ldots k_L}\frac{\partial}{\partial k_i^\mu}\left(\tilde{q}_j^\mu \frac{1}{P_1^{a_1}\ldots P_N^{a_N}}\right) = 0$$

$\Rightarrow$ Linear relations between Feynman integrals
- Can easily be adopted to gradient-flow integrals
- Additional new relations for gradient-flow integrals: [Artz, RH, Lange, Neumann, Prausa '19]

$$\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f \partial_{t_f} F(t_f,\ldots) = F(t_f^{\mathrm{up}},\ldots) - F(0,\ldots)$$

Laporta algorithm

- Schematically integration-by-parts read

$$0 = (d - a_1)I(a_1, a_2, a_3) + (a_1 - a_2)I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2)I(a_1 + 1, a_2, a_3 - 1)$$

- Rarely possible to find general solution like

$$I(a_1, a_2, a_3) = a_1 I(a_1 - 1, a_2, a_3) + (d + a_1 - a_2)I(a_1, a_2 - 1, a_3) + 2a_3 I(a_1, a_2, a_3 - 1)$$

- Instead set up system of equations and solve it [Laporta 2000] :
  - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$\begin{aligned} 0 &= (d-1)I(1,1,1) + I(2,1,0), \\ 0 &= (d-2)I(2,1,1) + I(3,0,1) - I(3,1,0), \\ &\vdots \end{aligned}$$

  - Solve with Gaussian elimination

$\Rightarrow$ Express integrals through significantly smaller number of master integrals

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e.g. NNLO chromo-magnetic dipole operator:
O(4000) integrals reduced to 13 master integrals

  - Solve with Gaussian elimination

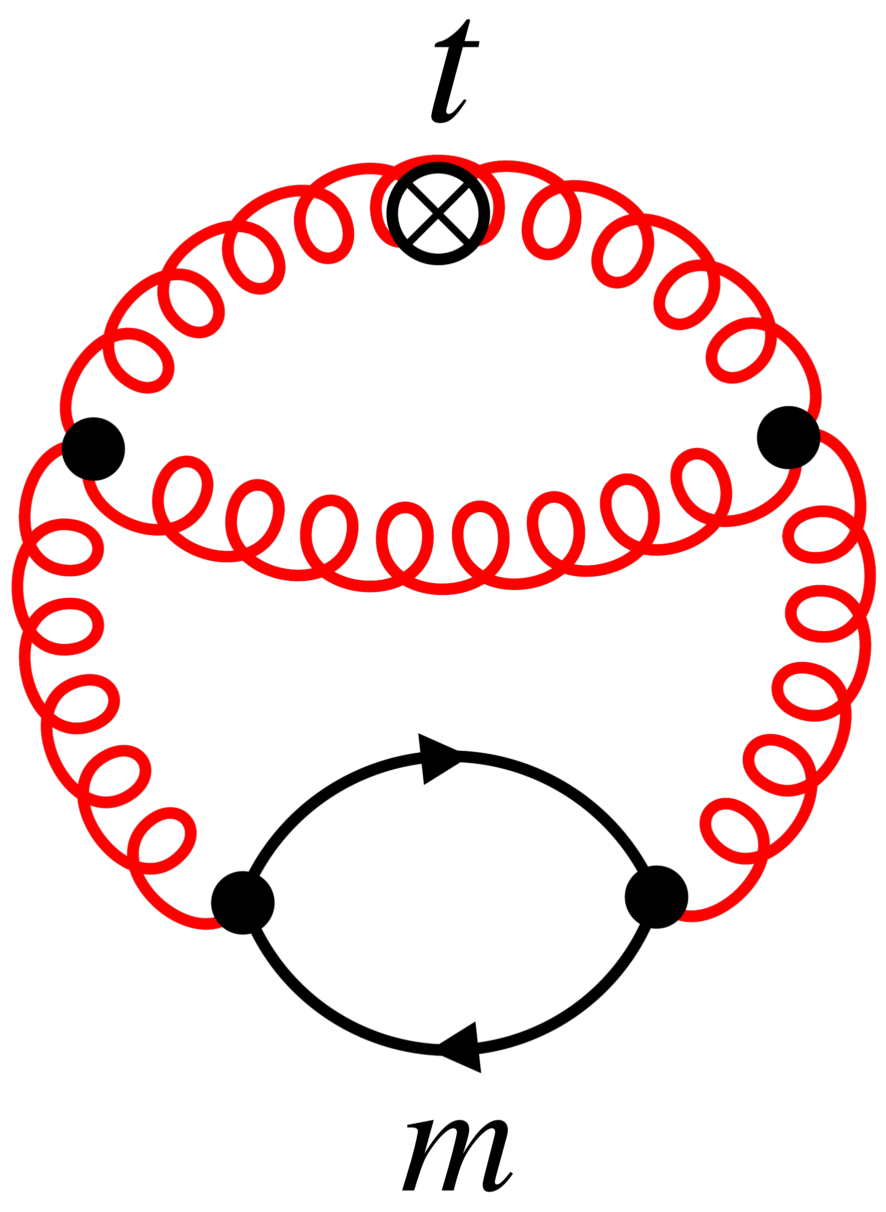
$\Rightarrow$ Express integrals through significantly smaller number of master integrals

Approximate solution of gradient flow integrals

example:

$\langle G_{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

result for $m \neq 0$?



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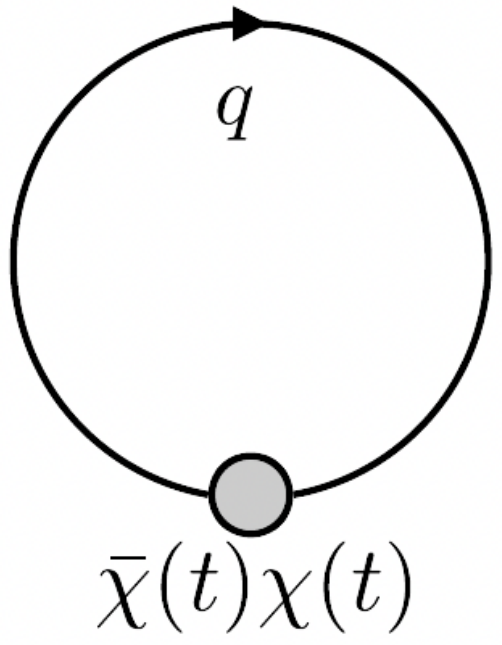
Strategy of regions [Beneke, Smirnov '97]

example:

$$S(t) \equiv \langle \bar{\chi}(t)\chi(t)\rangle = -\frac{3m}{8\pi^2 t^2} f(m^2, t)$$

[RH '21]



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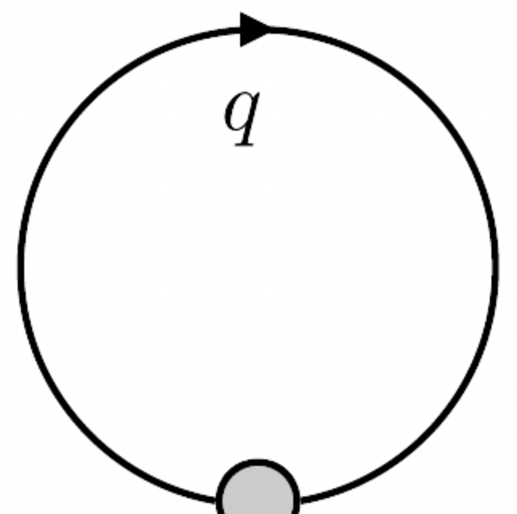

$$f(m^2, t) \equiv t \int_k \frac{e^{-tk^2}}{k^2 + m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0, m^2 t)$$

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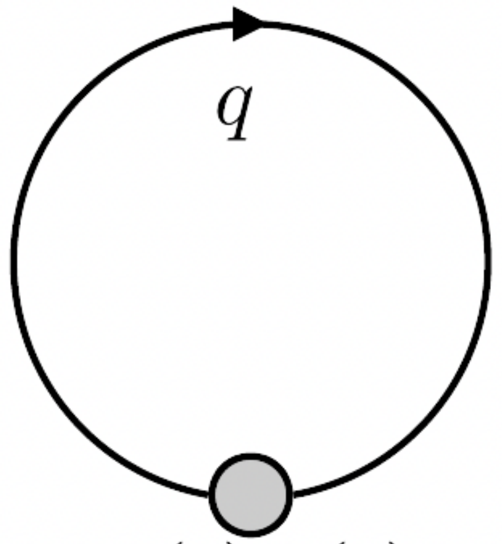

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$$\Gamma(s, x) = \int_x^\infty \mathrm{d}u\, u^{s-1} e^{-u}$$

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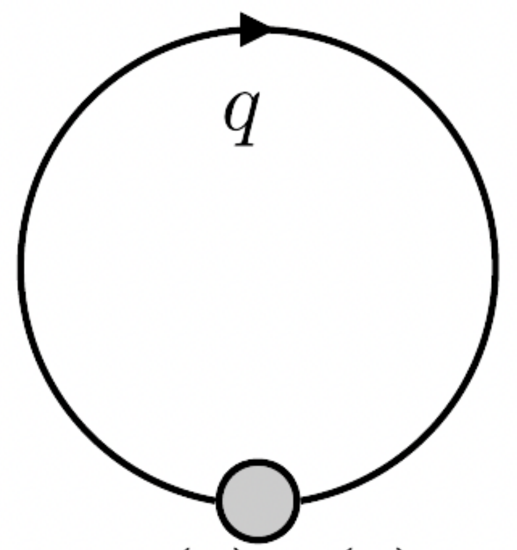

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$m^2 t \ll 1 \;\Leftrightarrow\; 1/t \gg m^2$

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"naive" expansion:

$$f^{(ii)}(m^2,t) = t\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\int_k \frac{k^{2n}}{k^2+m^2} = m^2 t\left[-\frac{1}{\epsilon} - 1 + \gamma_\mathrm{E} + \ln m^2\right] e^{m^2 t}$$

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assumes

$k^2 t \ll 1$

$k^2 \ll 1/t$

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$$f^{(i)}(m^2,t) = t \sum_{n=1}^{\infty} (-m^2)^{n-1} \int_k \frac{e^{-tk^2}}{k^{2n}} = \sum_{n=1}^{\infty} (-m^2)^{n-1} t^{n-1+\epsilon} \frac{\Gamma(D/2-n)}{\Gamma(D/2)}$$

$$= 1 + m^2 t \left(\frac{1}{\epsilon} + \ln t + 1\right) e^{m^2 t} - (m^2 t)^2 - \frac{3}{4}(m^2 t)^2 + \dots .$$