

Gradient flow and the short flow-time expansion

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01 Oct 2024

Lattice Meets Continuum

Siegen (30 Sep — 3 Oct 2024)

The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$
$$B_\mu(t = 0, x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t, x) = \mathcal{D}^2 \chi(t, x)$$
$$\chi(t = 0, x) = \psi(x)$$

Narayanan, Neuberger 2006

Lüscher 2009

Lüscher 2010

Lüscher, Weisz 2011

Lüscher 2013

Schematically...

$$\frac{\partial}{\partial t} B_\mu(t) = \mathcal{D}_\nu G_{\nu\mu}(t)$$

$$\mathcal{D}_\mu = \partial_\mu - iT^a g_0 B_\mu^a(t)$$

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$$\frac{\partial}{\partial t} B = \mathcal{D} G$$
$$\mathcal{D} = \partial - g_0 B$$

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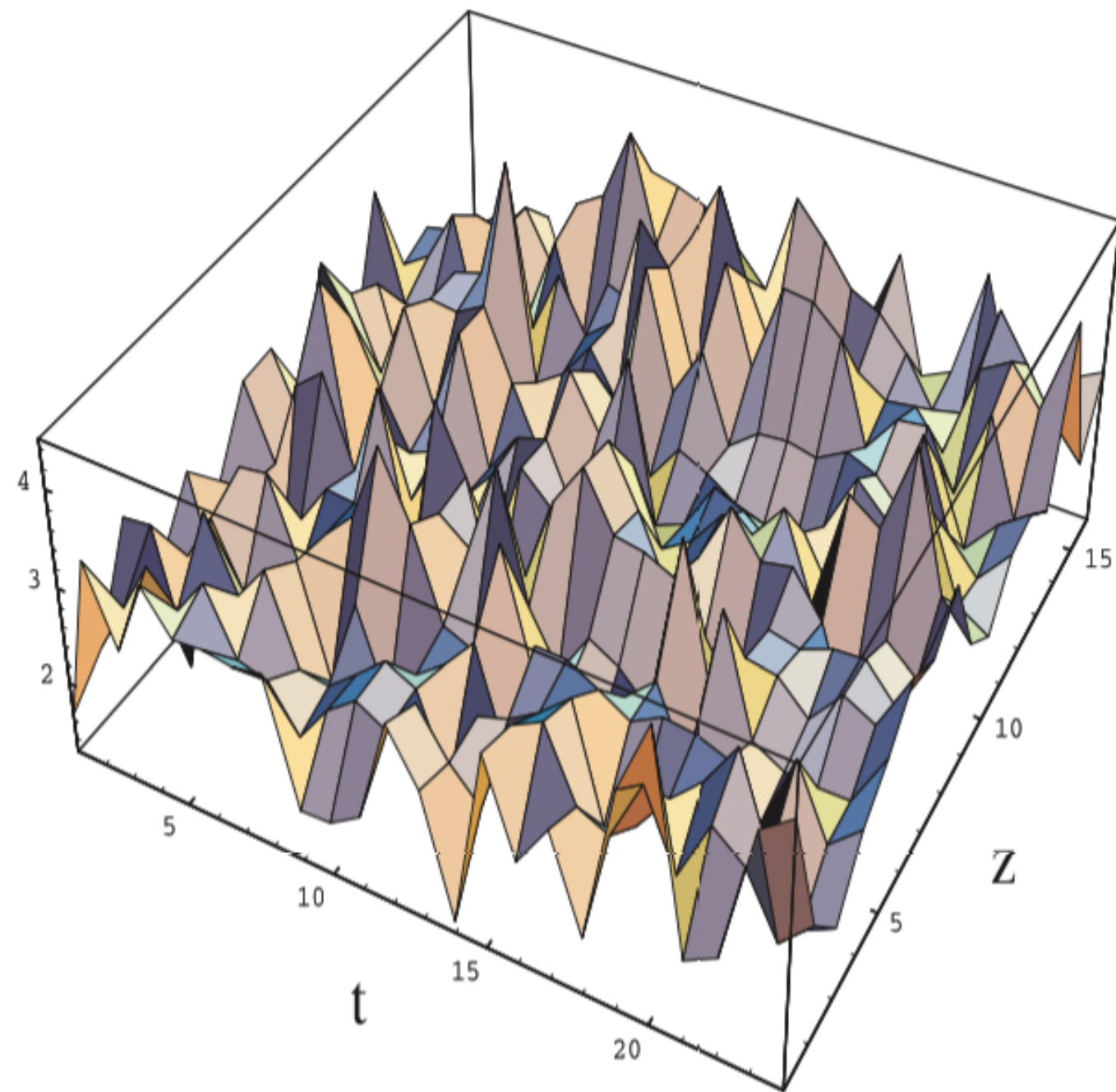
$$\partial_t B \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$$

cf. heat equation:

$$\partial_t u(t, \mathbf{x}) = \Delta u(t, \mathbf{x})$$

Lattice QCD

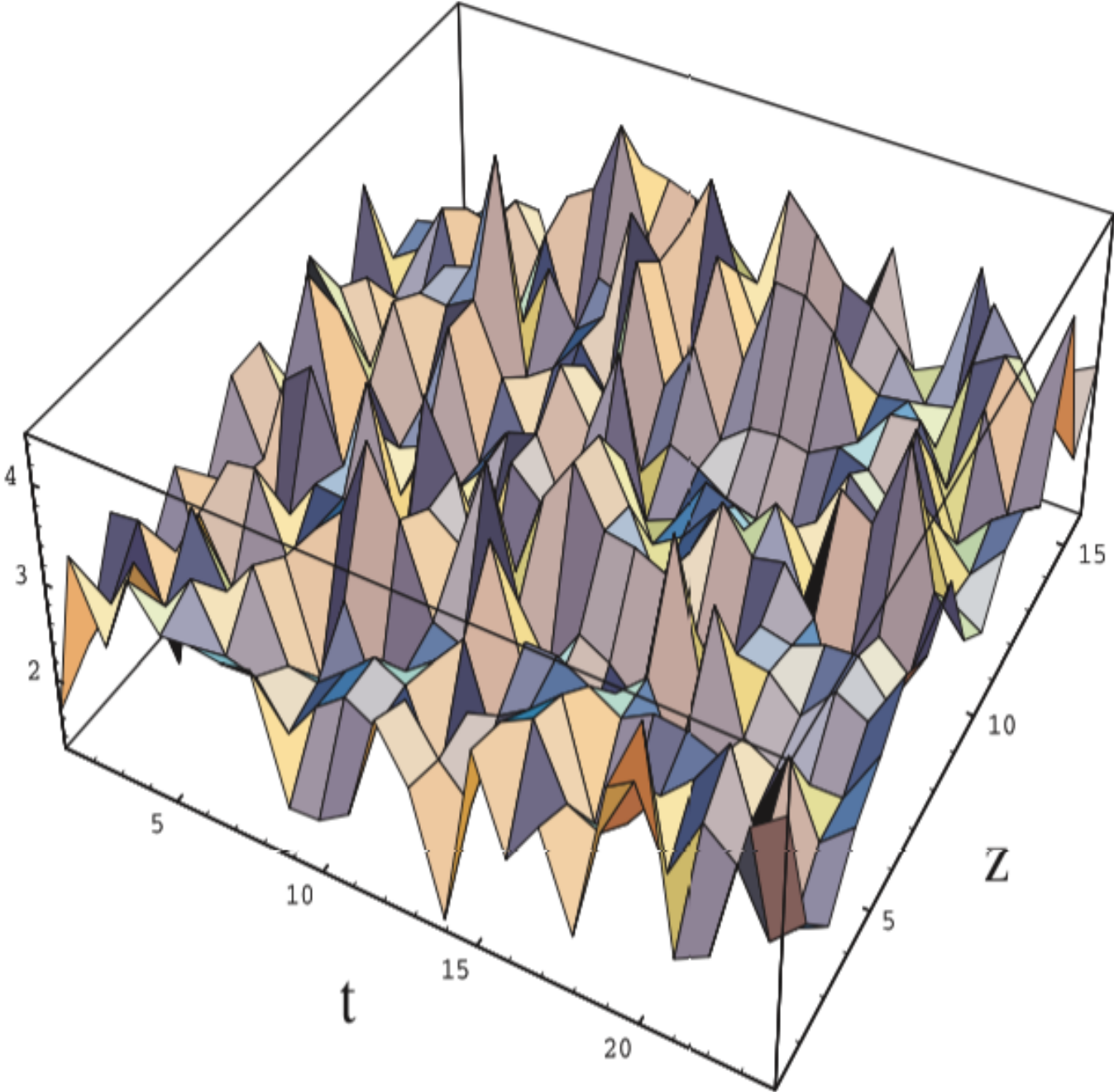
quantum fluctuations:



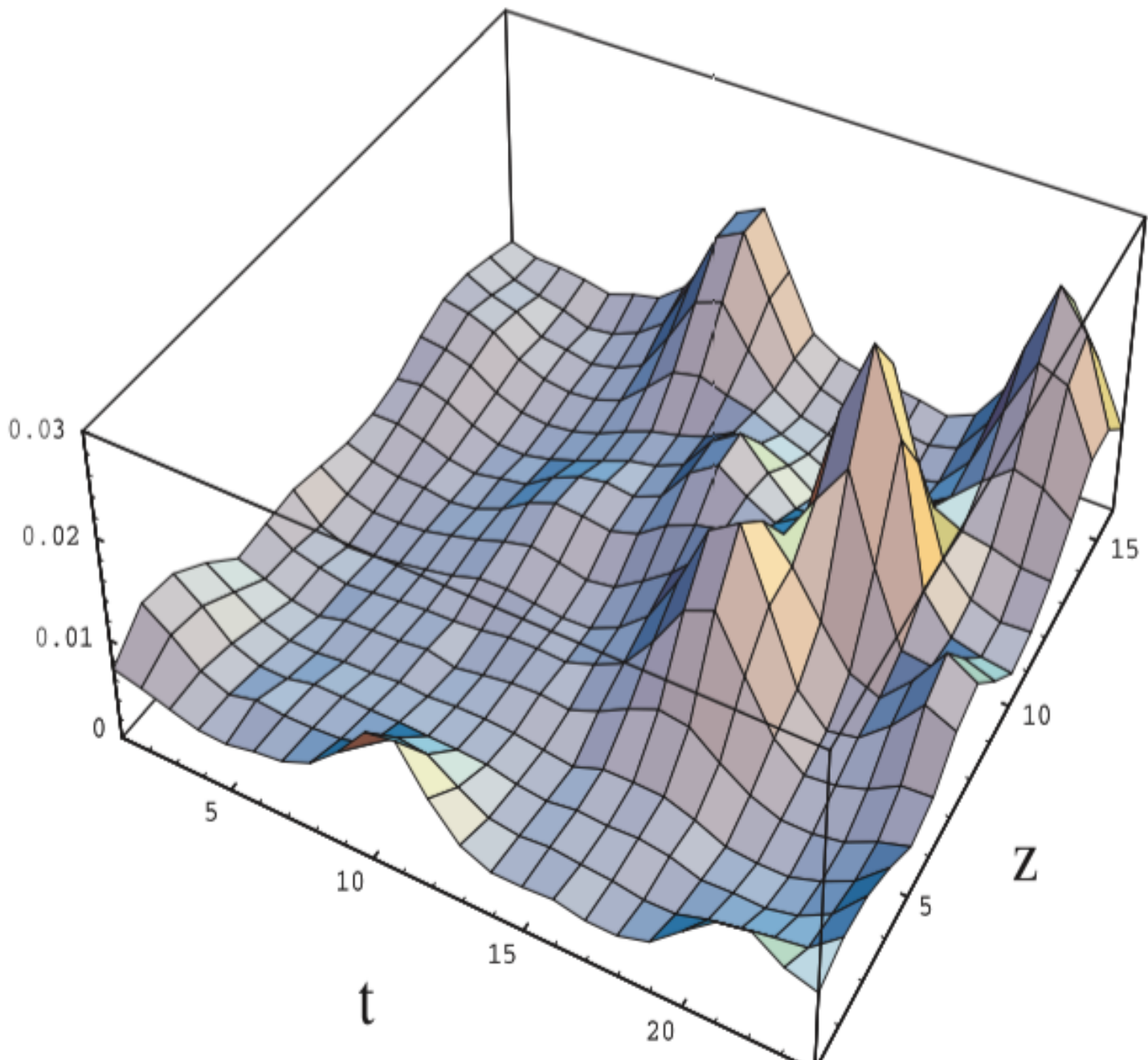
Engel 2009

Lattice QCD

quantum fluctuations:



“smearing”:



Engel 2009

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Perturbative solution

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perturbative ansatz: $B = g_0 B_1 + g_0^2 B_2 + \dots$

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momentum space: $\tilde{B}_1(t) = e^{-tp^2} \tilde{A}(p)$

$$\tilde{B}_2(t, p) = \int_0^t ds \int d^4 q K(t, s, p, q) A(p) A(p - q)$$

$$K(t, s, p, q) \sim \exp[-tp^2 - 2sq(q - p)]$$

etc.

Exponential damping in momentum integrals!

5-dimensional field theory

flowed gauge field:

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$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

$$\begin{aligned}\mathcal{L}_B &\sim \int_0^\infty dt \mathbf{L}_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right) \\ \mathcal{L}_\chi &\sim \int_0^\infty dt \bar{\lambda} (\partial_t - \mathcal{D}^2) \chi + \text{h.c.}\end{aligned}$$

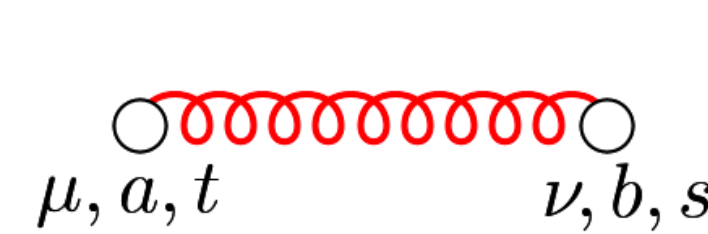
Lüscher, Weisz 2011
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Perturbative approach

$$\mathcal{L}_B \sim \int_0^\infty dt \mathbf{L}_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

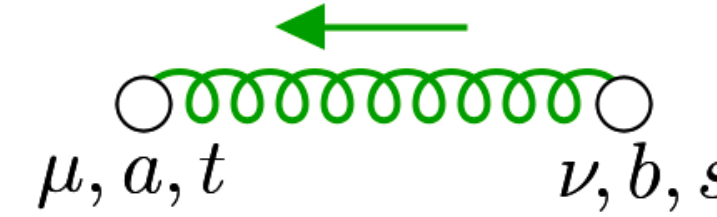
Perturbative approach

$$\mathcal{L}_B \sim \int_0^\infty dt \mathbf{L}_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$



$$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T B_\mu^a(t, x) B_\nu^b(s, 0) | 0 \rangle$$



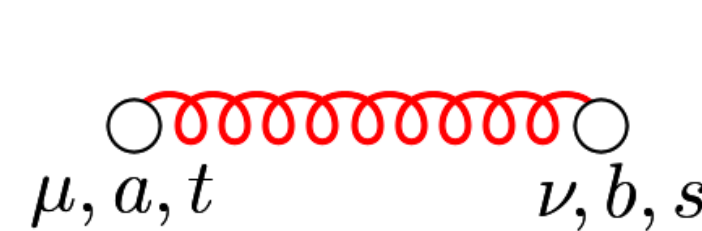
$$\delta_{ab} \delta_{\mu\nu} \theta(t-s) e^{-(t-s)p^2}$$

“gluon flow line”

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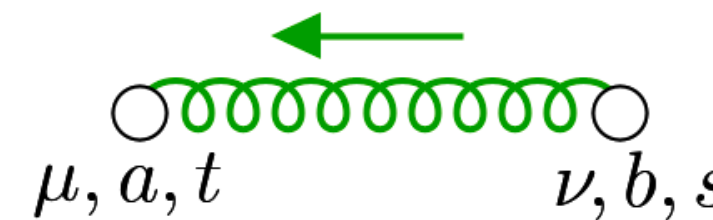
Perturbative approach

$$\mathcal{L}_B \sim \int_0^\infty dt L_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$



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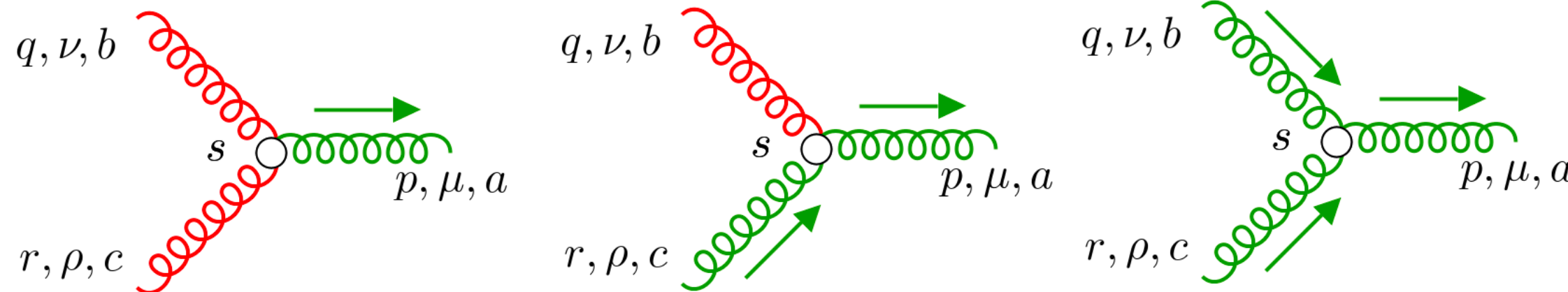
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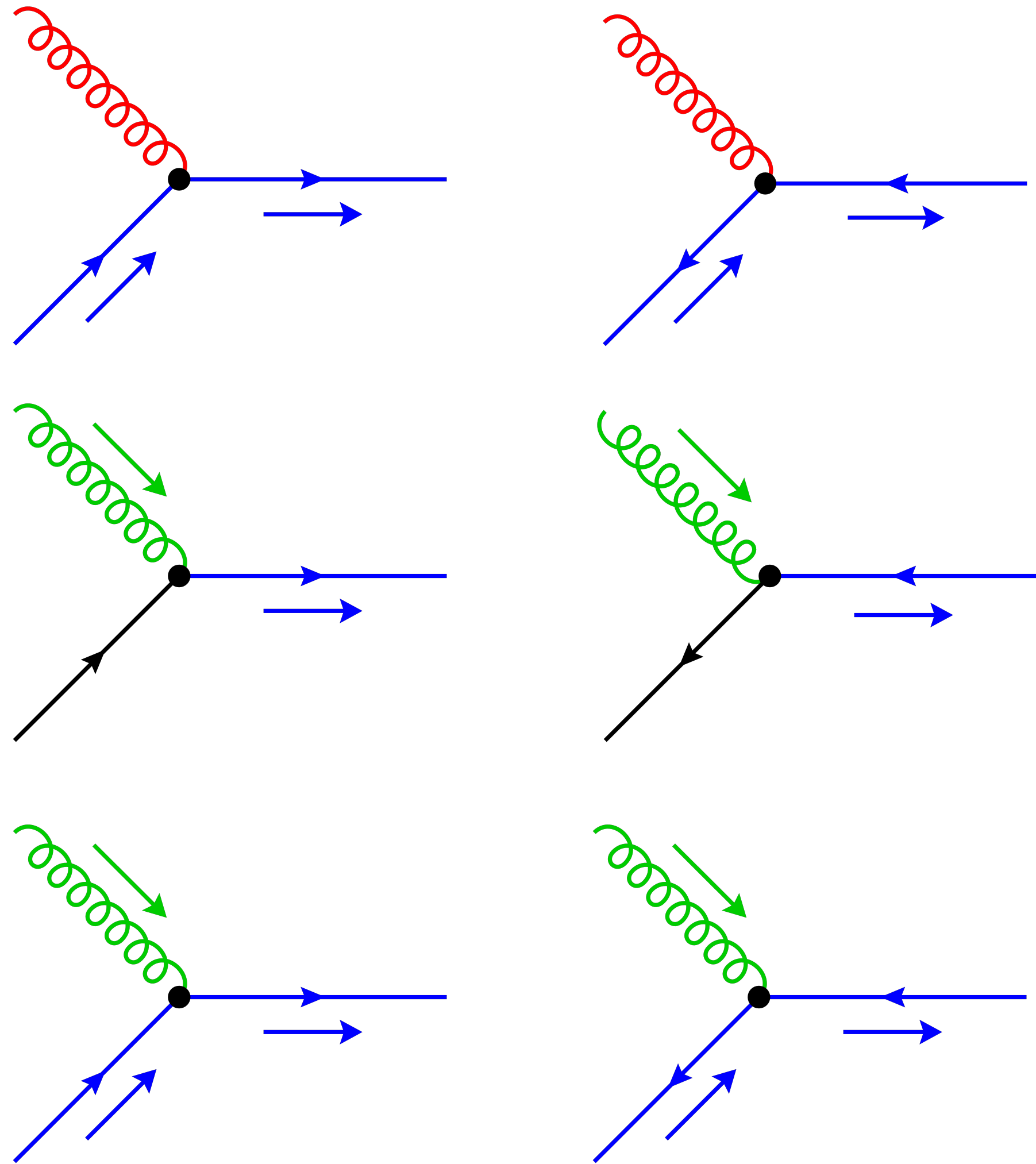
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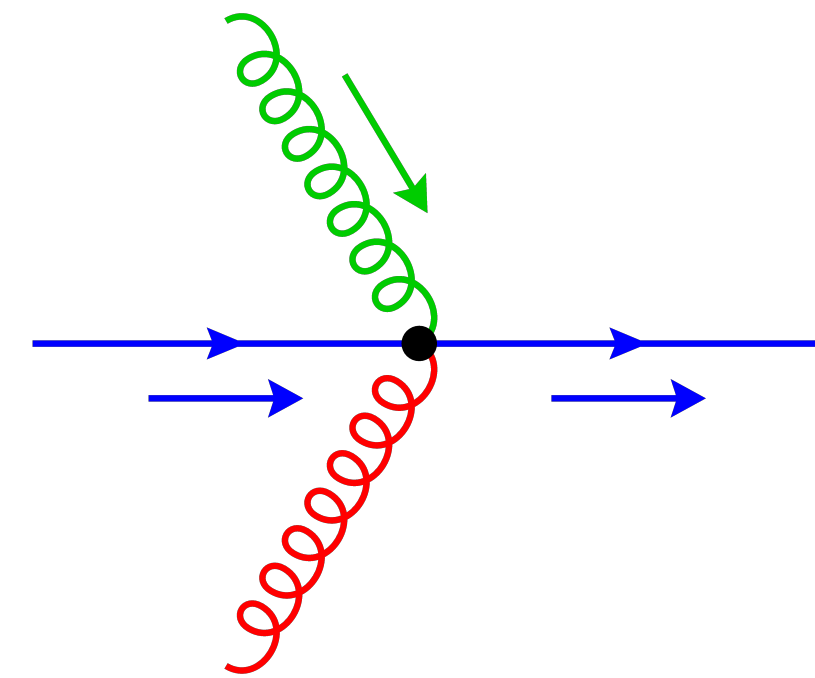
$$-ig f^{abc} \int_0^\infty ds \left(\delta_{\nu\rho} (r-q)_\mu + 2\delta_{\mu\nu} q_\rho - 2\delta_{\mu\rho} r_\nu + (\kappa - 1)(\delta_{\mu\rho} q_\nu - \delta_{\mu\nu} r_\rho) \right)$$

+ 4-gluon vertex

Perturbative approach



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etc.

Renormalization

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

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⇒ renormalization of
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flowed field renormalization:

$$B_\mu^{\text{R}}(t, x) = B_\mu(t, x)$$

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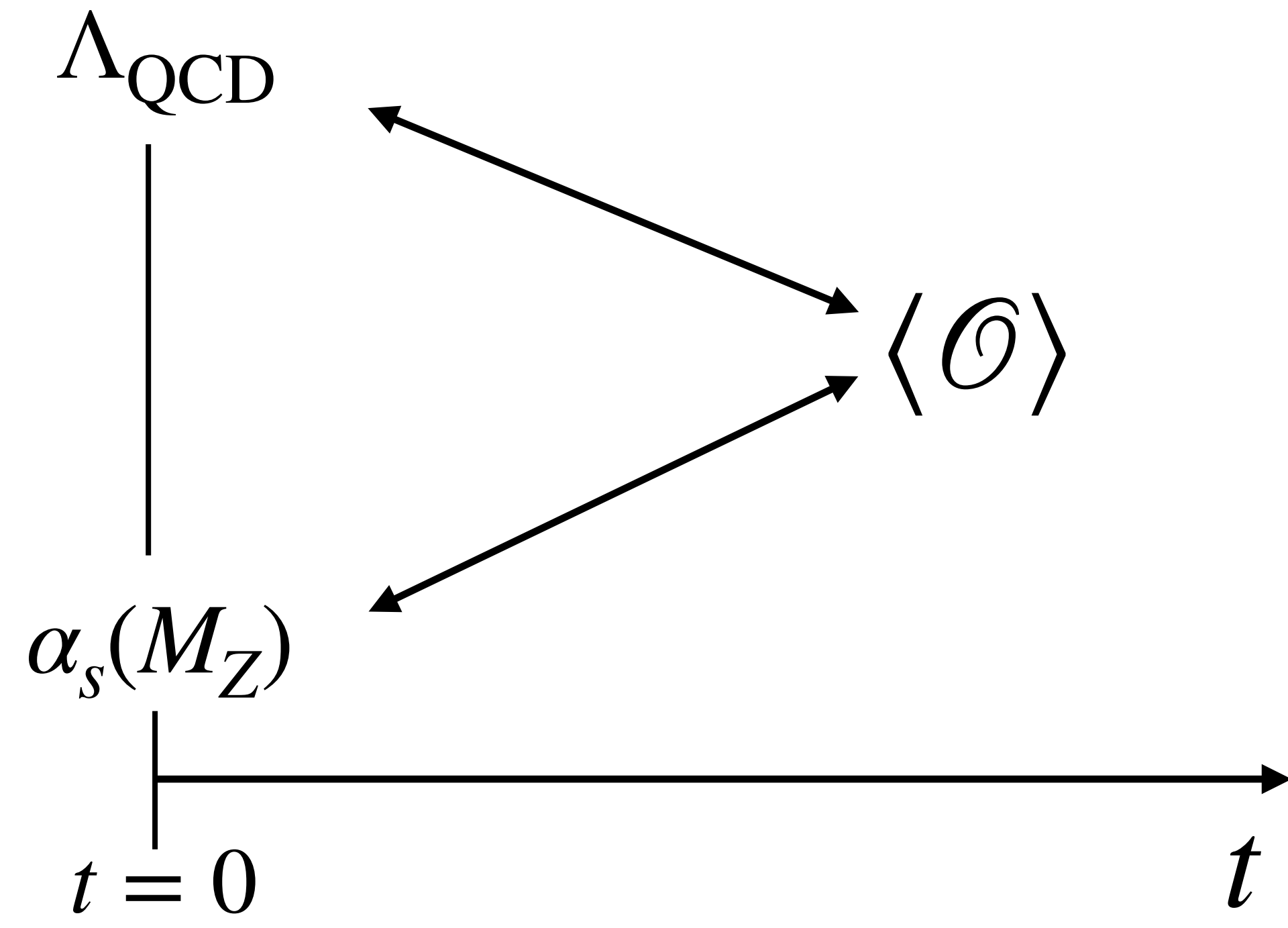
$$B_\mu^{\text{R}}(t, x) = B_\mu(t, x)$$

$$\chi^{\text{R}}(t, x) = \mathring{Z}_\chi^{1/2} \chi(t, x)$$

$$\mathring{Z}_\chi \langle \bar{\chi}(t) \mathcal{D} \chi(t) \rangle \Big|_{m=0} \equiv \text{LO}$$

NLO: Suzuki, Makino (2014)

NNLO: Artz, RH, Lange, Neumann, Prausa (2019)

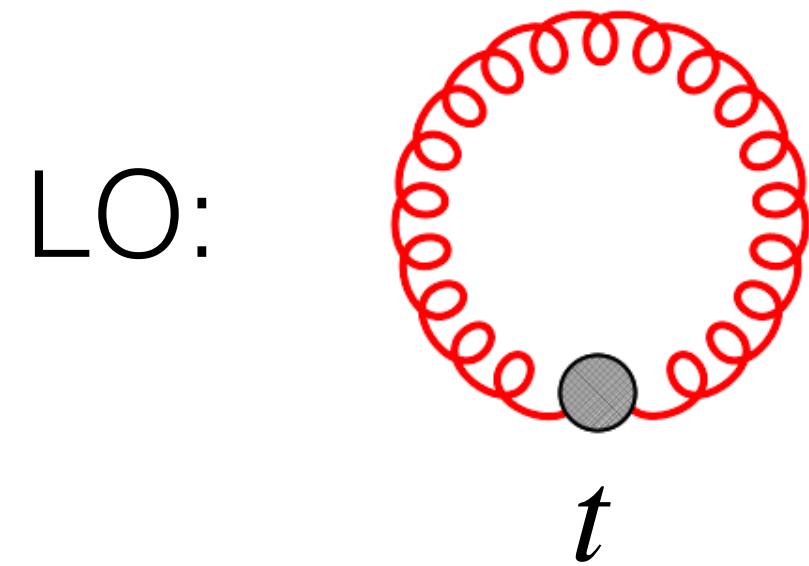


Let's calculate

$$\langle E(t) \rangle \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$

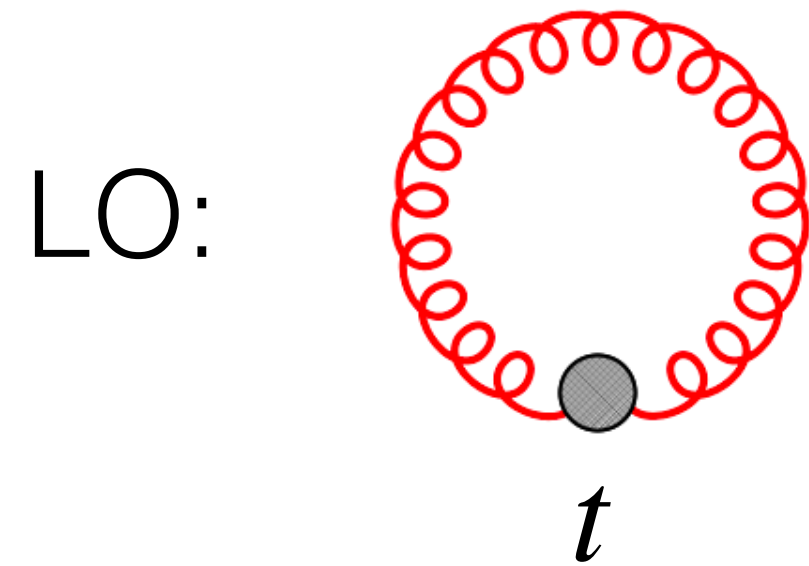
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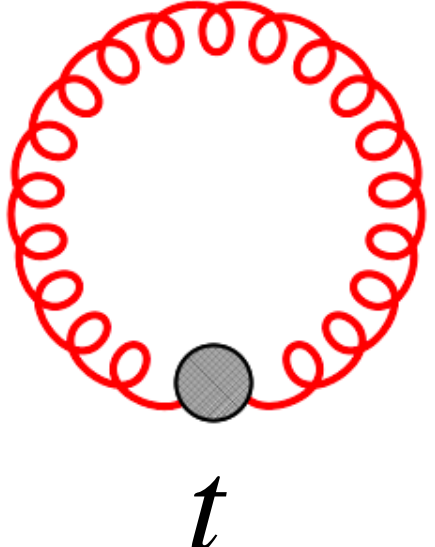
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LO:  $\sim \int d^D p e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

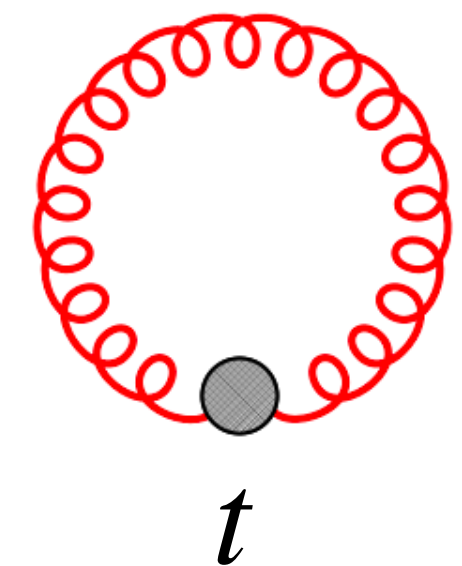
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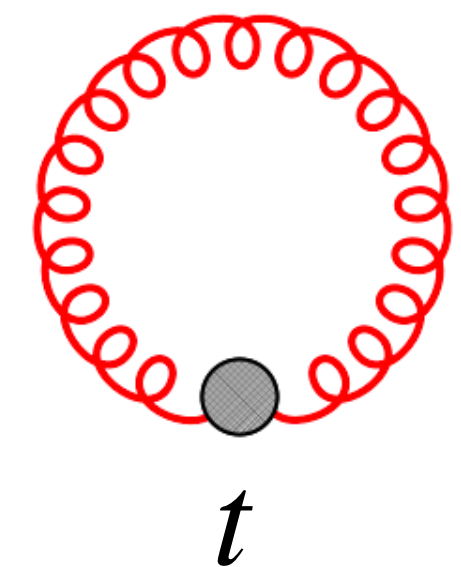
explicitly: $\langle E(t) \rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$

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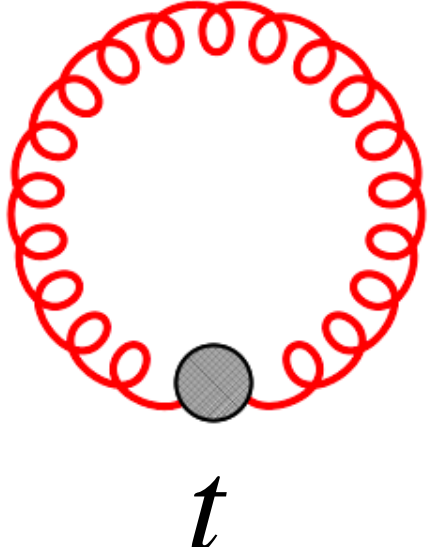
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→ measure α_s on the lattice?

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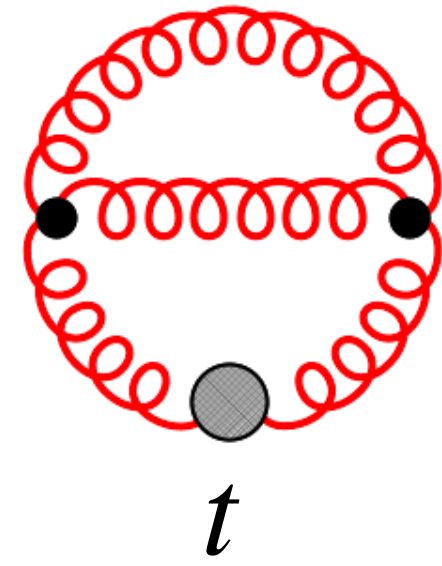
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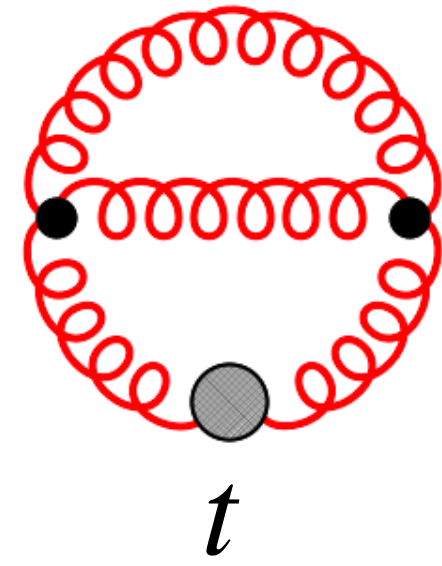
$$\alpha_s = \alpha_s(\mu)$$

Higher orders

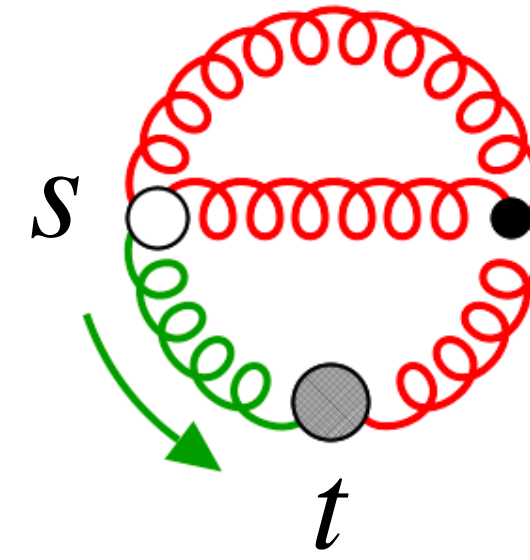


$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4 k^2 (p-k)^2}$$

Higher orders

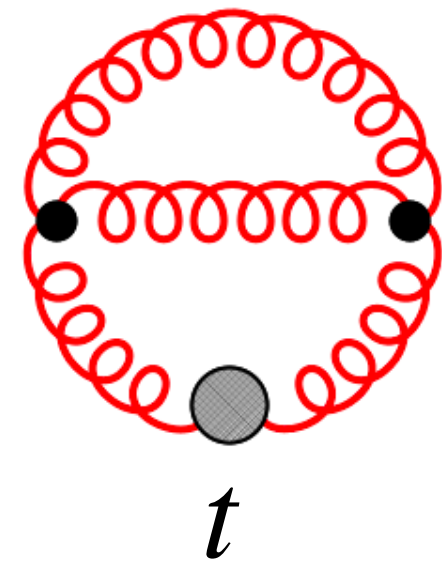


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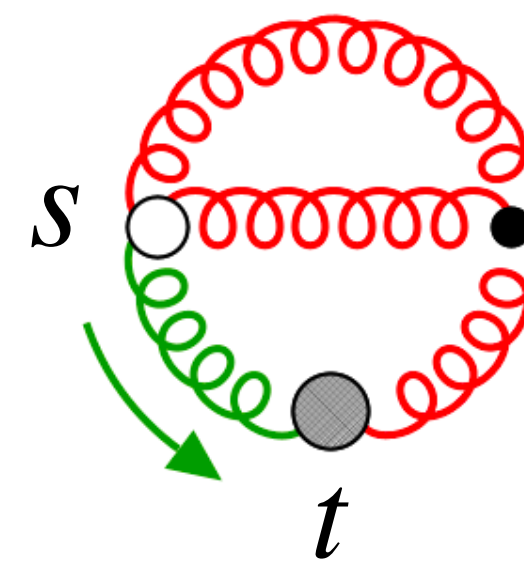


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

Higher orders



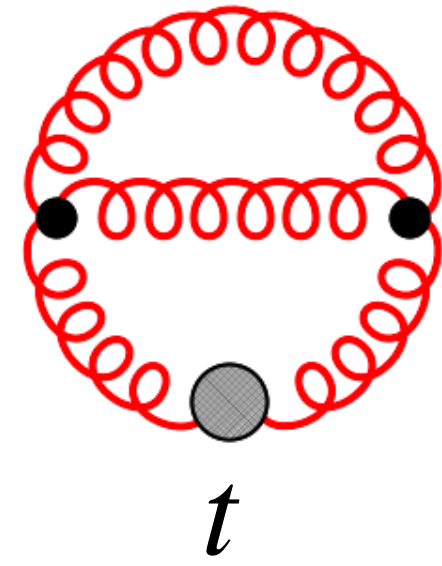
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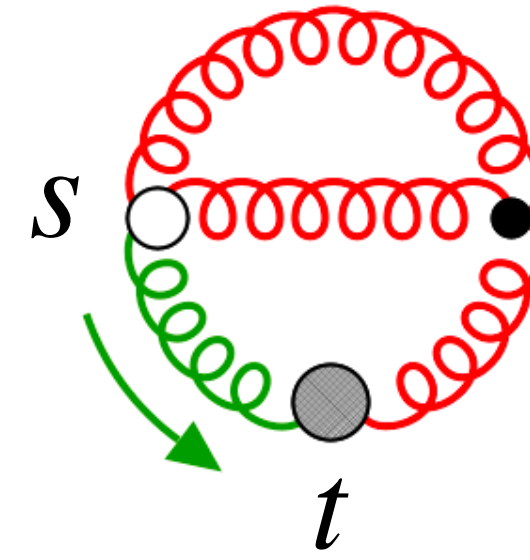
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- generalized loop integrals

Higher orders



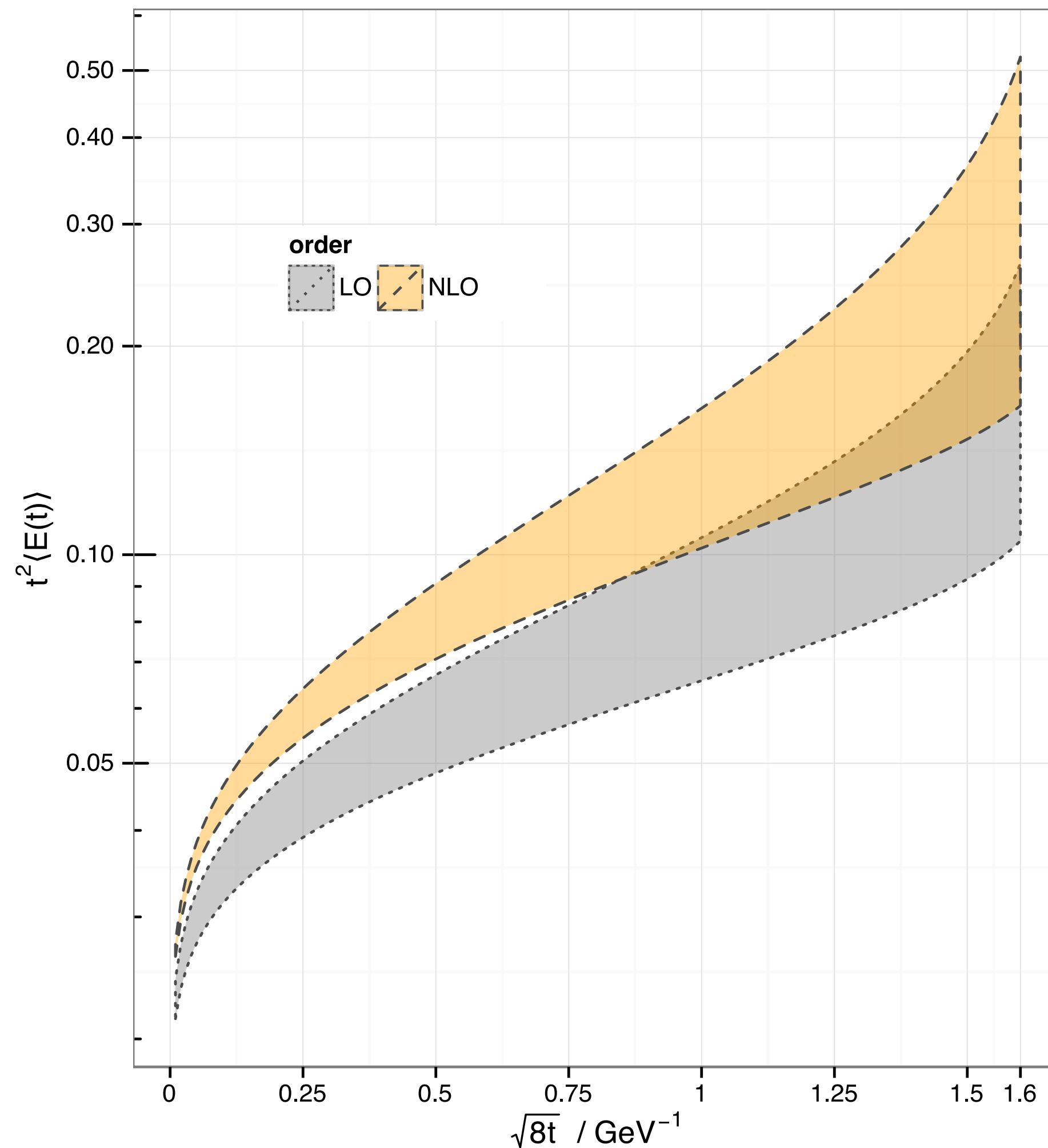
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- generalized loop integrals
- integration over flow-time parameters

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) \right] \quad \text{Lüscher 2010}$$



$$k_1 = \left(\frac{52}{9} + \frac{22}{3} \ln 2 - 3 \ln 3 \right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative
accuracy on α_s : $\pm 3-5\%$

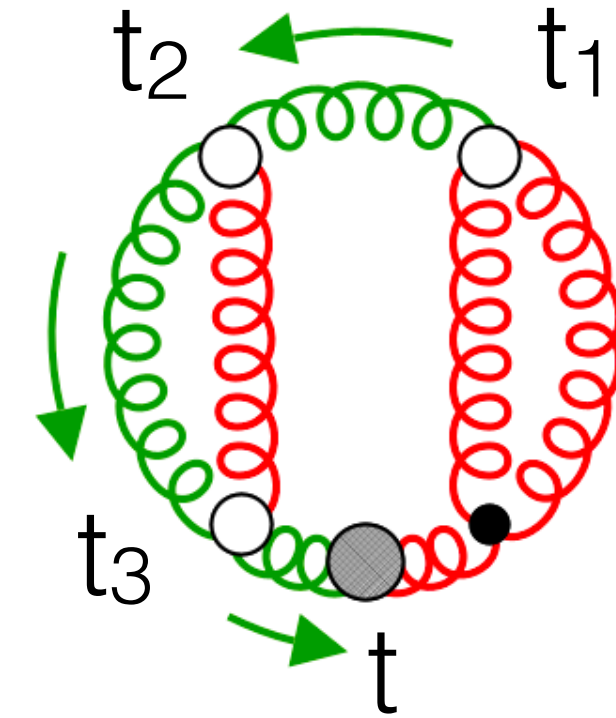
PDG: $\pm 1\%$

Three-loop calculation

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



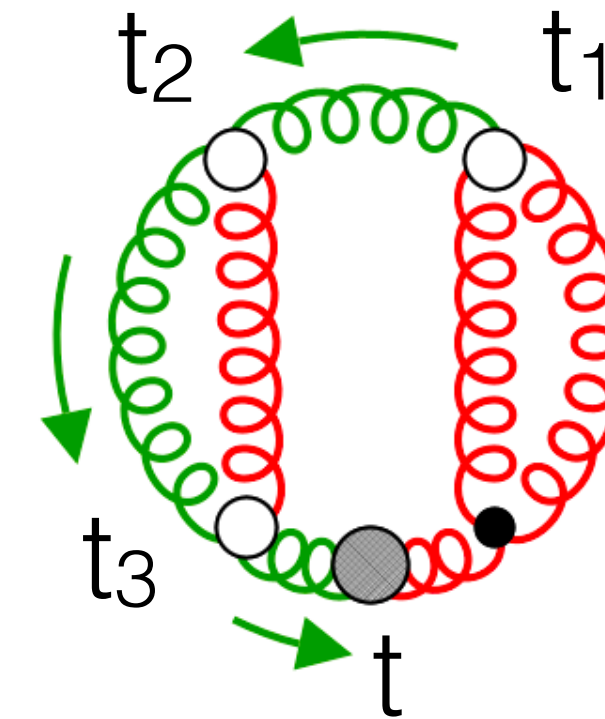
Three-loop calculation

The usual problems:

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The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals



Artz, RH, Lange, Neumann, Prausa '19

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation:

qgraf Nogueira 1993

diagram analyzation:

q2e/exp RH, Seidensticker, Steinhauser 1997

→ tapir/exp Gerlach, Herren, Lang 2022

algebraic manipulations:

FORM Vermaseren 2000, ...

reduction to masters:

Kira ⊗ FireFly Usovitsch, Uwer, Maierhöfer 2017

Chetyrkin, Tkachov 1981

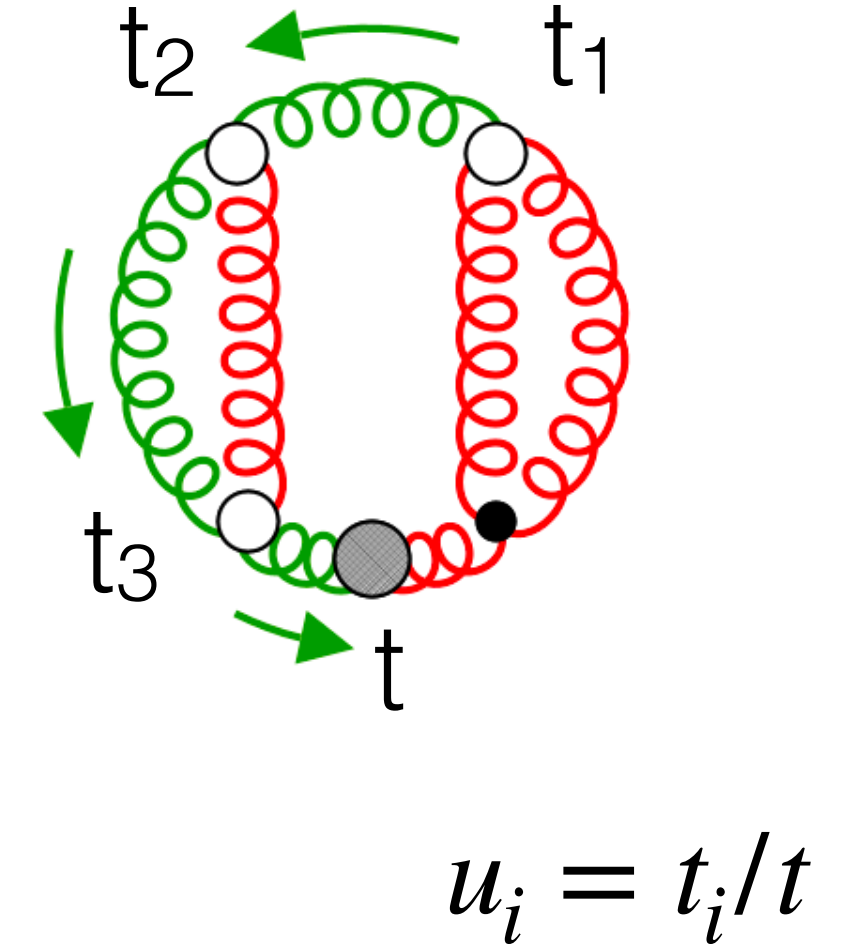
Laporta 2000

⊗ Klappert, Klein, Lange 2019

Three-loop calculation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\}) =$$

$$= \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \int d^D p_1 d^D p_2 d^D p_3 \frac{\exp \left[-t \left(a_1(u) p_1^2 + \dots + a_6(u) p_6^2 \right) \right]}{(p_1^2)^{b_1} \dots (p_6^2)^{b_6}}$$



IbP identities: $\frac{\partial}{\partial p_i} \cdot p_j I(\mathbf{c}, \mathbf{a}, \mathbf{b}) = D \delta_{ij} I(\mathbf{c}, \mathbf{a}, \mathbf{b}) + \sum I(\mathbf{c}', \mathbf{a}, \mathbf{b}')$

$$\frac{\partial}{\partial u_i} I(\mathbf{c}, \mathbf{a}, \mathbf{b}) = I(\mathbf{c}', \mathbf{a}(u=1), \mathbf{b}') - I(\mathbf{c}', \mathbf{a}(u=0), \mathbf{b}')$$

[Artz, RH, Lange, Neumann, Prausa 2019]

Huge systems of linear equations, solved by “master integrals”.

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sector decomposition:

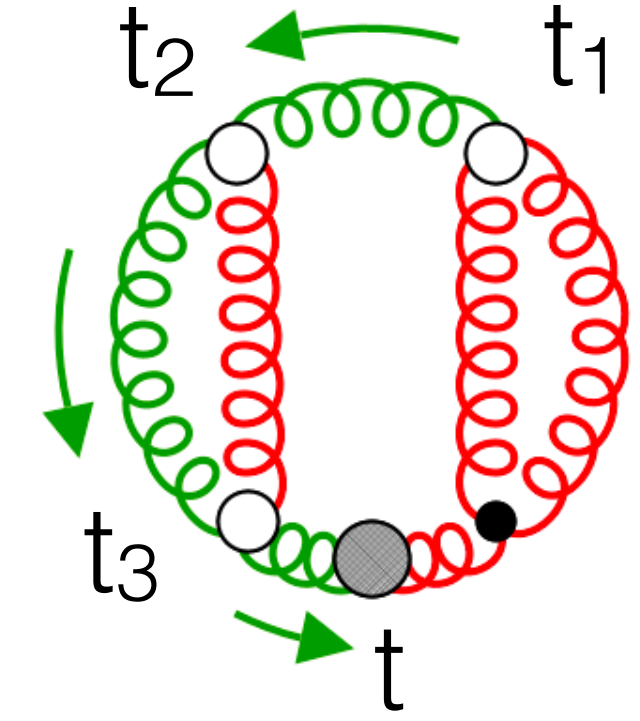
Binoth, Heinrich 2002

$$\int d^D k \int d^D p \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p)^2} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \dots$$

Numerical evaluation

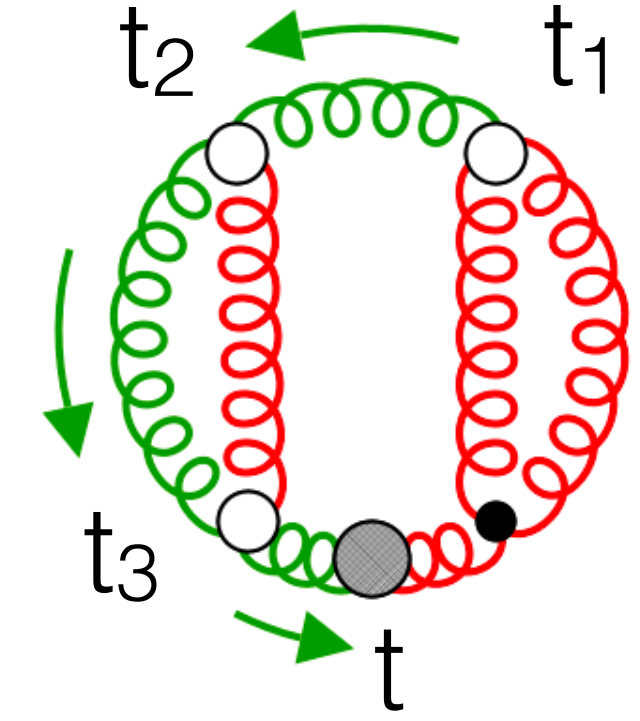
$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\}) =$$

$$= \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \int d^D p_1 d^D p_2 d^D p_3 \frac{\exp \left[-t \left(a_1(u) p_1^2 + \dots + a_6(u) p_6^2 \right) \right]}{(p_1^2)^{b_1} \dots (p_6^2)^{b_6}}$$



Numerical evaluation

$$\begin{aligned}
 I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\}) &= \\
 &= \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \int d^D p_1 d^D p_2 d^D p_3 \frac{\exp \left[-t \left(a_1(u) p_1^2 + \dots + a_6(u) p_6^2 \right) \right]}{(p_1^2)^{b_1} \dots (p_6^2)^{b_6}}
 \end{aligned}$$



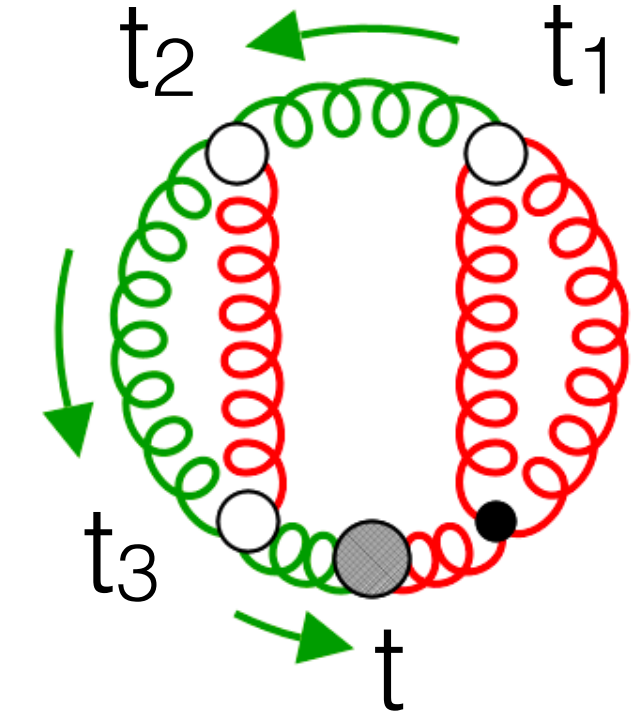
Schwinger parameters:

$$\frac{1}{(p^2)^b} \sim \int_0^\infty dx x^{b-1} e^{-xp^2}$$

Numerical evaluation

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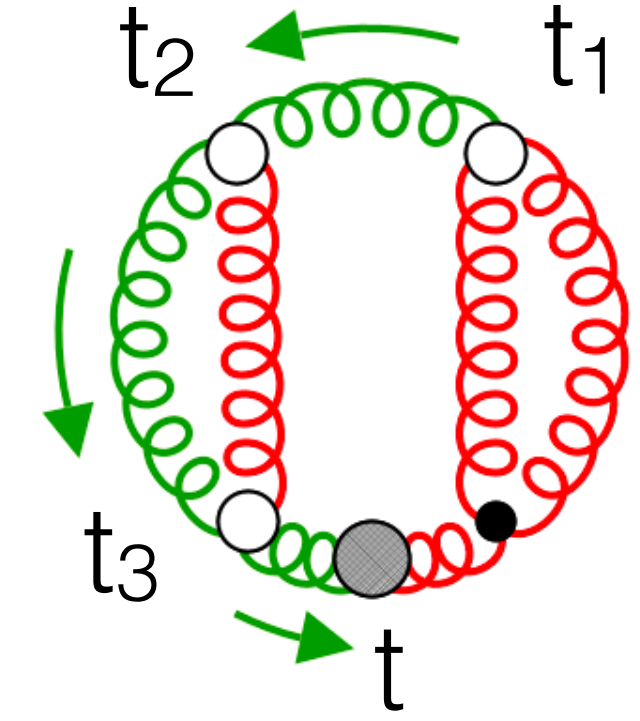
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$$\sim \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \left(\prod_{j=1}^6 \int_0^\infty dx_j x_j^{b_j-1} \right) \int d^D p_1 d^D p_2 d^D p_3 \exp \left[-t \mathbf{p}^T A(x, u) \mathbf{p} \right]$$

Numerical evaluation

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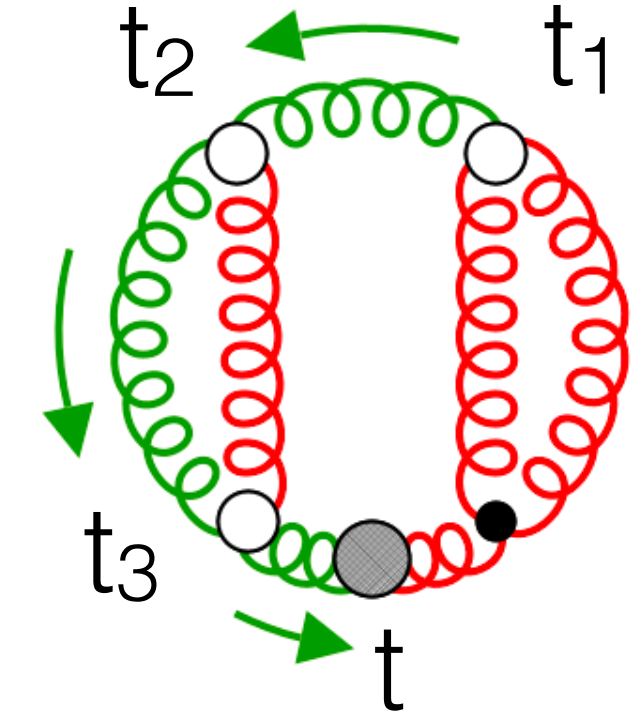
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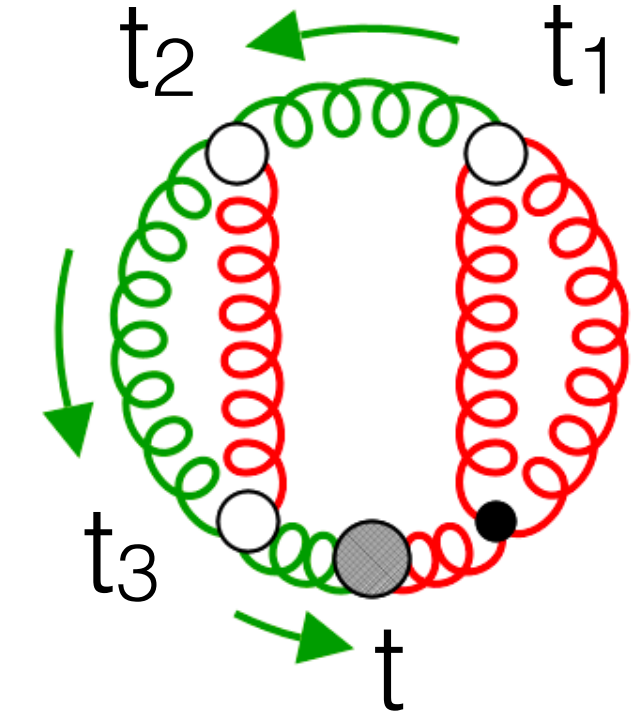
$$\frac{1}{(p^2)^b} \sim \int_0^\infty dx x^{b-1} e^{-xp^2} \quad \left(\begin{array}{c} \text{map} \\ \rightarrow \end{array} \int_0^1 dx \dots \right)$$

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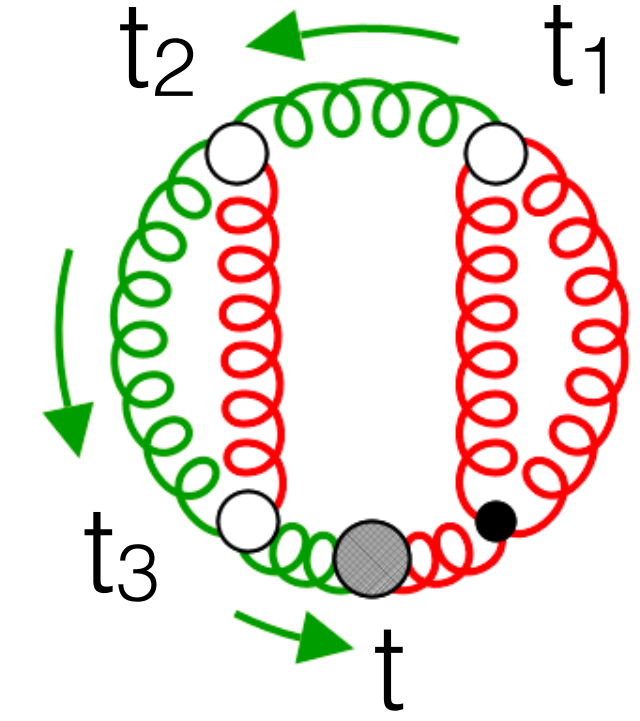
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Binoth, Heinrich (2000)

Implementation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\})$$

$$c_1 = c_2 = 0$$

$$a_1 = u_1 u_2, \quad a_2 = u_2, \quad a_3 = u_2 - u_1 u_2$$

$$a_4 = 1, \quad a_5 = 1 + u_1 u_2, \quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

$$b_2 = b_3 = b_5 = b_6 = 0$$

Implementation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\})$$

$$\begin{aligned} c_1 = c_2 = 0 \\ a_1 = u_1 u_2, \quad a_2 = u_2, \quad a_3 = u_2 - u_1 u_2 \\ a_4 = 1, \quad a_5 = 1 + u_1 u_2, \quad a_6 = 1 - u_2 \\ b_1 = b_4 = 1 \\ b_2 = b_3 = b_5 = b_6 = 0 \end{aligned}$$

ftint RH, Nellopoulos, Olsson, Wesle '24

(based on pySecDec)

Heinrich, Magerya, Kerner, Jones, ...

Implementation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\})$$

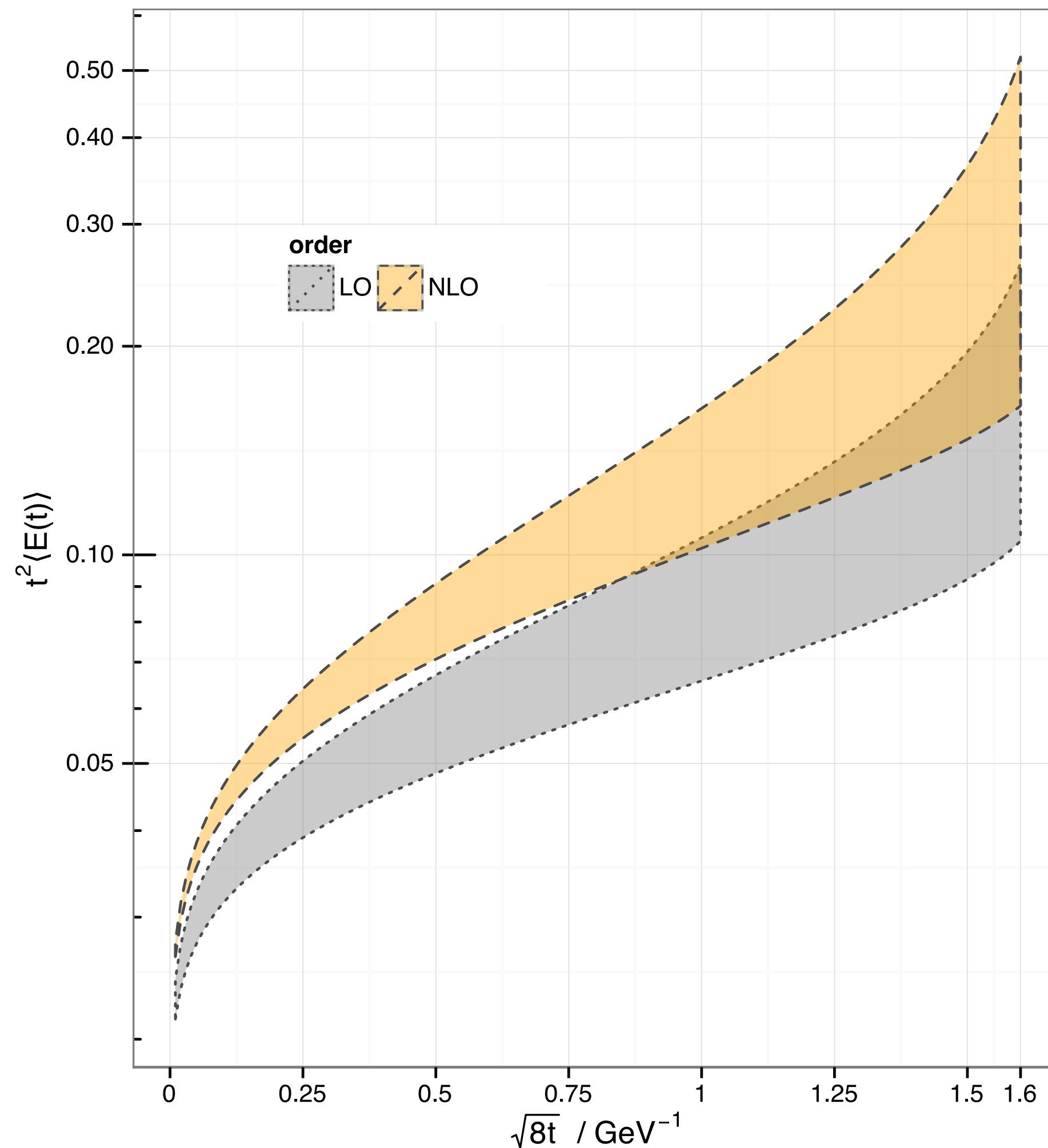
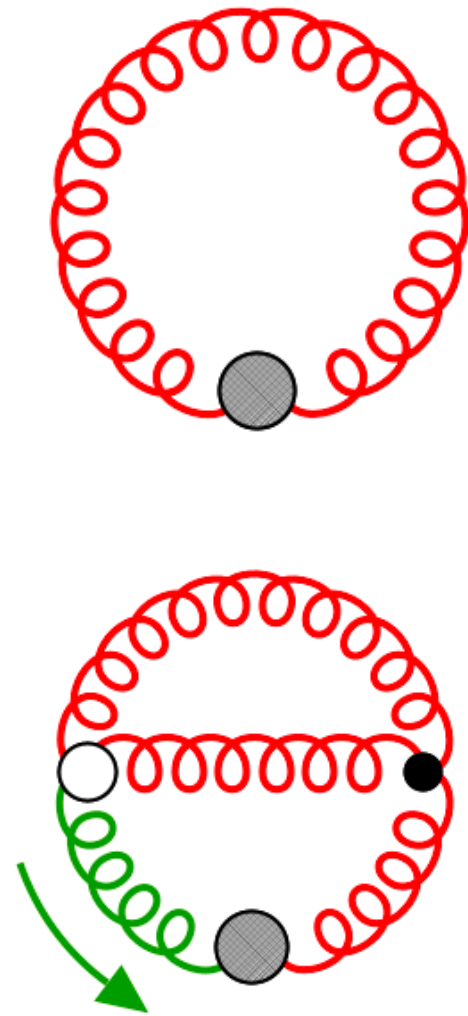
$$\begin{aligned} c_1 &= c_2 = 0 \\ a_1 &= u_1 u_2, \quad a_2 = u_2, \quad a_3 = u_2 - u_1 u_2 \\ a_4 &= 1, \quad a_5 = 1 + u_1 u_2, \quad a_6 = 1 - u_2 \\ b_1 &= b_4 = 1 \\ b_2 &= b_3 = b_5 = b_6 = 0 \end{aligned}$$

ftint RH, Nellopoulos, Olsson, Wesle '24
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```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}}, {1,0,0,1,0,0}] -> (  
+eps^-1*(+8.33333333333333343*10^-02+0.000000000000000000*10^+00*I)  
+eps^-1*(+1.4433895444086145*10^-15+0.000000000000000000*10^+00*I)*plusminus  
+eps^0*(+3.0238270284562663*10^-01+0.000000000000000000*10^+00*I)  
+eps^0*(+1.6918362746499228*10^-08+0.000000000000000000*10^+00*I)*plusminus  
+eps^1*(+6.5531010458012129*10^-01+0.000000000000000000*10^+00*I)  
+eps^1*(+3.7857260802916662*10^-08+0.000000000000000000*10^+00*I)*plusminus  
)
```

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) \right]$$

Lüscher 2010



$$k_1 = \left(\frac{52}{9} + \frac{22}{3} \ln 2 - 3 \ln 3 \right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

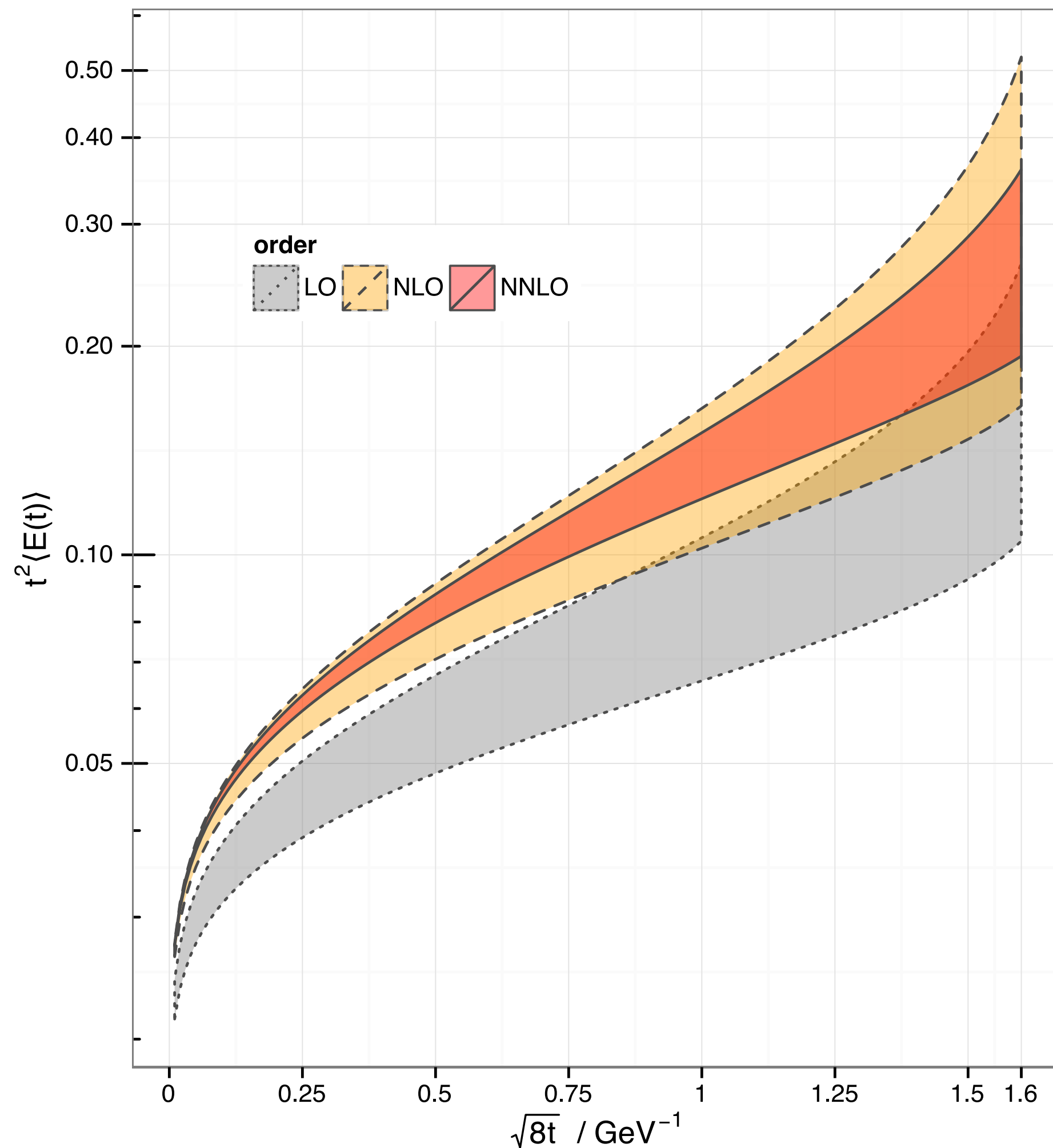
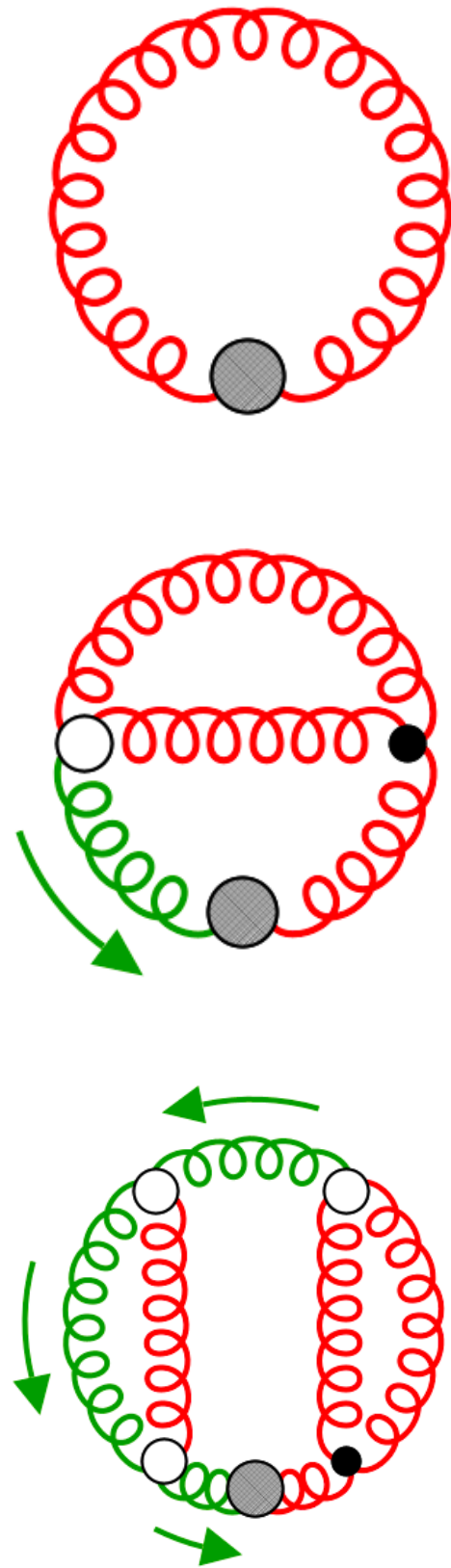
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

resulting perturbative
accuracy on α_s : $\pm 3-5\%$

PDG: $\pm 1\%$

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right]$$

RH, Neumann 2016



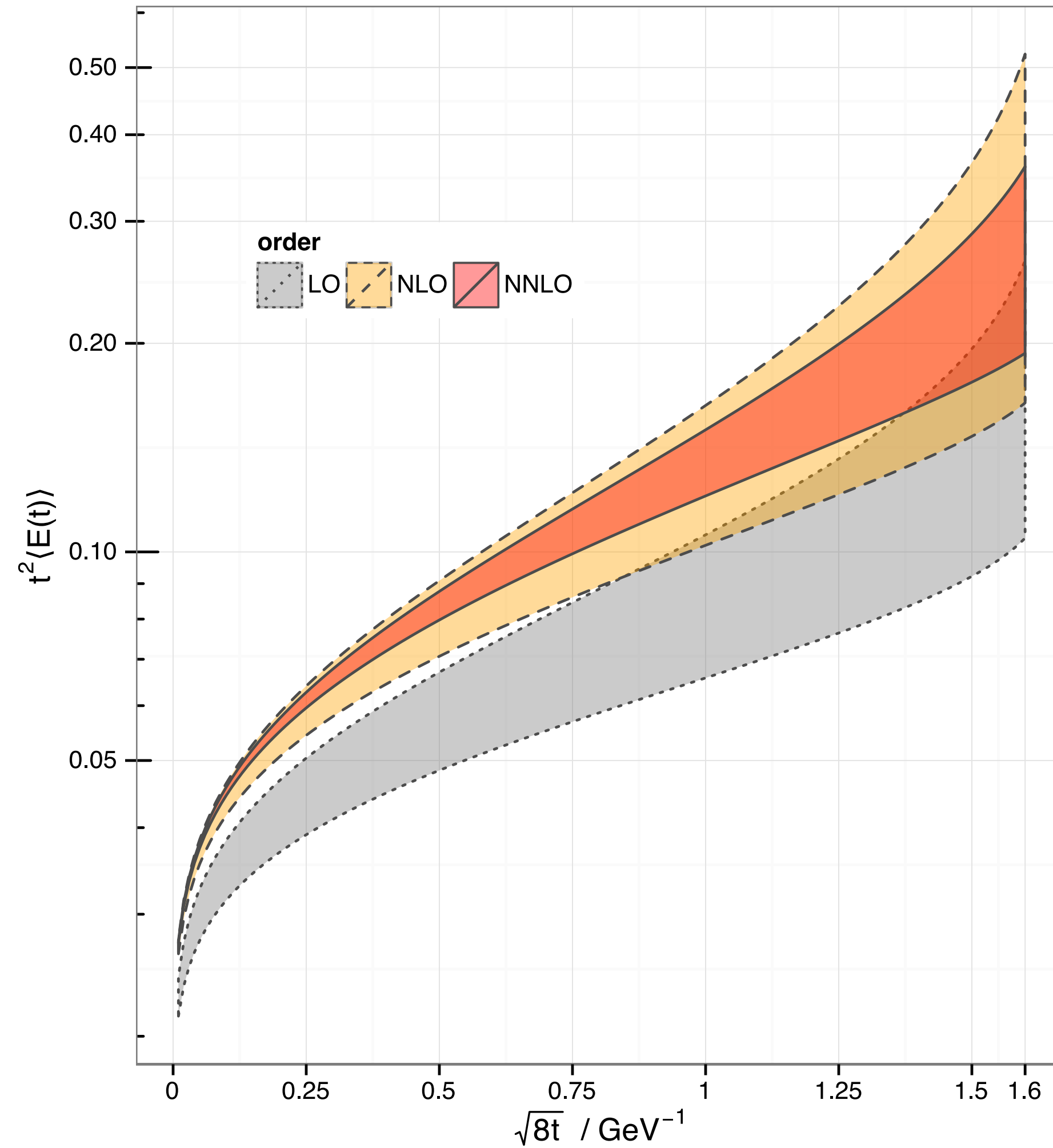
resulting perturbative
accuracy on α_s : $O(1\%)$

PDG: $\pm 1\%$

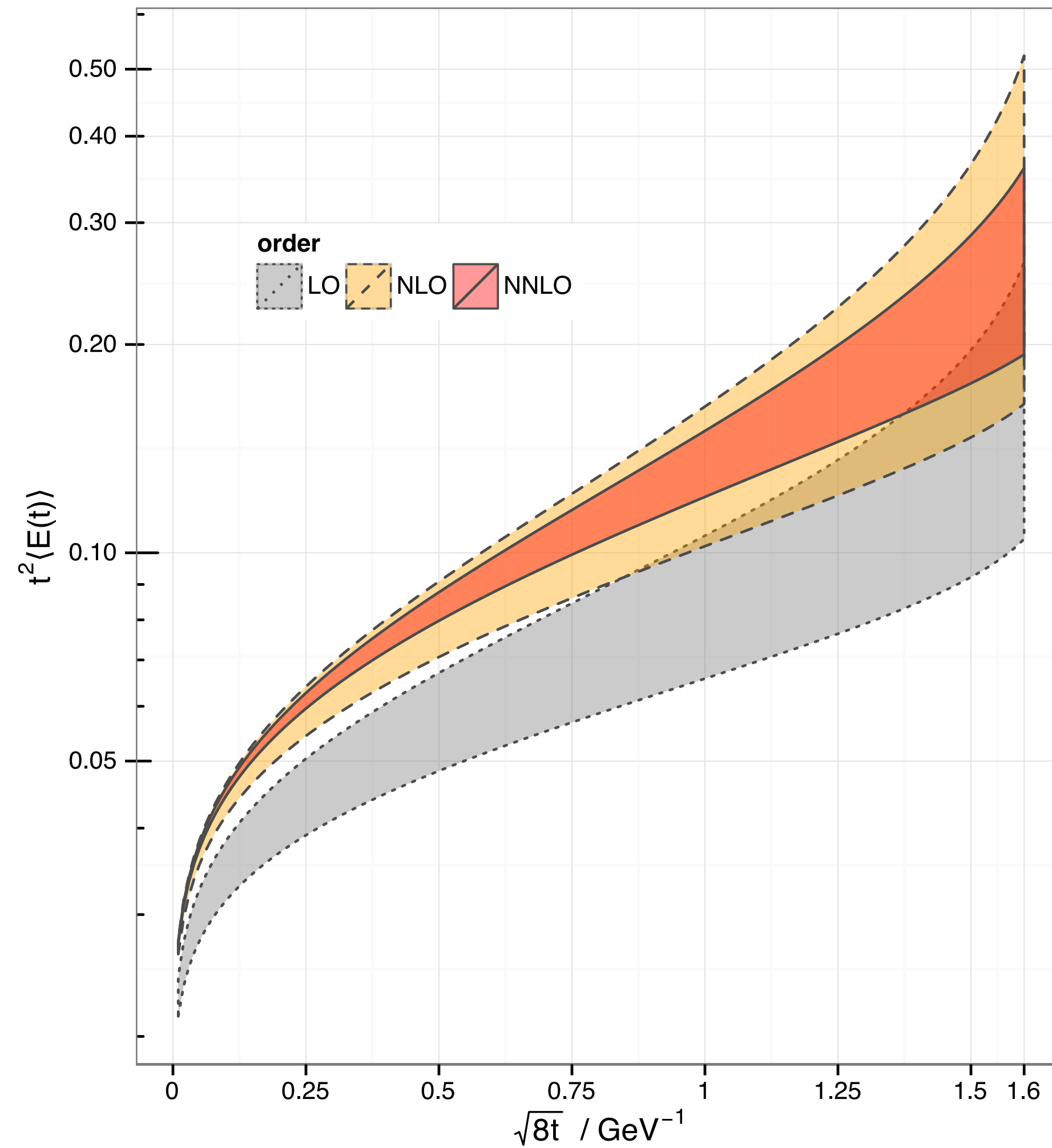
Derive $\alpha_s(m_Z)$

q_8	$t^2 \langle E(t) \rangle \cdot 10^4$							
	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

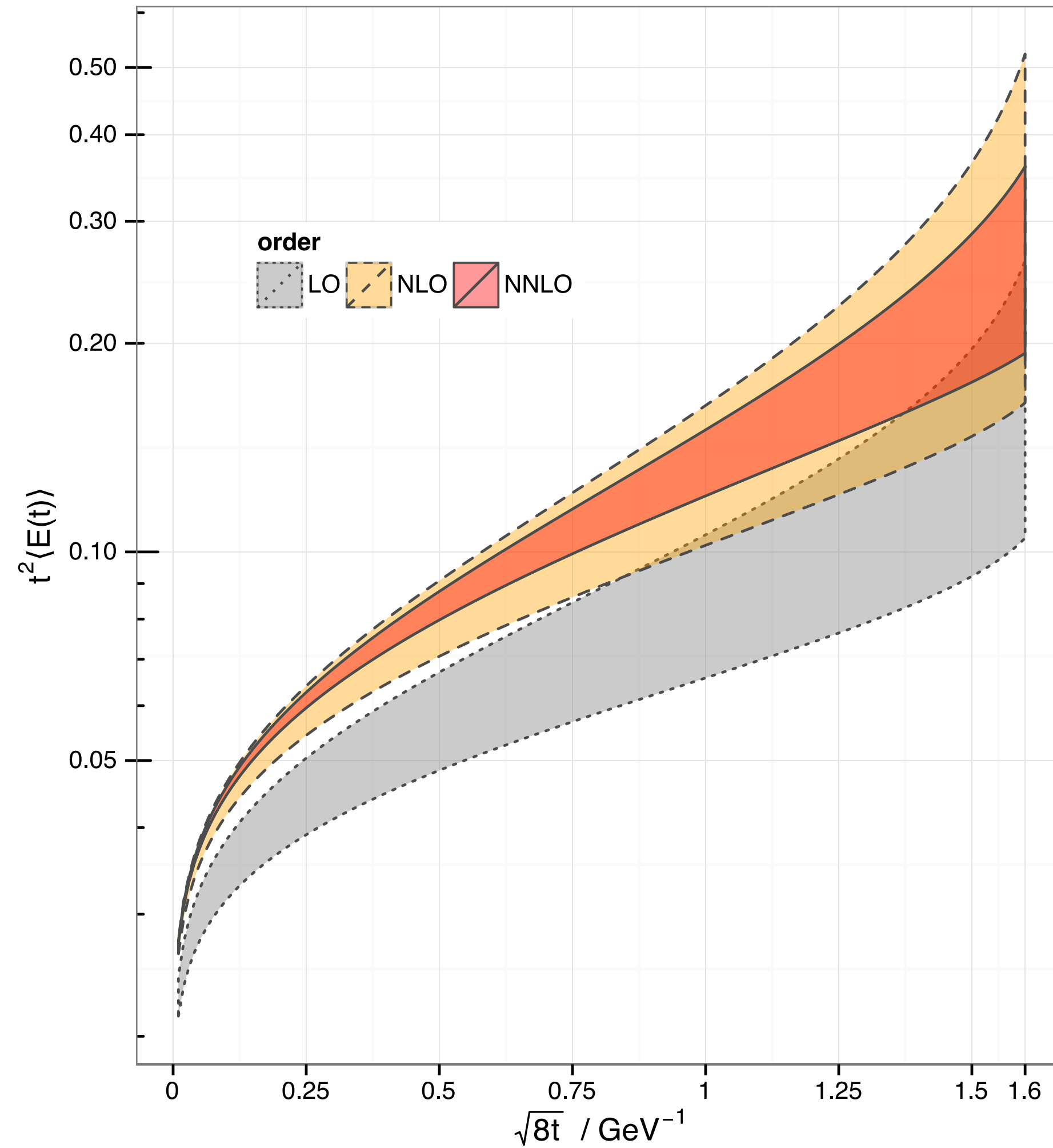
$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right]$$



$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right] \equiv \frac{3}{4\pi} \hat{\alpha}_s(t) = \frac{3}{4} \hat{a}_s(t)$$

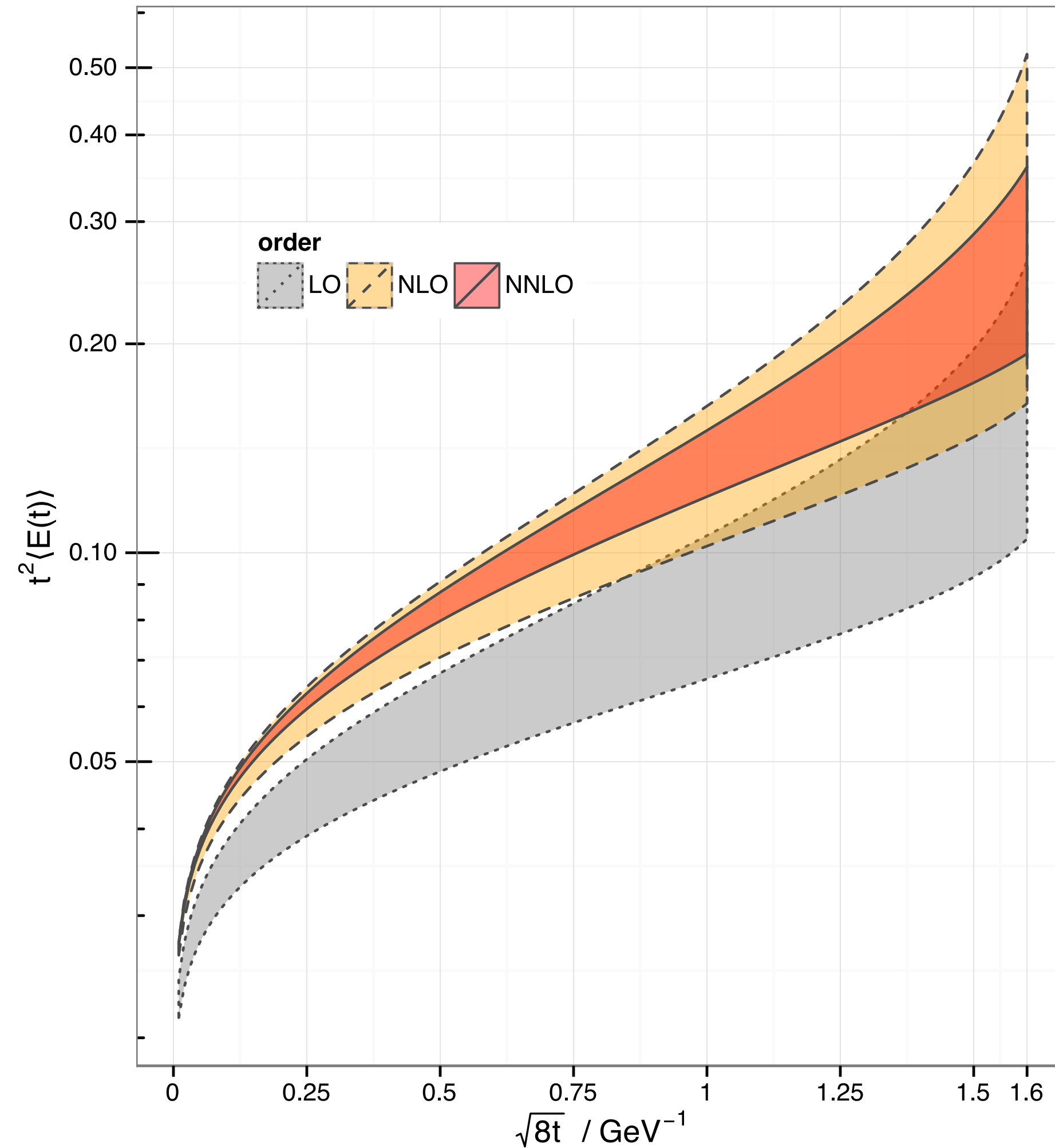


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$$t \frac{d}{dt} \hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

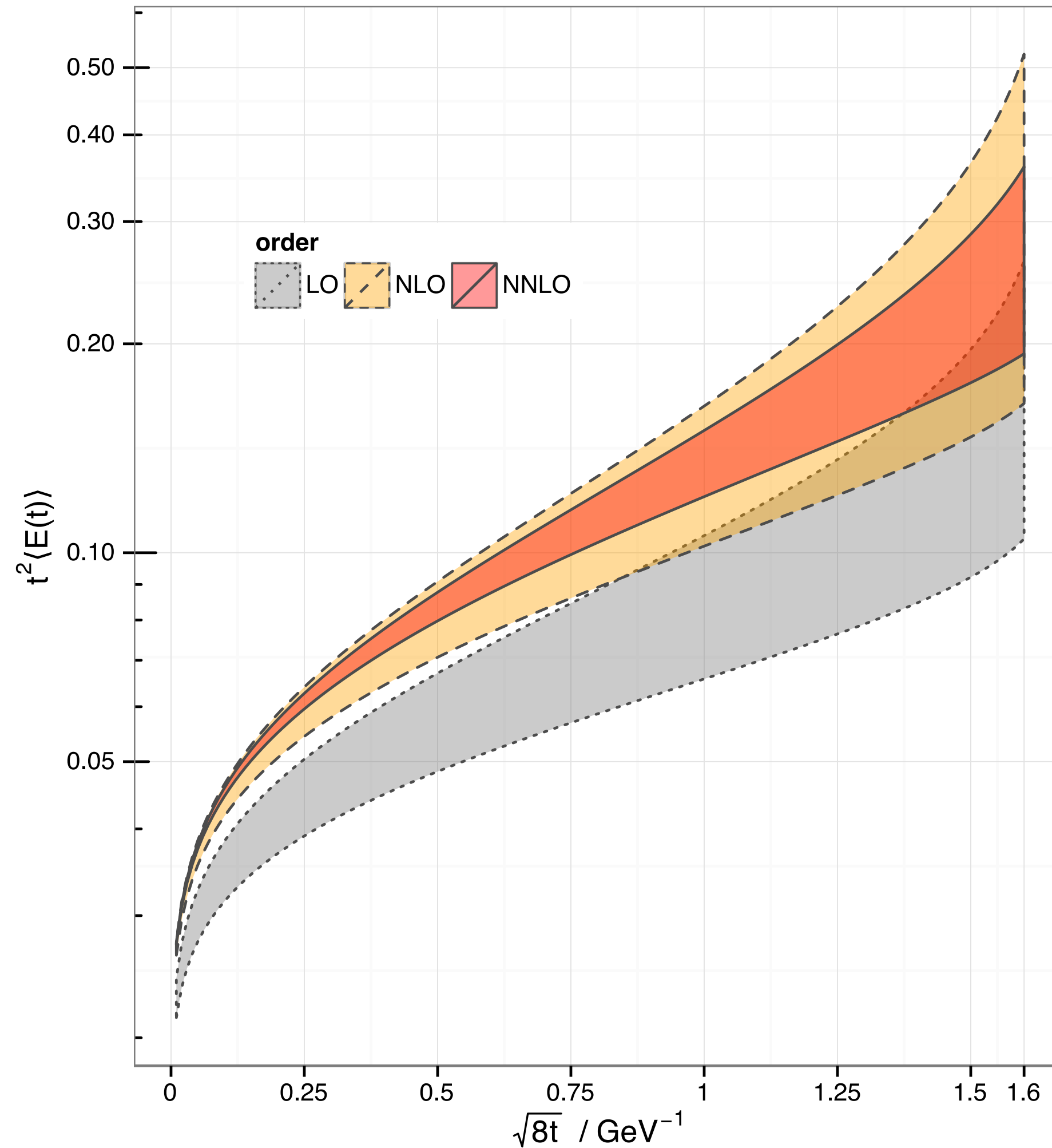
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$$t \frac{d}{dt} \hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

$$= \hat{a}_s^2 \left[\hat{\beta}_0 + \hat{a}_s \hat{\beta}_1 + \hat{a}_s^2 \hat{\beta}_2 + \dots \right]$$

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right] \equiv \frac{3}{4\pi} \hat{\alpha}_s(t) = \frac{3}{4} \hat{a}_s(t)$$

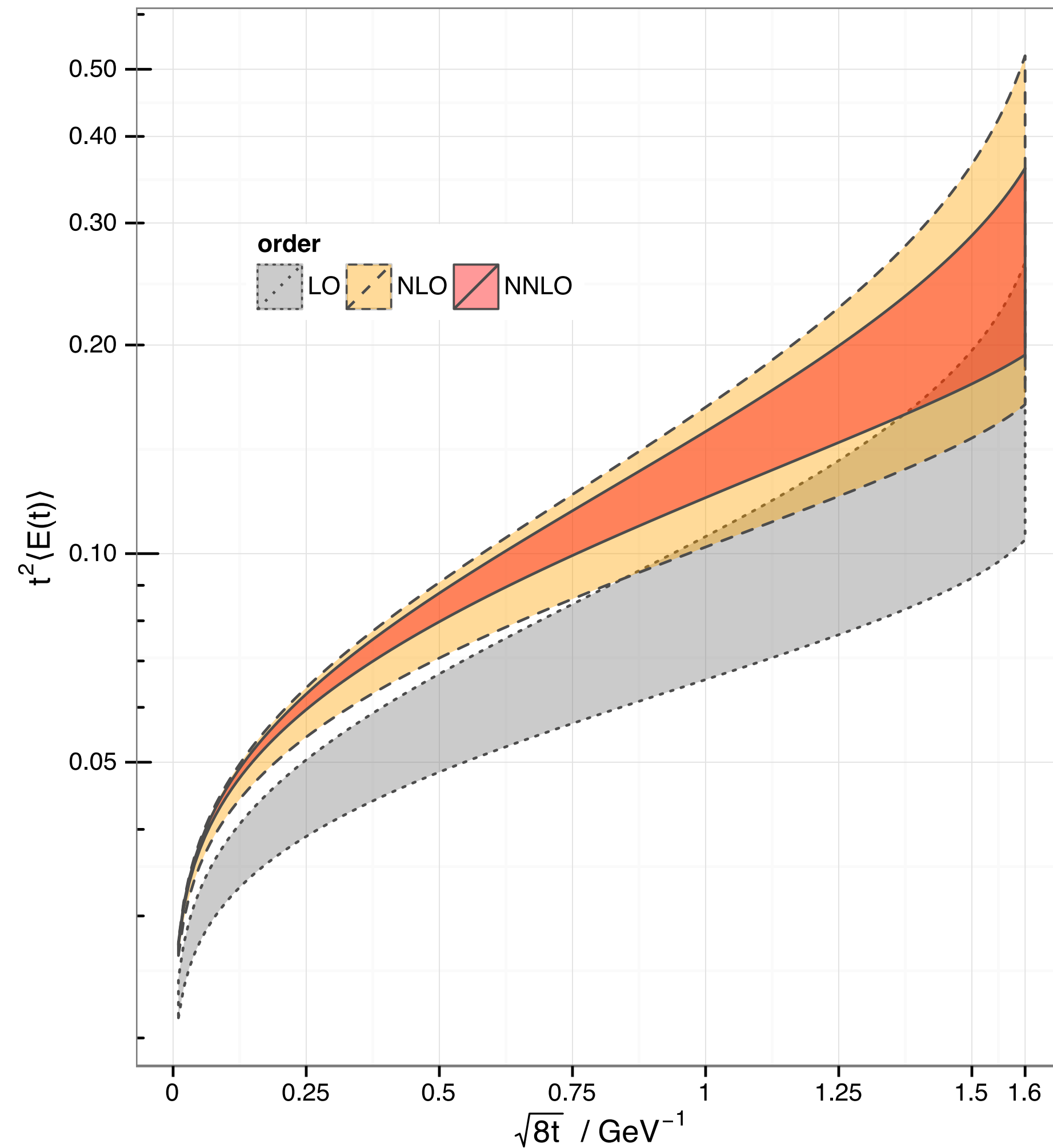


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universal

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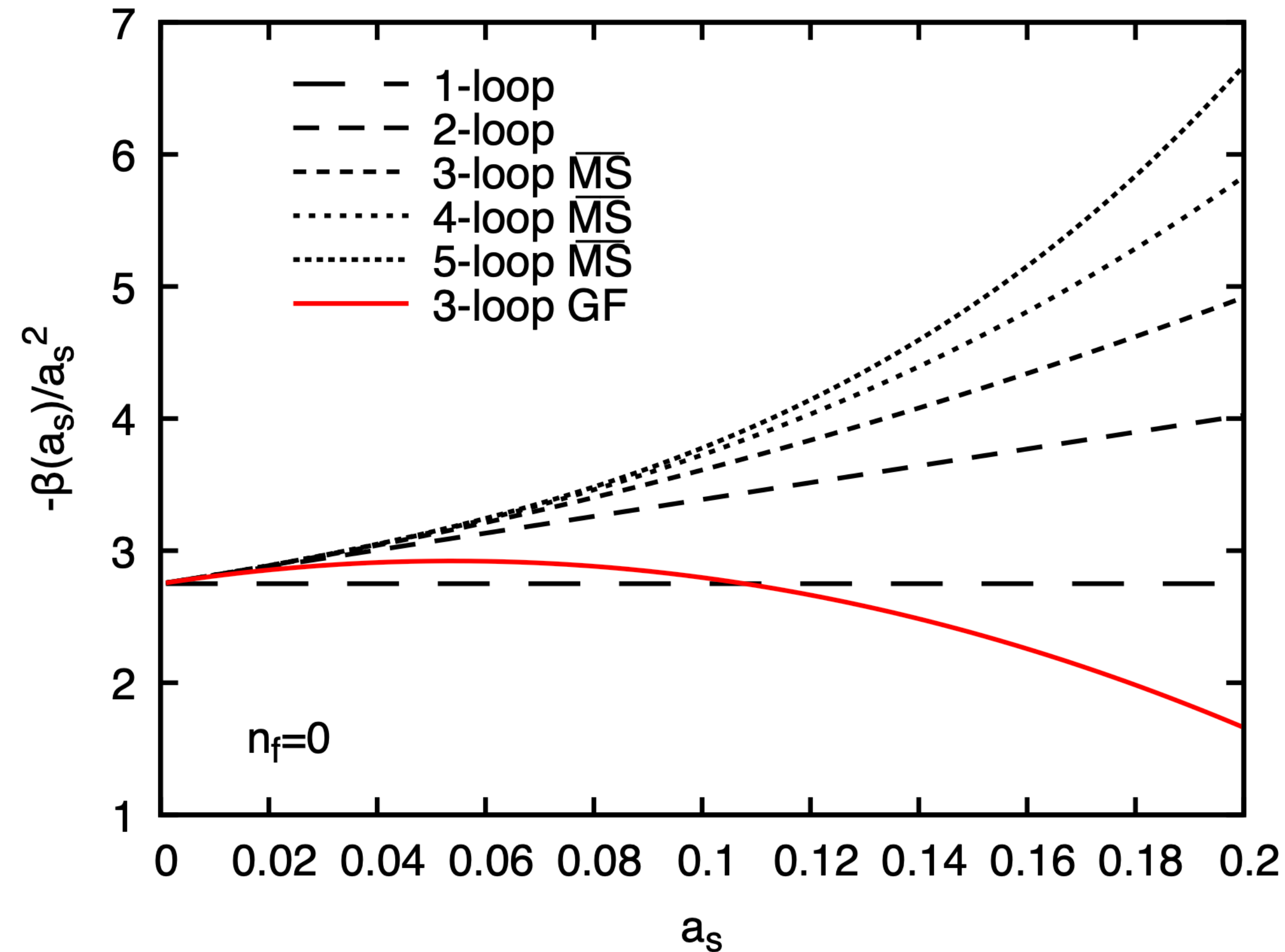
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universal

GF specific
depends on k_2

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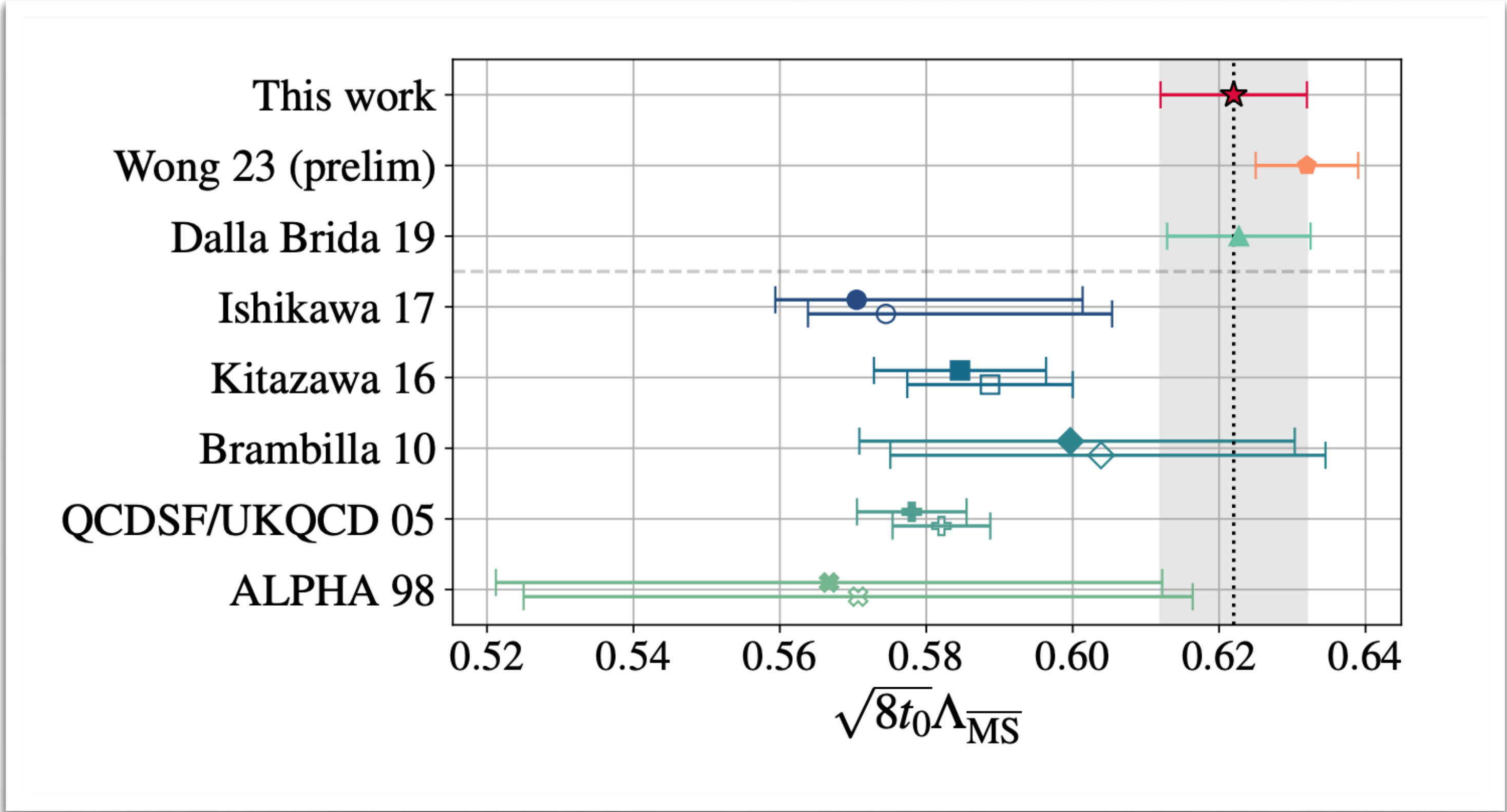
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universal

GF specific
depends on k_2

Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel (2023)

see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

Application to effective field theories

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation
theory

lattice

match
renormalization
schemes?

Application to effective field theories

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation
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Instead:

$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match
renormalization
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gradient flow
renormalization

Application to effective field theories

Observable:

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perturbation theory

lattice

match renormalization schemes?

Instead:

$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

gradient flow renormalization

Crucial observation:

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \text{ is UV finite} \\ \Rightarrow \lim_{a \rightarrow 0} \langle \tilde{\mathcal{O}}_n(t) \rangle \text{ exists!}$$

Lüscher, Weisz 2011

Small flow-time expansion

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

Small flow-time expansion

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small flow-time expansion:

Lüscher, Weisz '11

Suzuki '13

Lüscher '13

$$\tilde{\mathcal{O}}_n(t) \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \mathcal{O}_m$$

$$\tilde{C}_n(t) \xrightarrow{t \rightarrow 0} \sum_m C_m \zeta_{mn}^{-1}(t)$$

\Rightarrow need $\zeta_{nm}(t)$ for small t \Rightarrow perturbation theory

Determining $\zeta(t)$

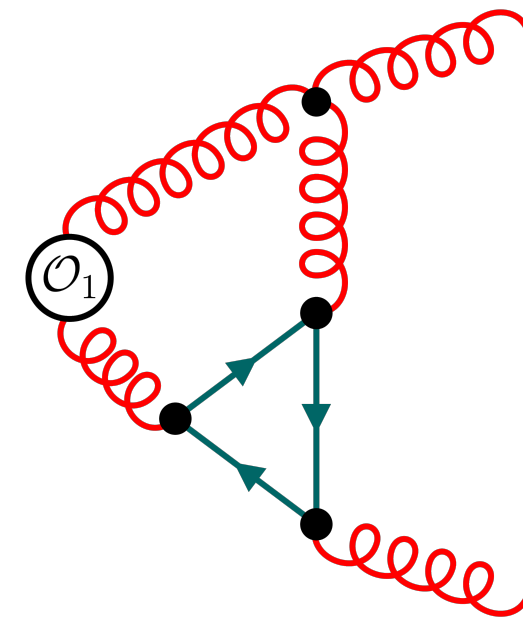
Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \langle \mathcal{O}_m \rangle$$

Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

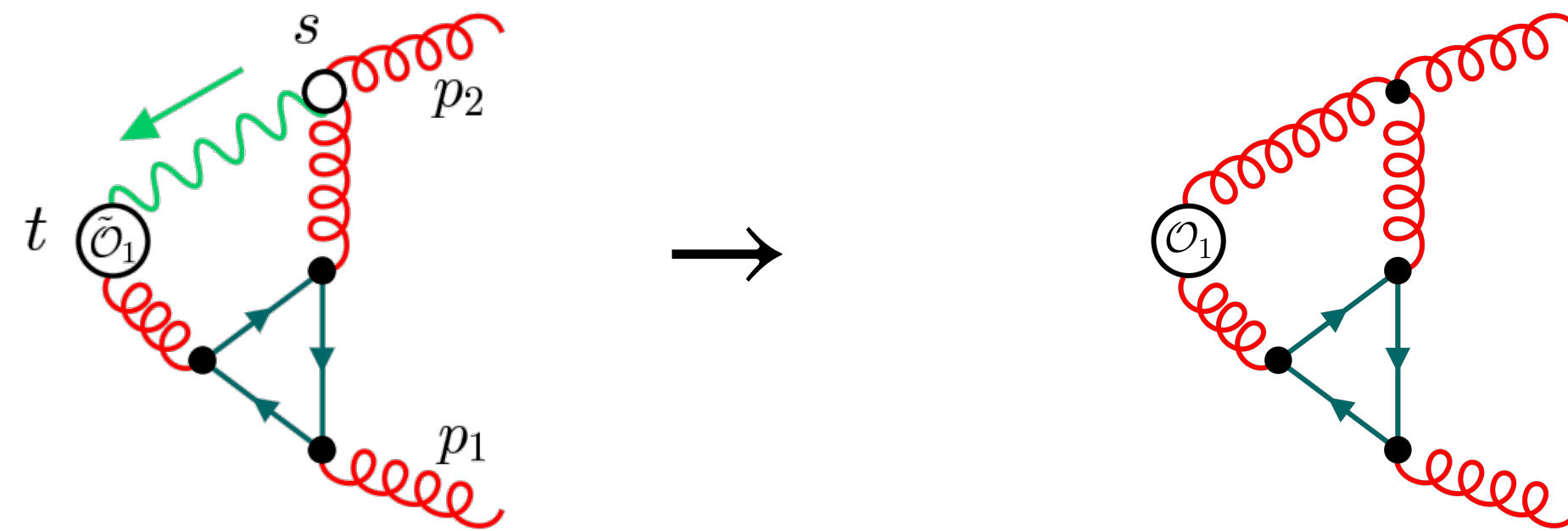
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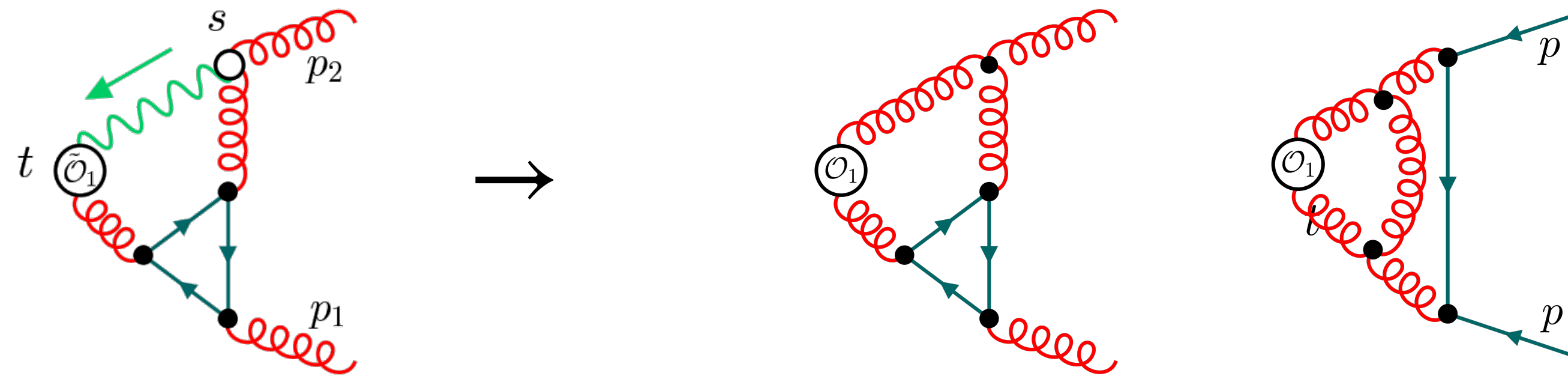
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Determining $\zeta(t)$

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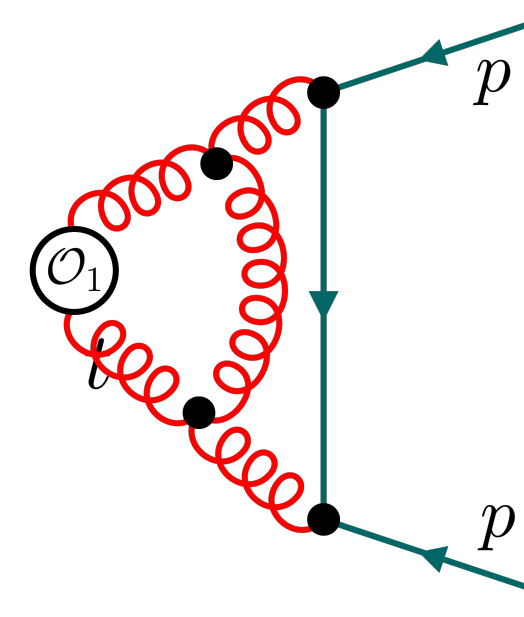
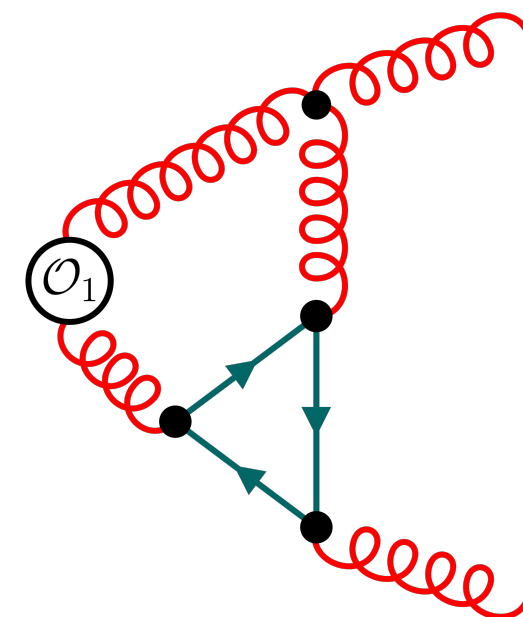
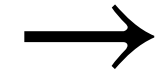
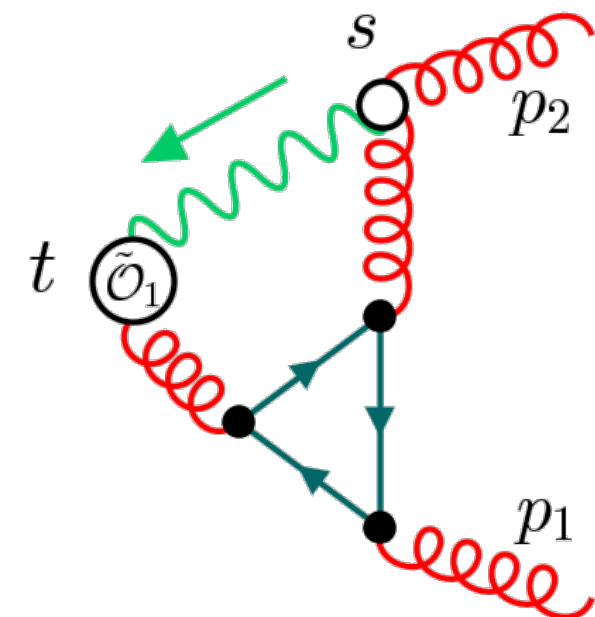
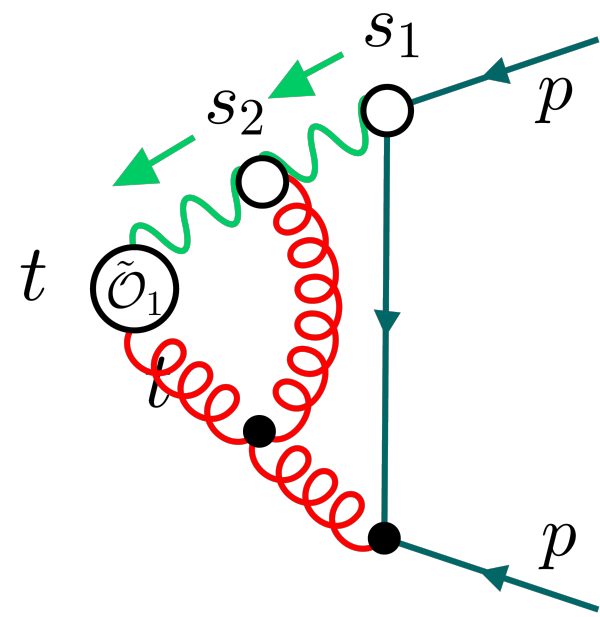
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Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

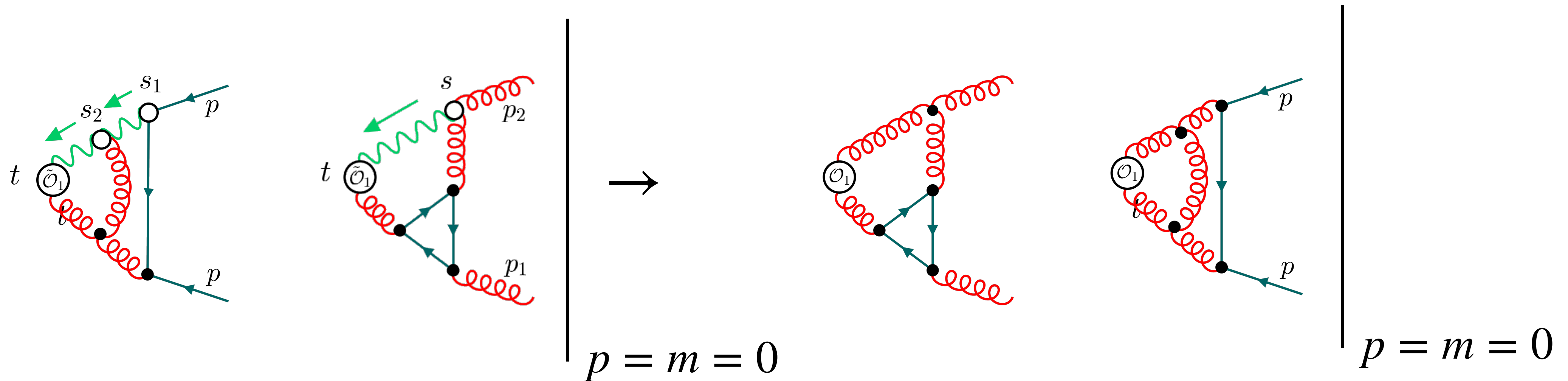
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Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

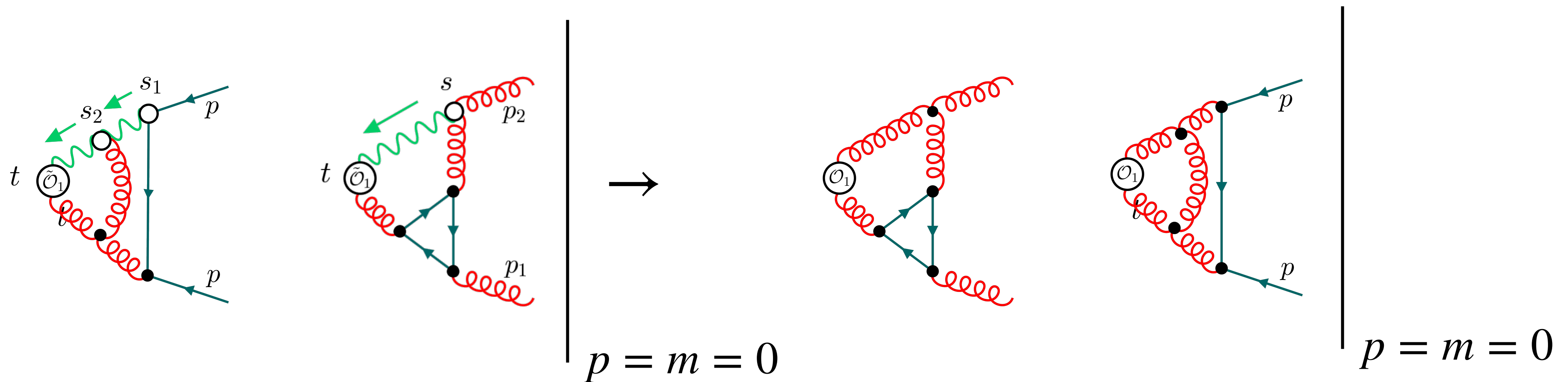
$$\langle \tilde{\mathcal{O}}_n(t) \rangle \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \langle \mathcal{O}_m \rangle$$



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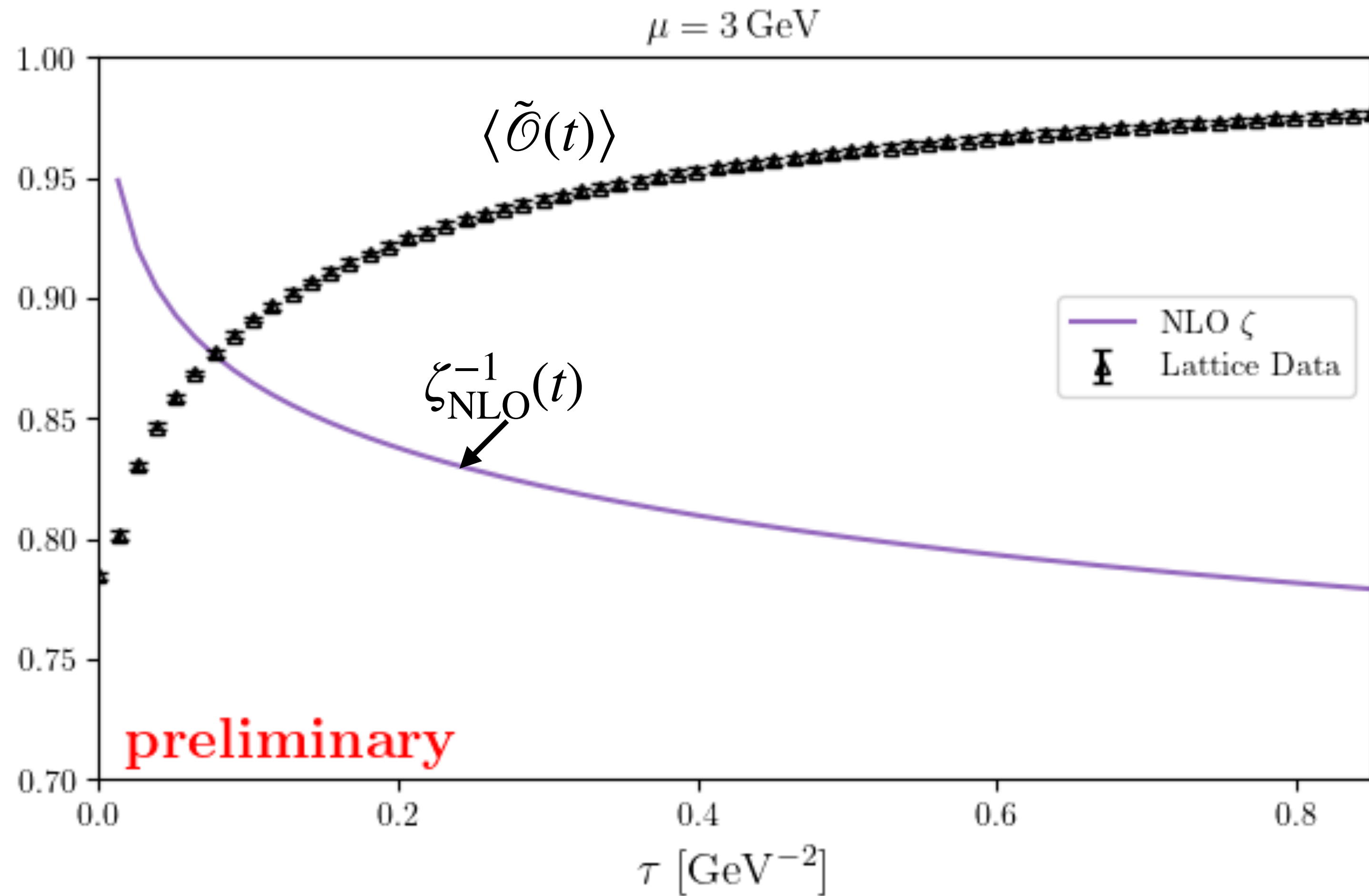
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only tree-level diagrams survive on r.h.s.

Gorishnii, Larin, Tkachov '83

Proof of principle

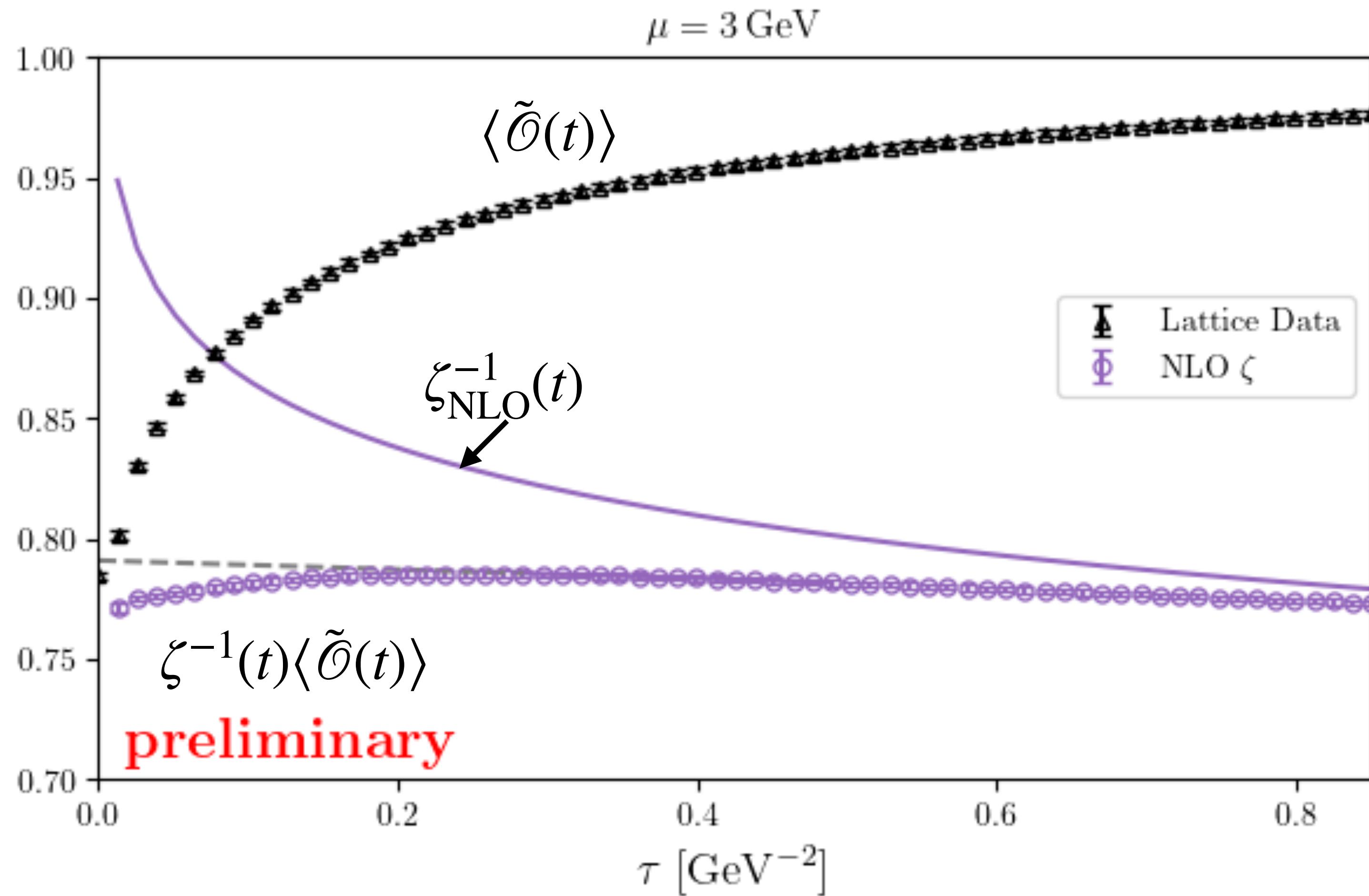


$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

$$= \sum_n C_n(\zeta^{-1}(t) \langle \tilde{\mathcal{O}}(t) \rangle)_n$$

→ see Fabian Lange's talk

Proof of principle

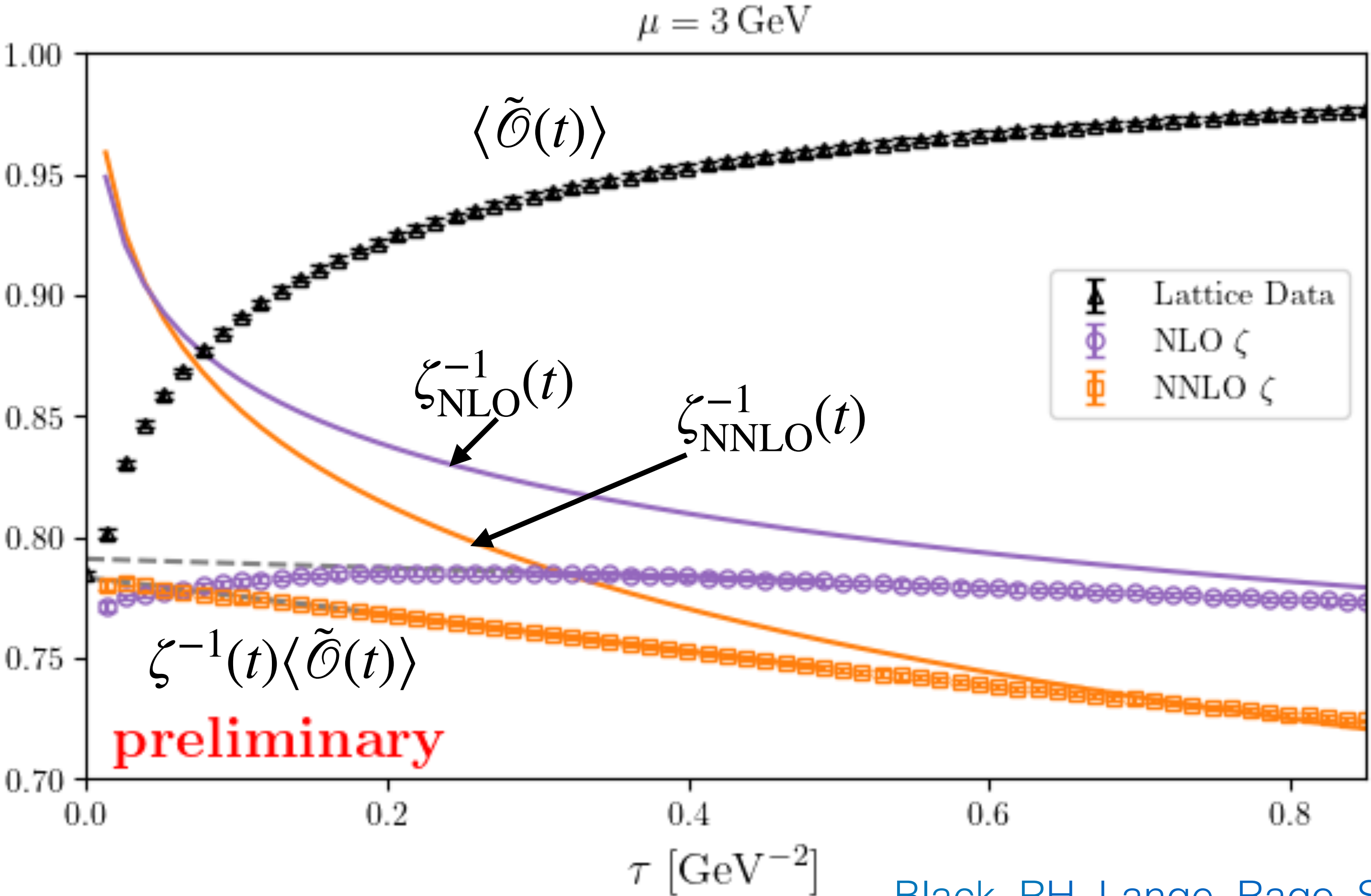


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Proof of principle

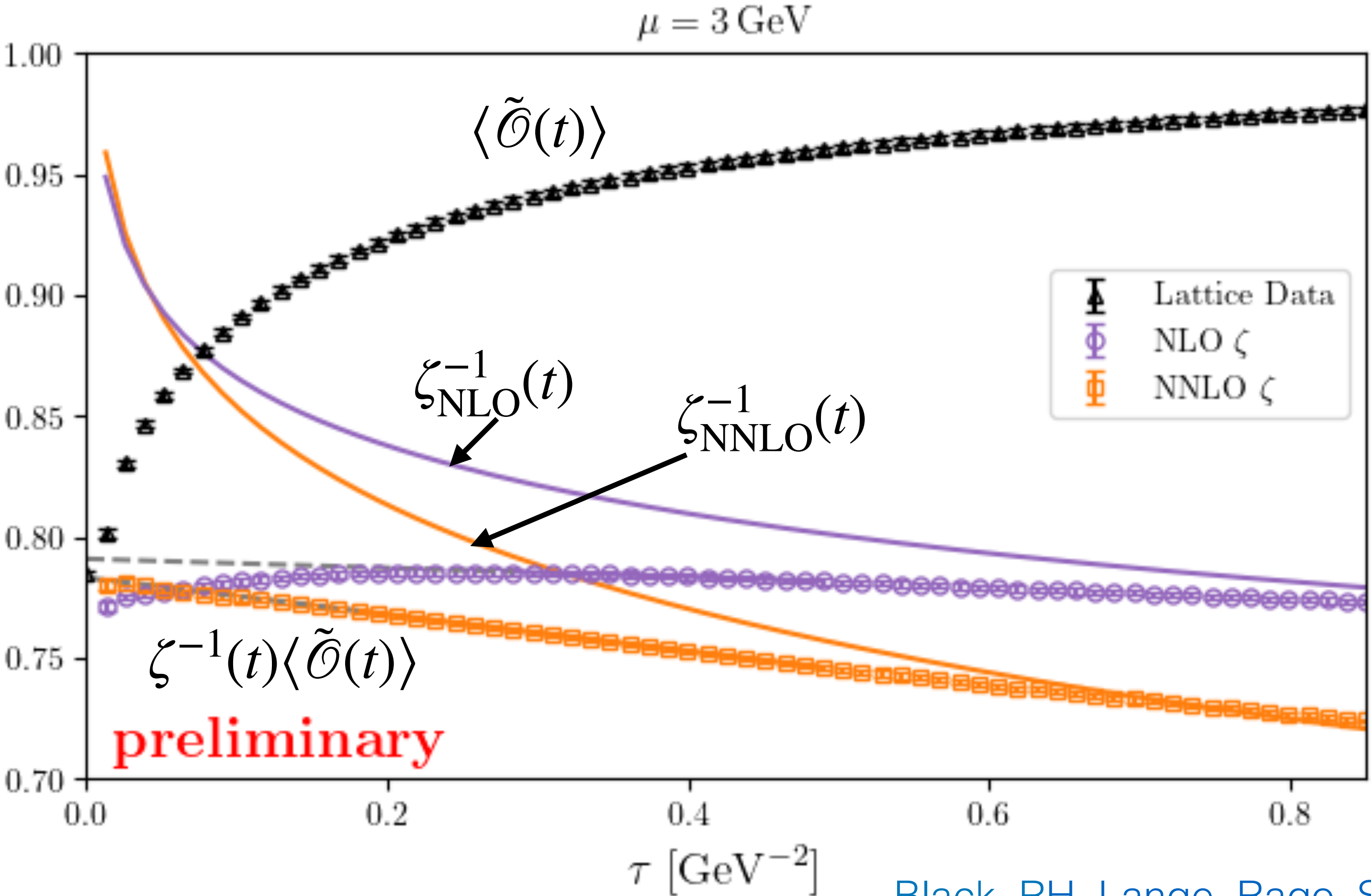


Black, RH, Lange, Rago, Shindler, Witzel (2023)

→ see Fabian Lange's talk



Proof of principle



Black, RH, Lange, Rago, Shindler, Witzel (2023)

→ see Fabian Lange's talk

$$\begin{aligned}
 R &= \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle \\
 &= \sum_n C_n(\zeta^{-1}(t) \langle \tilde{\mathcal{O}}(t) \rangle)_n \\
 &\quad + t \cdot (\dots)
 \end{aligned}$$



Energy momentum tensor

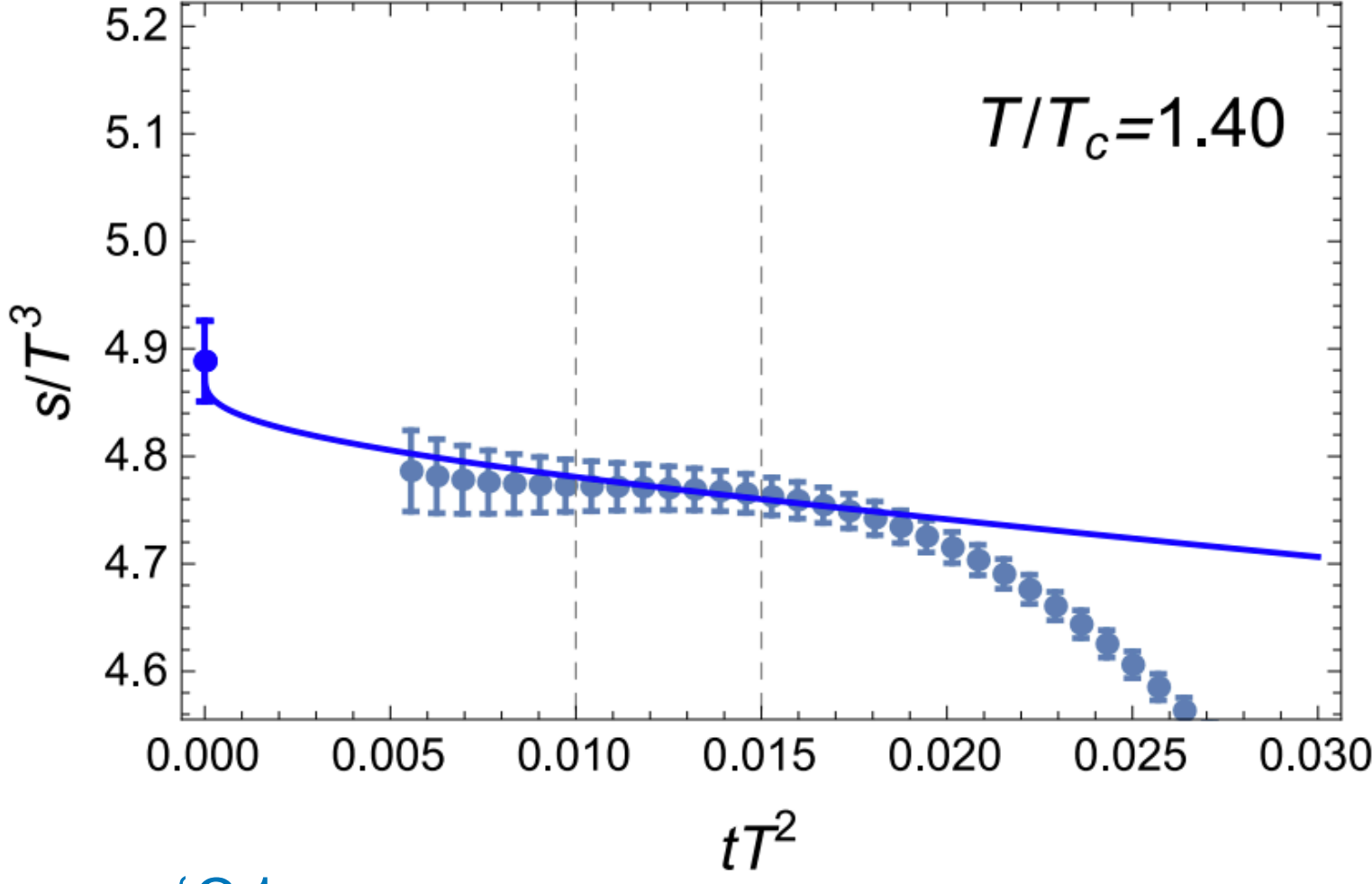
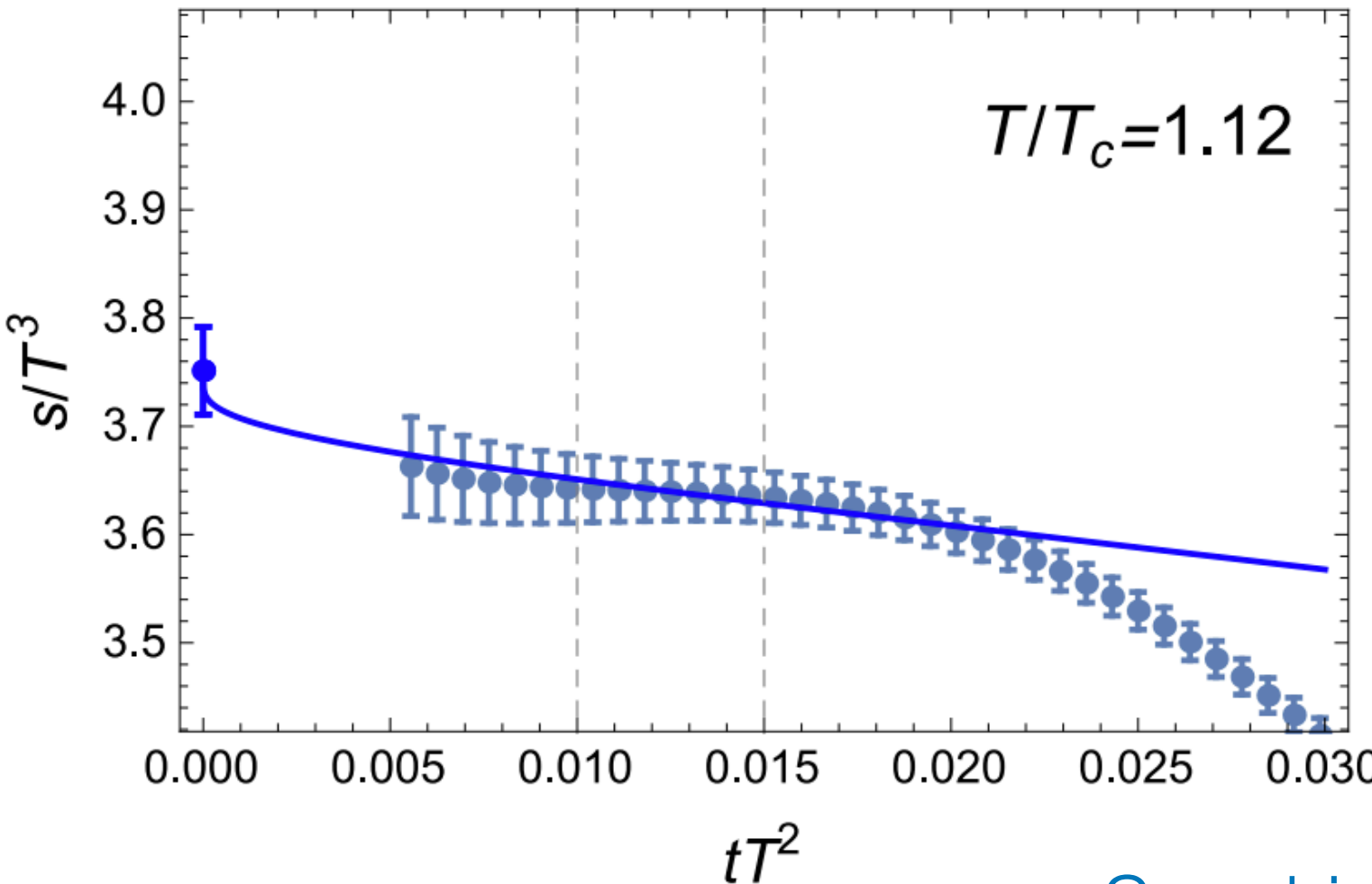
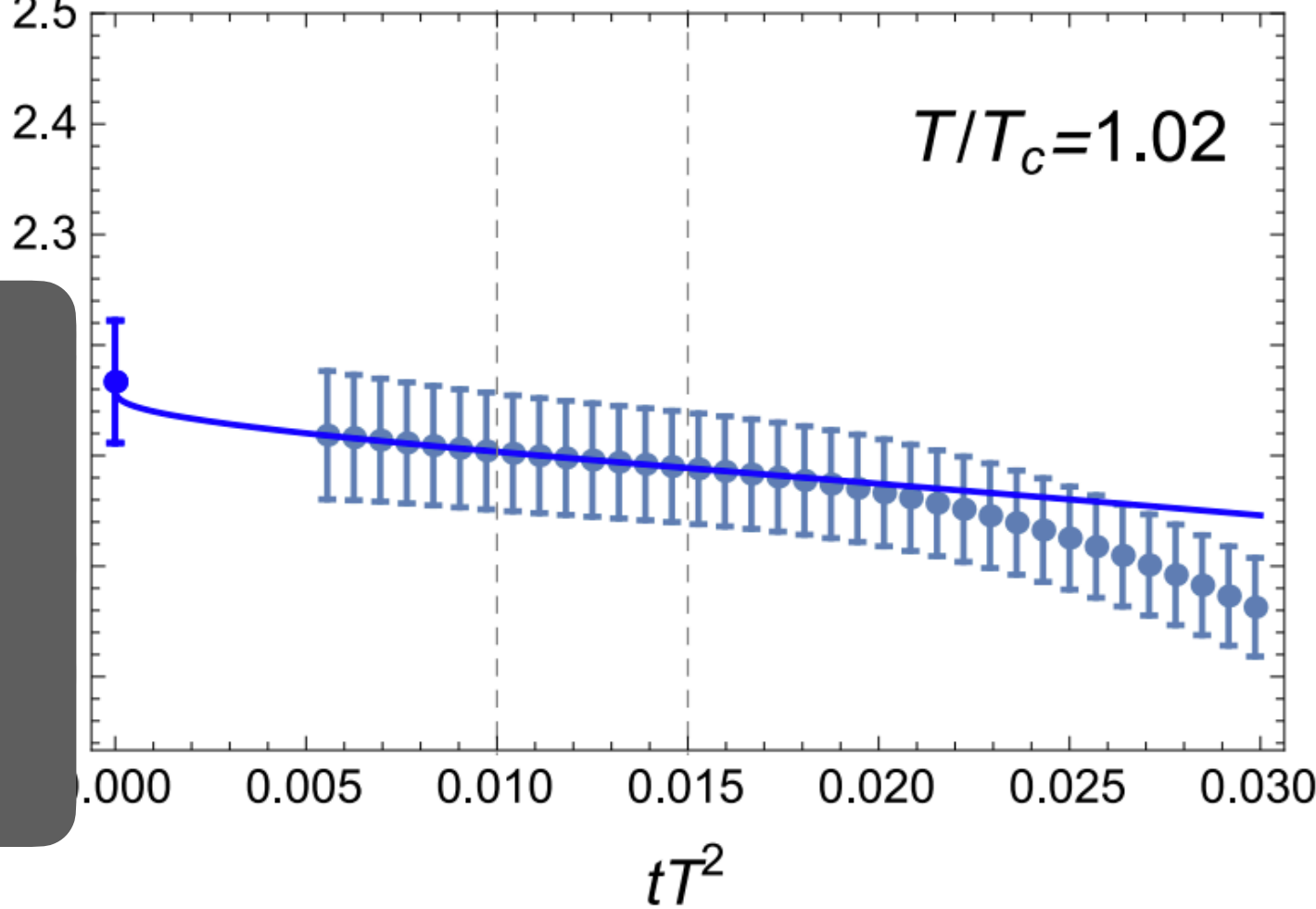
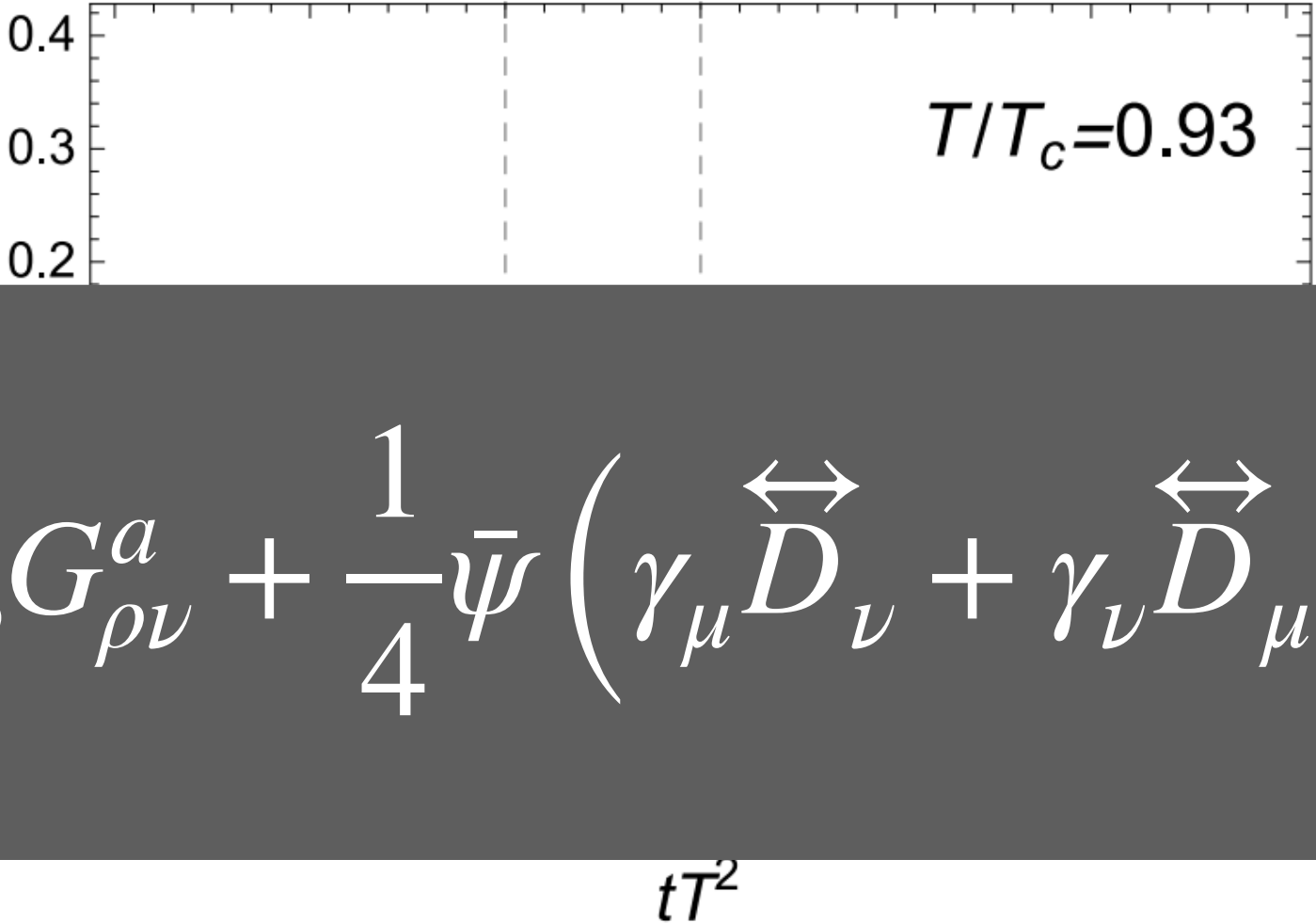
Suzuki (2013)
Suzuki, Makino (2014)

Entropy density:

$$T_{\mu\nu} = -\frac{1}{4}\delta_{\mu\nu}G_{\rho\sigma}^a G_{\rho\sigma}^a + G_{\mu\rho}^a G_{\rho\nu}^a + \frac{1}{4}\bar{\psi}\left(\gamma_\mu\overleftrightarrow{D}_\nu + \gamma_\nu\overleftrightarrow{D}_\mu\right)\psi$$

$$T_{\mu\nu} = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

NLO



Suzuki, Takaura '21

Energy momentum tensor

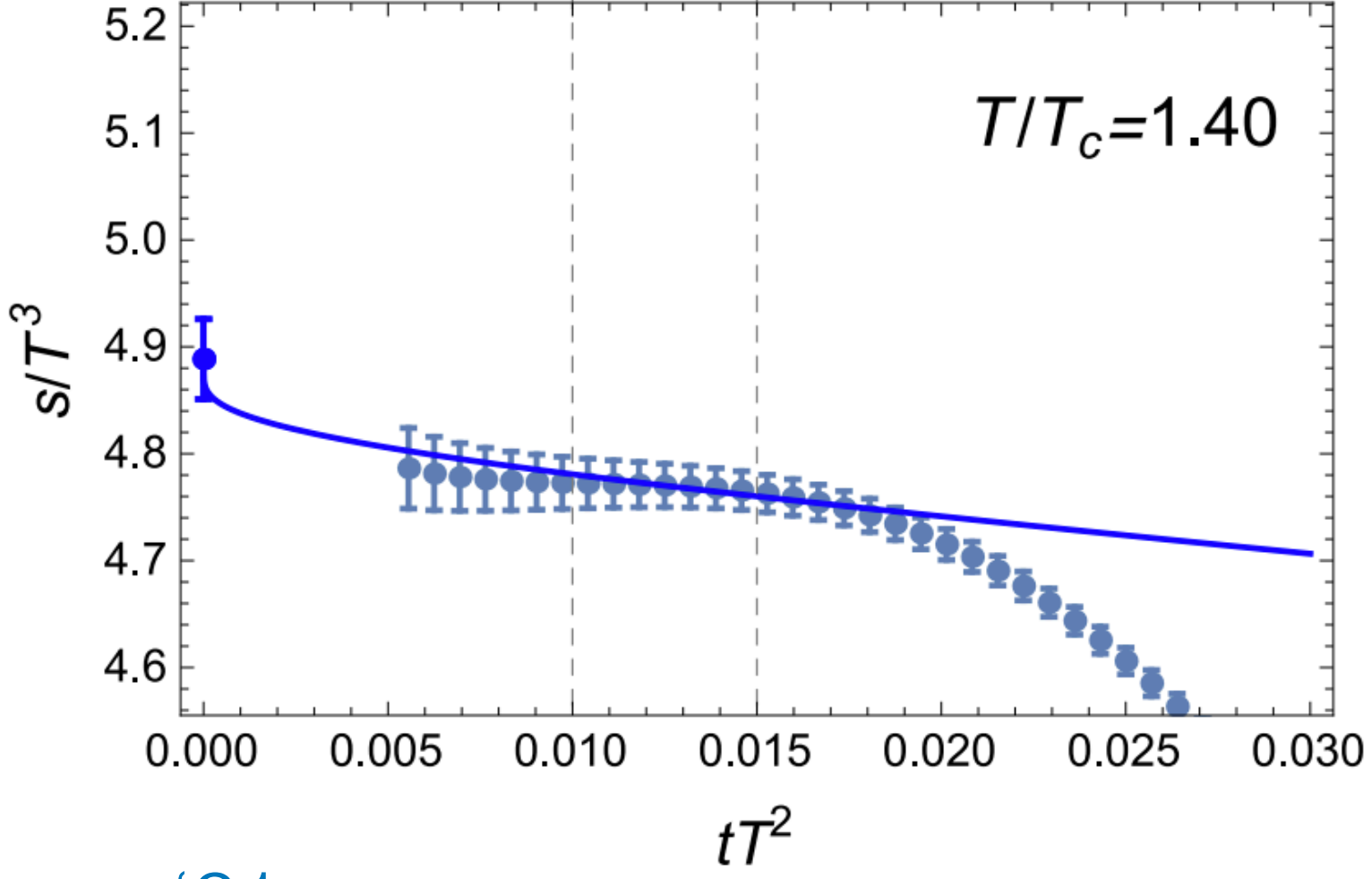
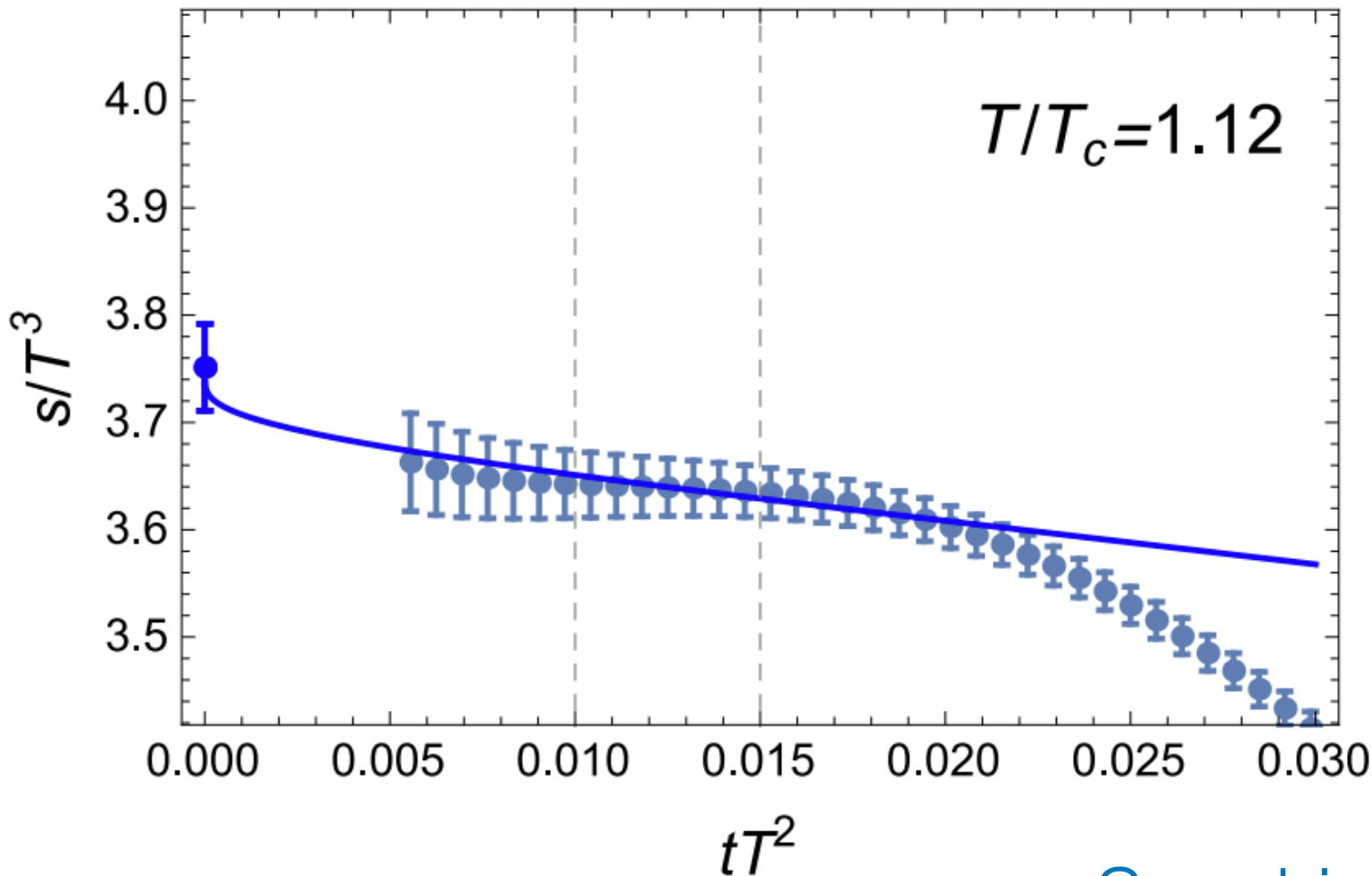
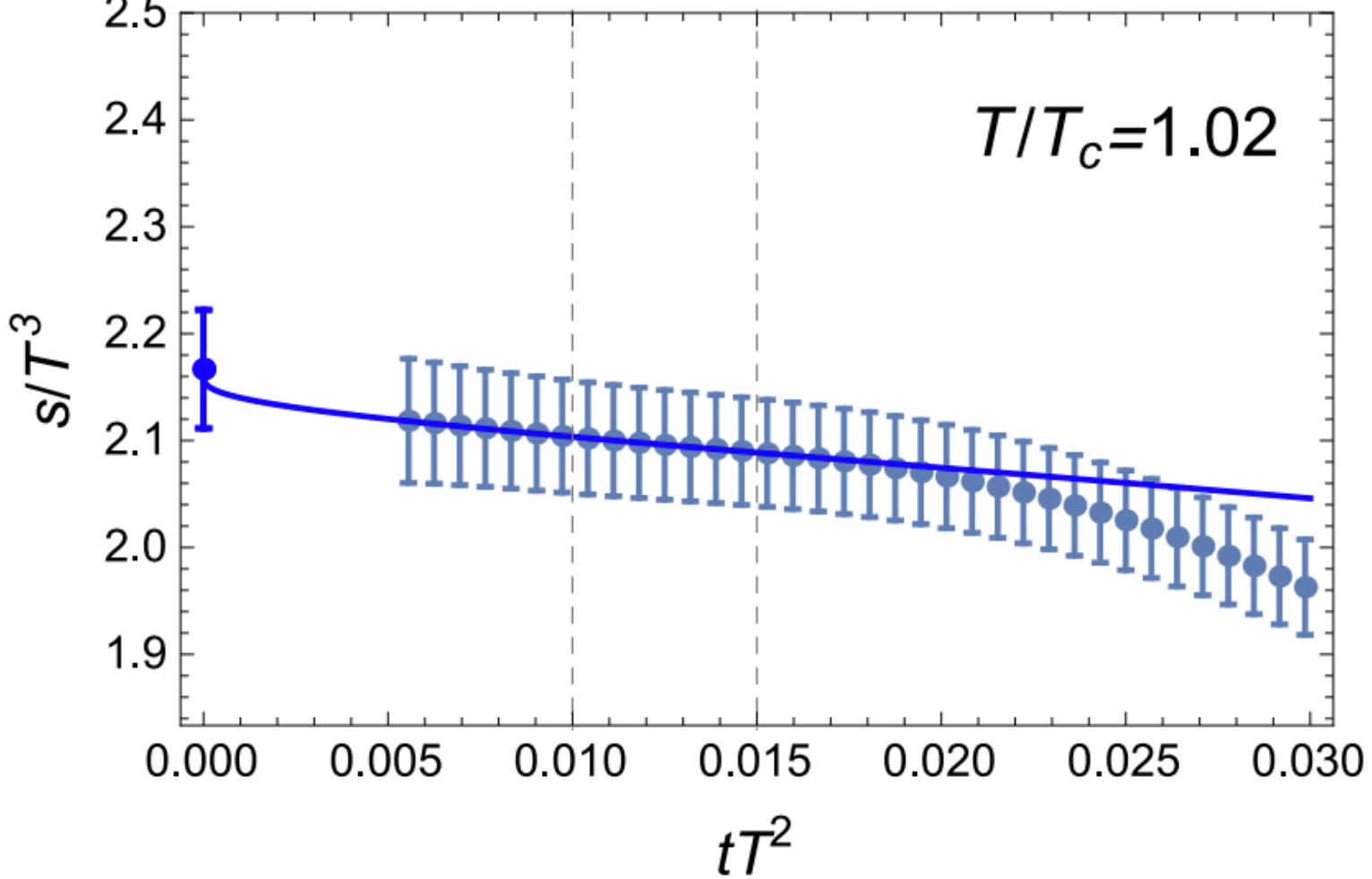
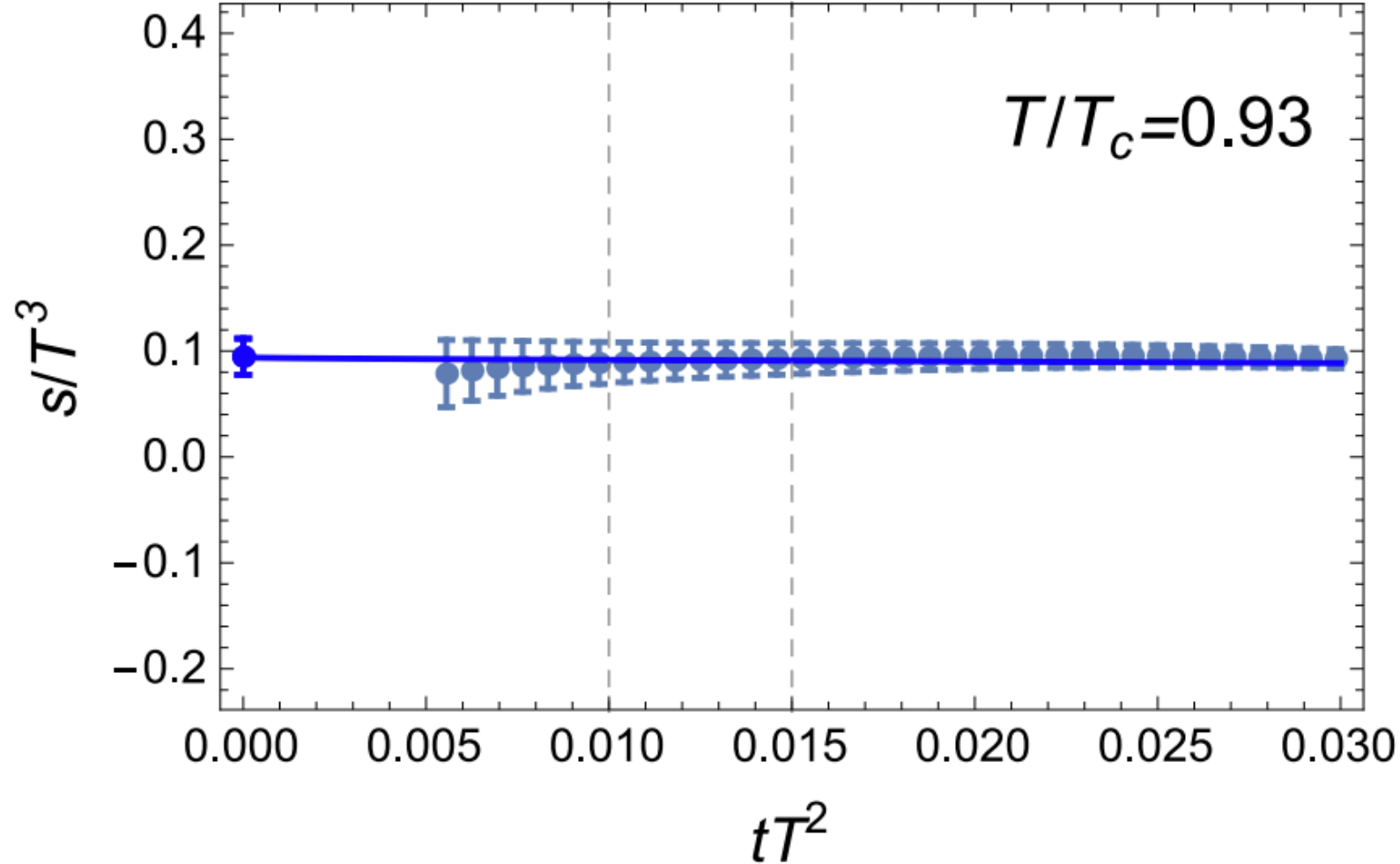
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Entropy density:

$$\varepsilon + p = -\frac{4}{3} \left\langle T_{00}(x) - \frac{1}{4} T_{\mu\mu}(x) \right\rangle$$

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NLO



Suzuki, Takaura '21

Energy momentum tensor

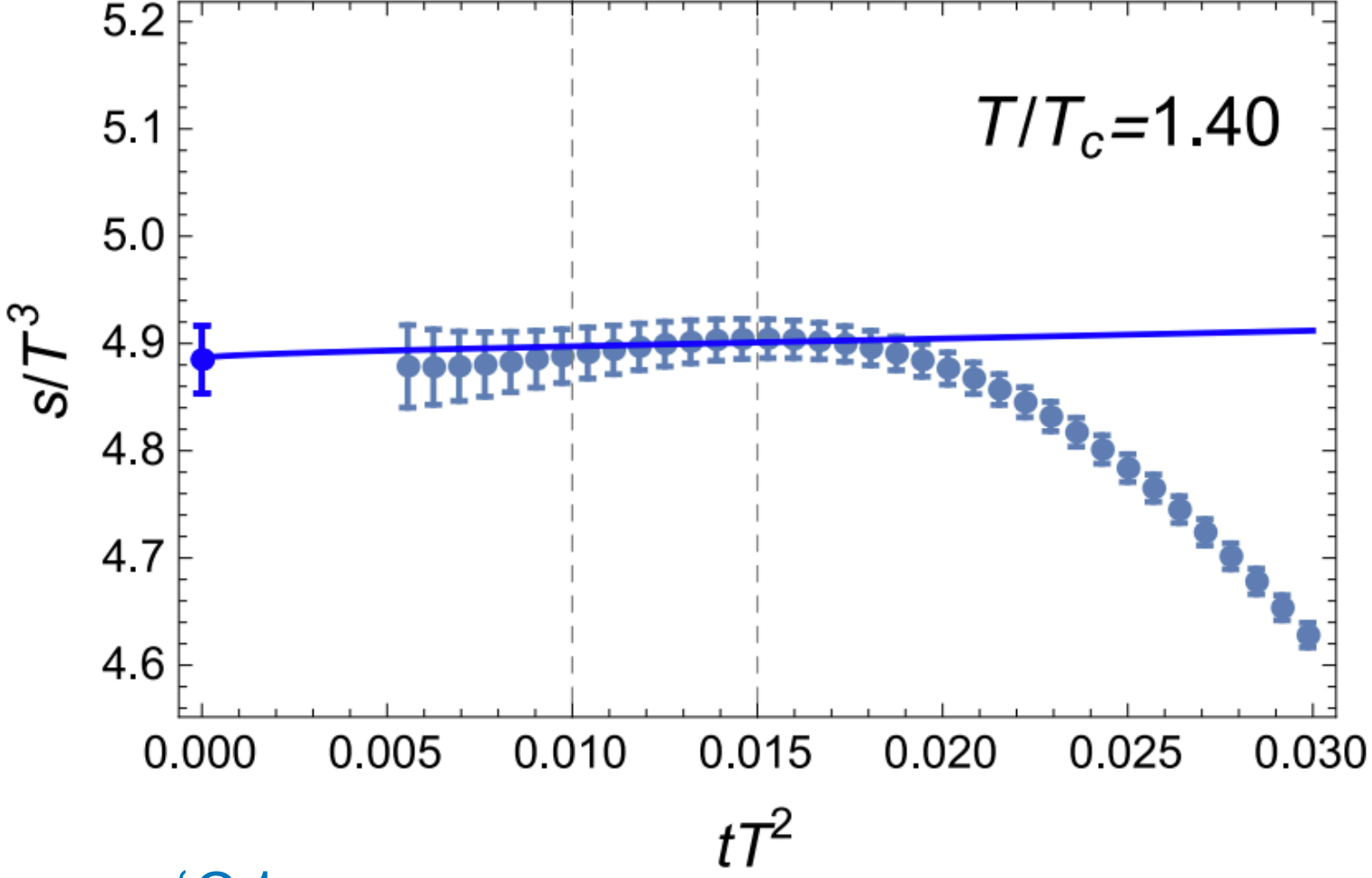
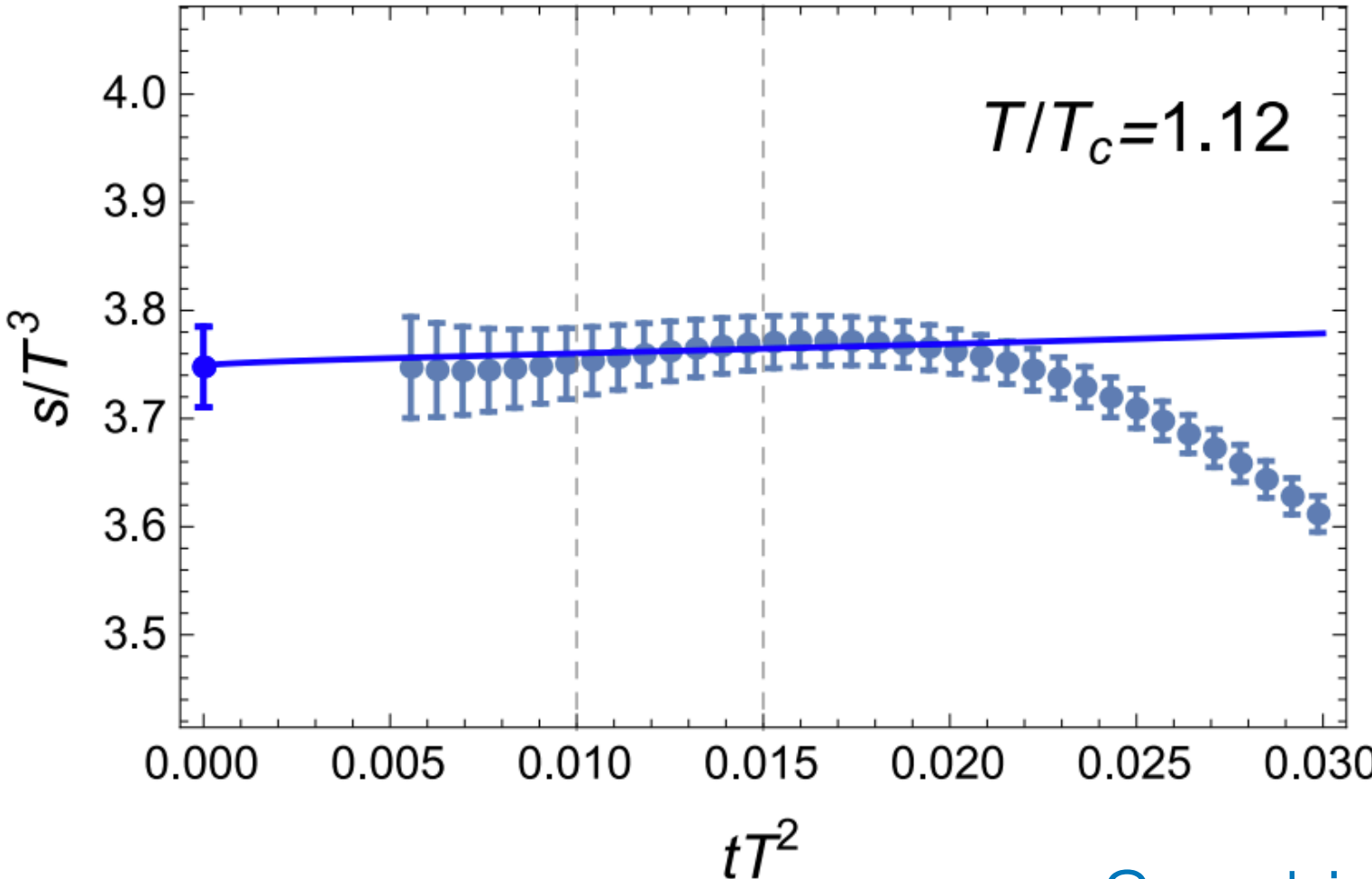
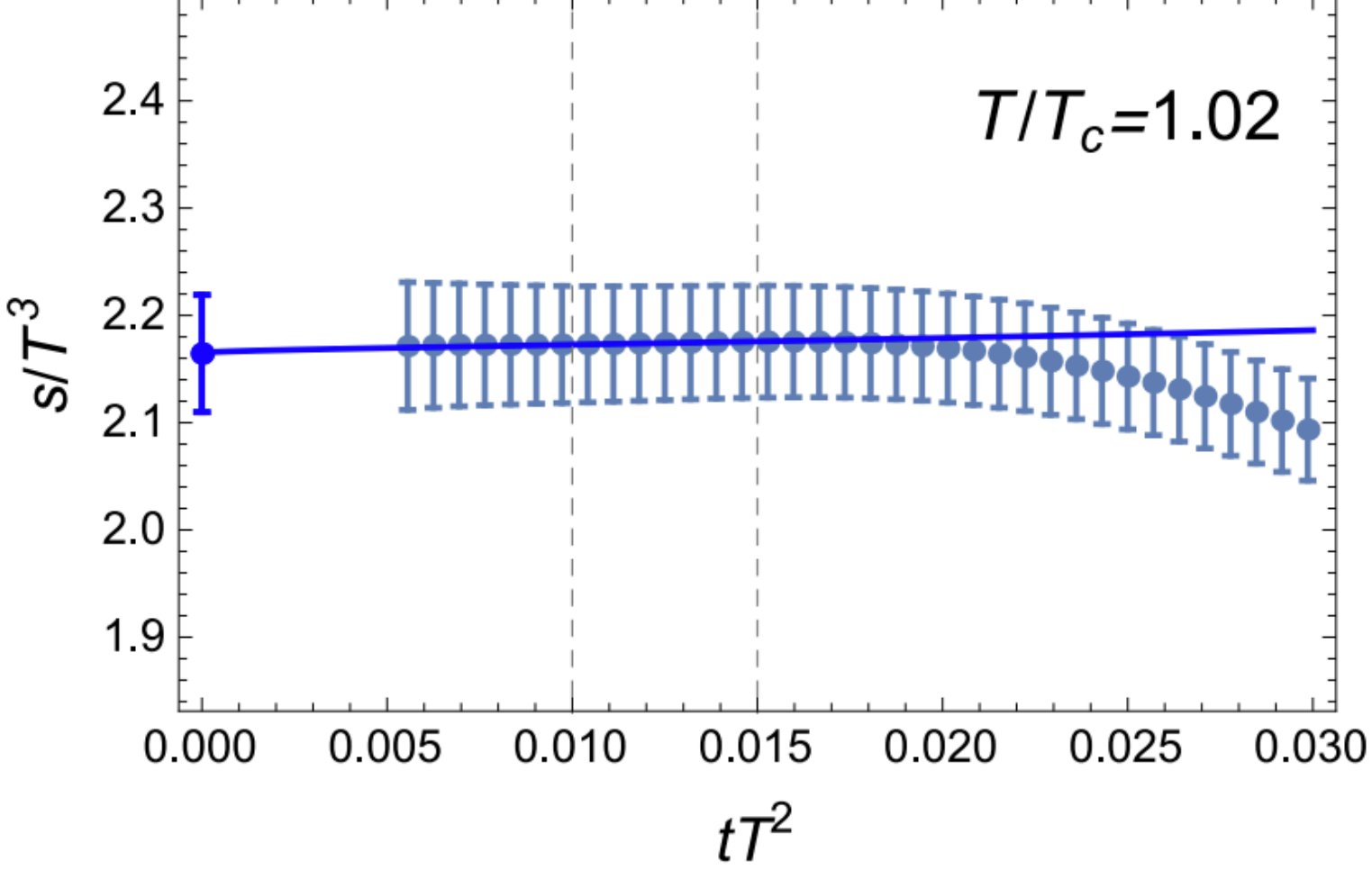
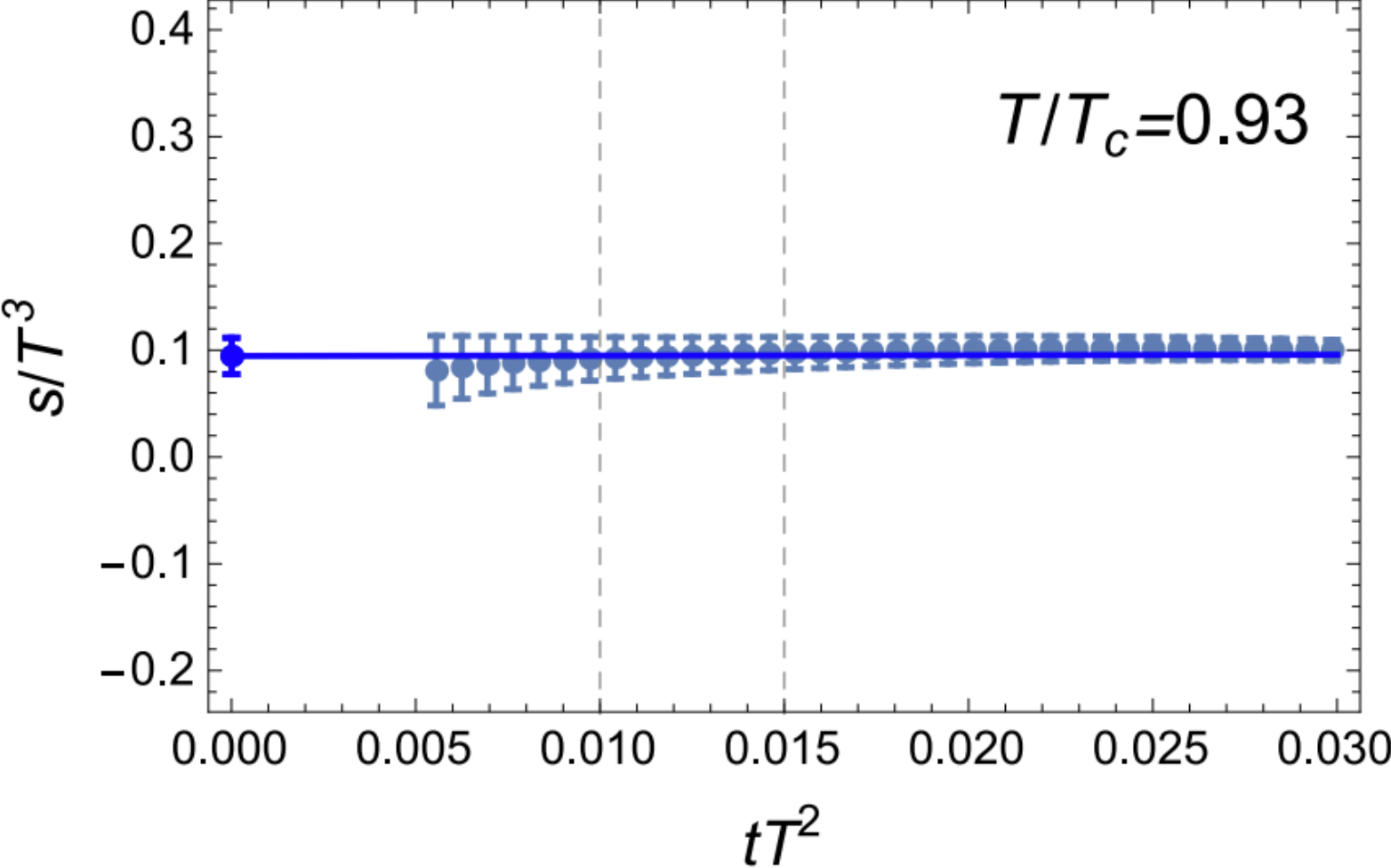
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NNLO



Suzuki, Takaura '21

Other applications

electric dipole operators

NLO: [Mereghetti, Monahan, Rizik, Shindler, Stoffer \(2022\)](#)
[Crosas, Monahan, Rizik, Shindler, Stoffer \(2023\)](#)
NNLO: [Borgulat, RH, Rizik, Shindler \(2022\)](#)

four-quark operators → see Fabian Lange's talk

NLO: [A. Suzuki, Tanaguchi, H. Suzuki, Kanaya \(2020\)](#)
NNLO: [RH, Lange \(2022\)](#)
[Black, RH, Lange, Rago, Shindler, Witzel \(2023, 2024\)](#)

LEFT operators → see Òscar Crosas' talk

moments of PDFs

NLO: [A. Shindler \(2024\)](#)

quark bilinears

NLO: [Hieda, Suzuki \(2016\)](#)
NNLO: [Borgulat, RH, Kohlen, Lange \(2023\)](#)

hadronic vacuum polarization

NNLO: [RH, Lange, Neumann \(2020\)](#)

$\overline{\text{MS}}$ renormalization of composite operators

$$\mathcal{L}_{\text{eff}} = \sum_n C_n \mathcal{O}_n$$

$\overline{\text{MS}}$ renormalization of composite operators

needs renormalization:

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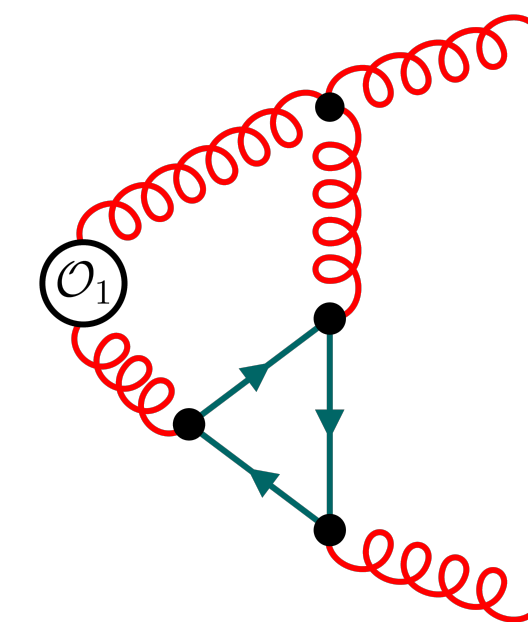
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compute suitable Green's functions



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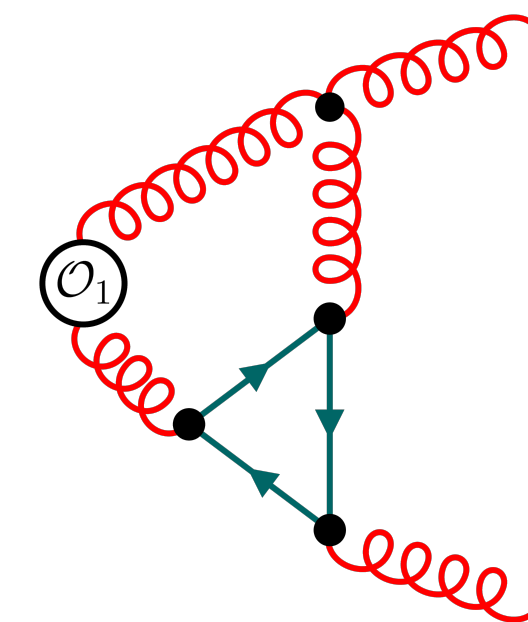
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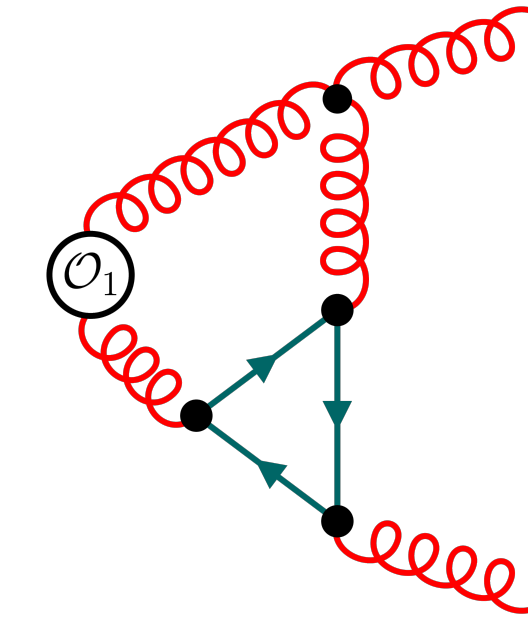
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problem: separate UV from IR divergences



$\overline{\text{MS}}$ renormalization of composite operators

compute suitable Green's functions
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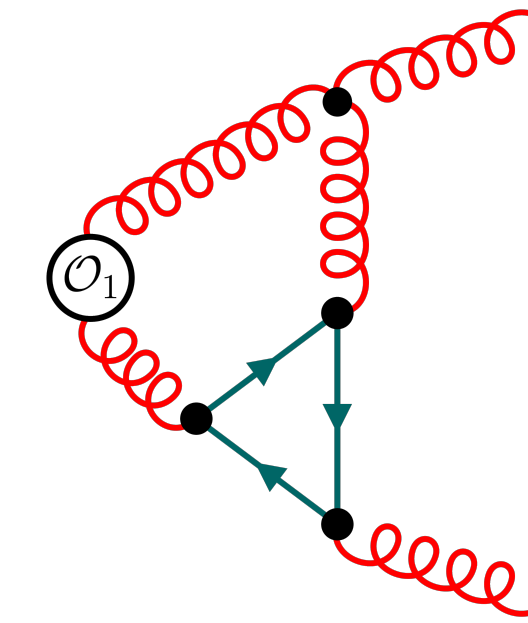
$\overline{\text{MS}}$ renormalization of composite operators

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problem: separate UV from IR divergences

In $\overline{\text{MS}}$, Z is independent of masses and momenta

But note:

$$\int d^D p p^\alpha \sim \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} = 0$$



$\overline{\text{MS}}$ renormalization of composite operators

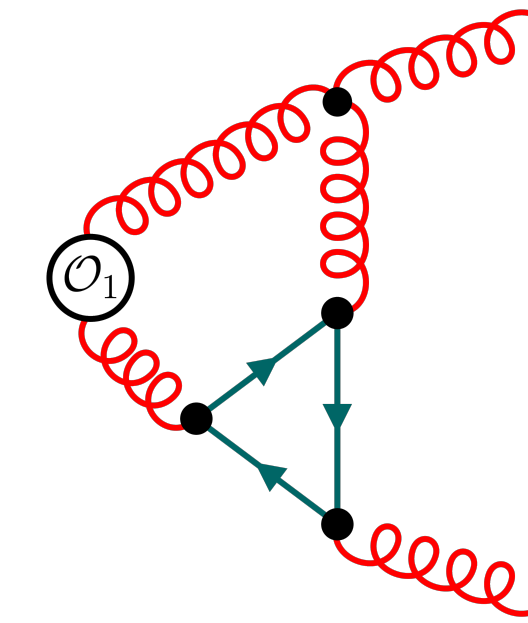
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Keep external momenta? How? Introduce (auxiliary?) masses? Gauge invariance?



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(SFTX)

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UV finite

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UV finite

\Rightarrow calculation of $\zeta(t)$ also determines Z in $\overline{\text{MS}}$ scheme!

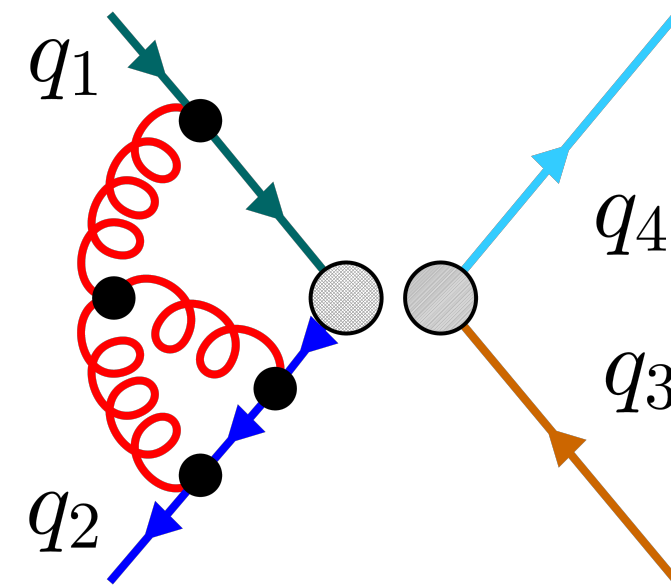
$\overline{\text{MS}}$ renormalization of composite operators

application: check for $\Delta F = 2$ four-quark operators [Buras, Gorbahn, Haisch, Nierste \(2006\)](#)
[RH, Lange \(2022\)](#)

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$\overline{\text{MS}}$ renormalization of composite operators

scalar QCD:
$$\mathcal{L}_{\text{SQCD}} = \mathcal{L}_{\text{QCD}} + (D_\mu \phi)^\dagger (D_\mu \phi) - \frac{\lambda}{4} \phi^\dagger \phi$$

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flowed quark field:

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$$Z_\varphi \varphi^\dagger(t) \varphi(t) = \zeta_{11}(t) \frac{1}{t} \mathbf{1} + Z_{\phi^\dagger \phi} \zeta_{12}(t) \phi^\dagger \phi + \dots$$

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Gorishnii, Kataev, Larin (1987)

Borgulat, Felten, RH, Kohlen (in prep)

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The GF scheme

needs renormalization:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \sum_n C_n \mathcal{O}_n \\ &= \sum_n (CZ)_n (Z^{-1} \mathcal{O})_n = \sum_n C_n^{\text{R}} \mathcal{O}_n^{\text{R}}\end{aligned}$$

anomalous dimension:

$$\mu \frac{d}{d\mu} C_n = \gamma_{nm} C_m \qquad \gamma_{nm} = \mu \frac{d}{d\mu} \ln Z$$

gradient-flow scheme:

$$\mathcal{L}_{\text{eff}} = \sum_n \tilde{C}_n(t) \tilde{\mathcal{O}}_n(t) = \sum_n (C\zeta^{-1}(t))_n (\zeta(t) \mathcal{O})_n(t)$$

The GF scheme

needs renormalization:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \sum_n C_n \mathcal{O}_n \\ &= \sum_n (CZ)_n (Z^{-1} \mathcal{O})_n = \sum_n C_n^R \mathcal{O}_n^R\end{aligned}$$

anomalous dimension:

$$\mu \frac{d}{d\mu} C_n = \gamma_{nm} C_m \quad \gamma_{nm} = \mu \frac{d}{d\mu} \ln Z$$

gradient-flow scheme:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \sum_n \tilde{C}_n(t) \tilde{\mathcal{O}}_n(t) = \sum_n (C\zeta^{-1}(t))_n (\zeta(t) \mathcal{O})_n(t) \\ t \frac{d}{dt} \tilde{C}(t) &= \tilde{\gamma} \tilde{C}(t) \quad \tilde{\gamma} = t \frac{d}{dt} \ln \zeta(t)\end{aligned}$$

RH, Lange, Neumann (2020)

see also Hasenfratz, Monahan, Rizik, Shindler, Witzel (2021)

Conclusions and Outlook

- Gradient flow provides ideal basis for combining lattice and perturbation theory
- Many perturbative tools can be adapted
- Several proofs of principle already available
- Full potential still to be explored