Applications of the Gradient Flow

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RWTH Aachen University

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Loops and Legs in Quantum Field Theory
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Motivation

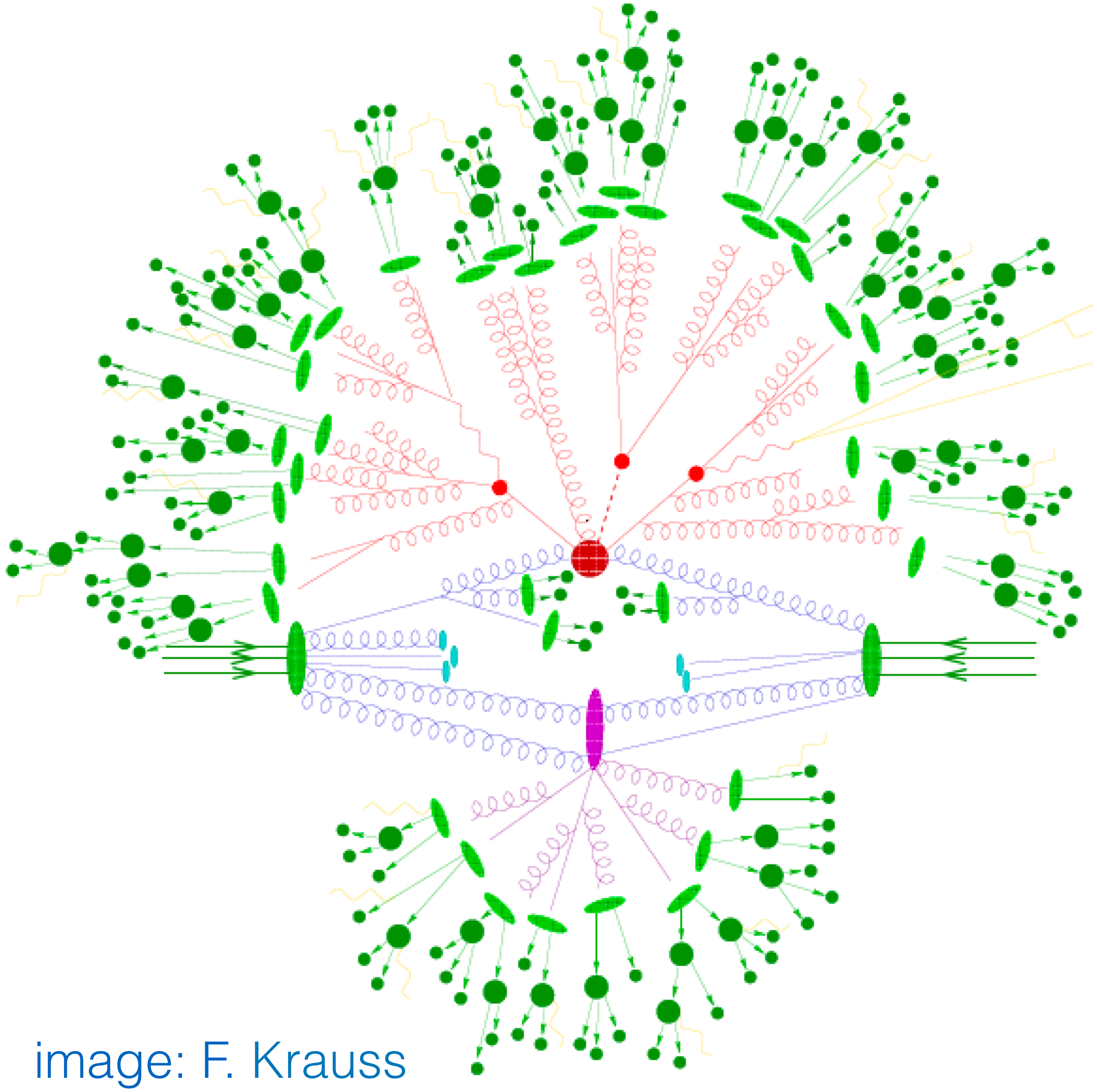

image: F. Krauss

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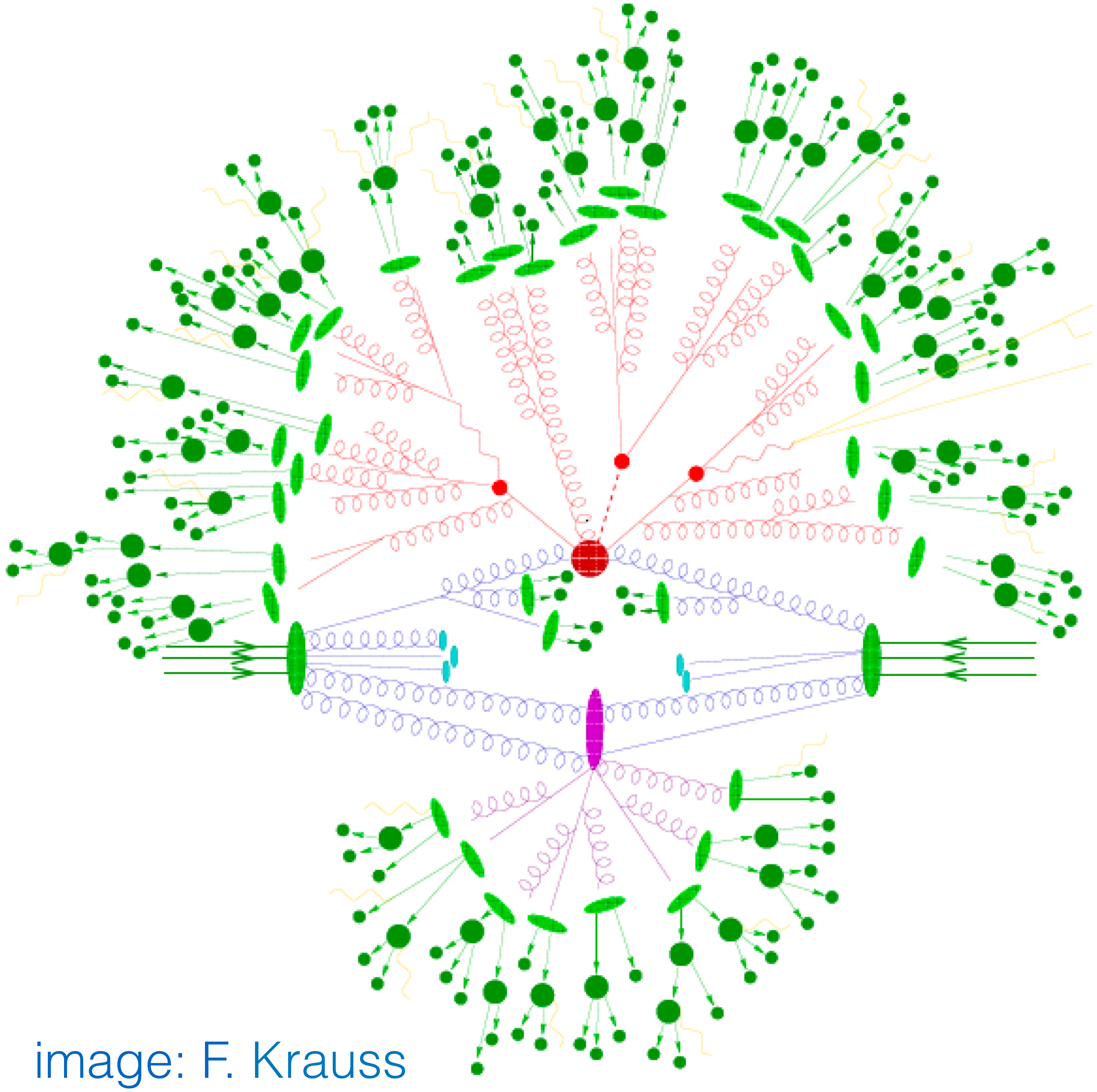

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perturbative contributions:
first principles

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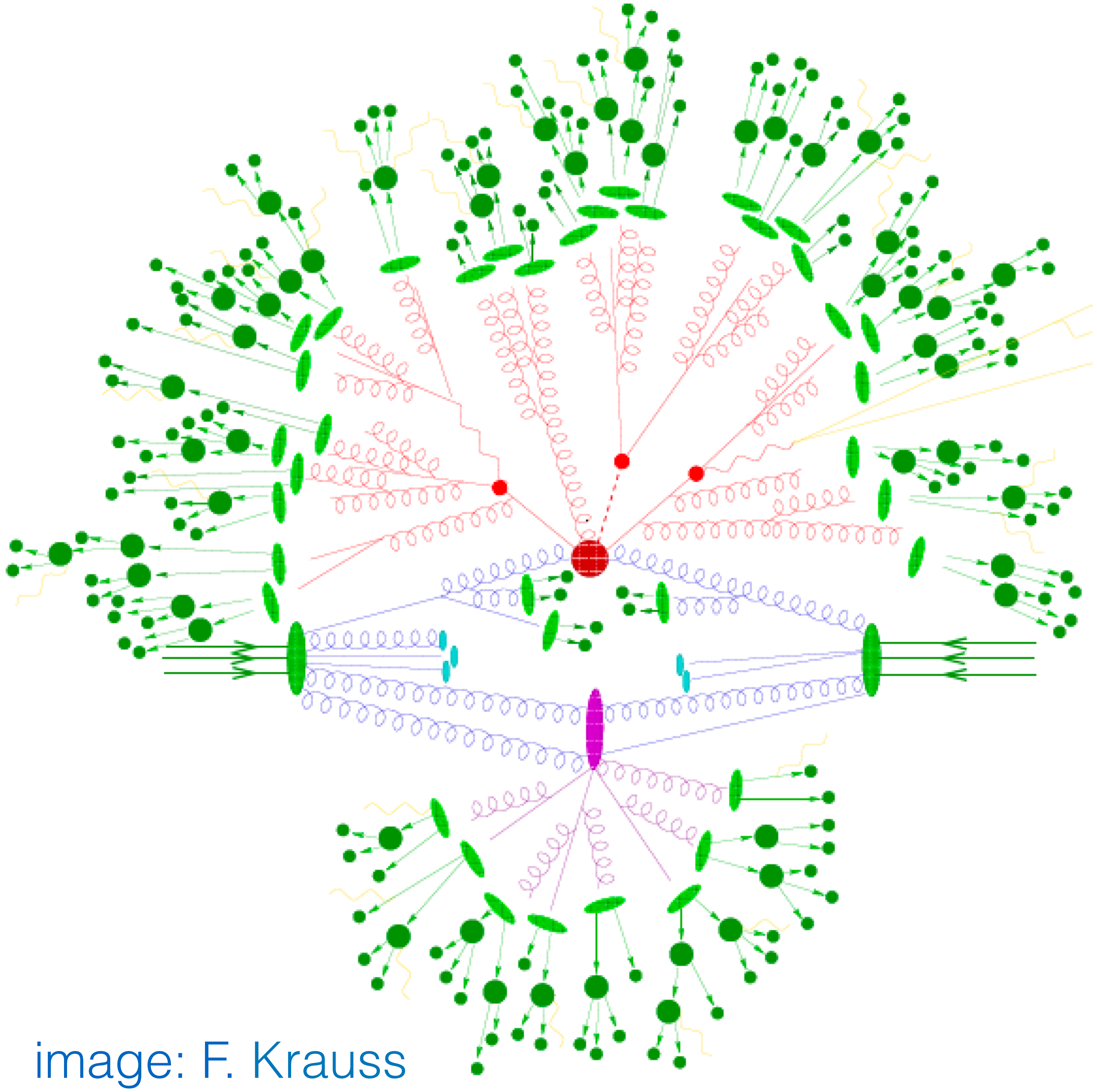

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perturbative contributions:
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non-perturbative contributions:
phenomenological fits

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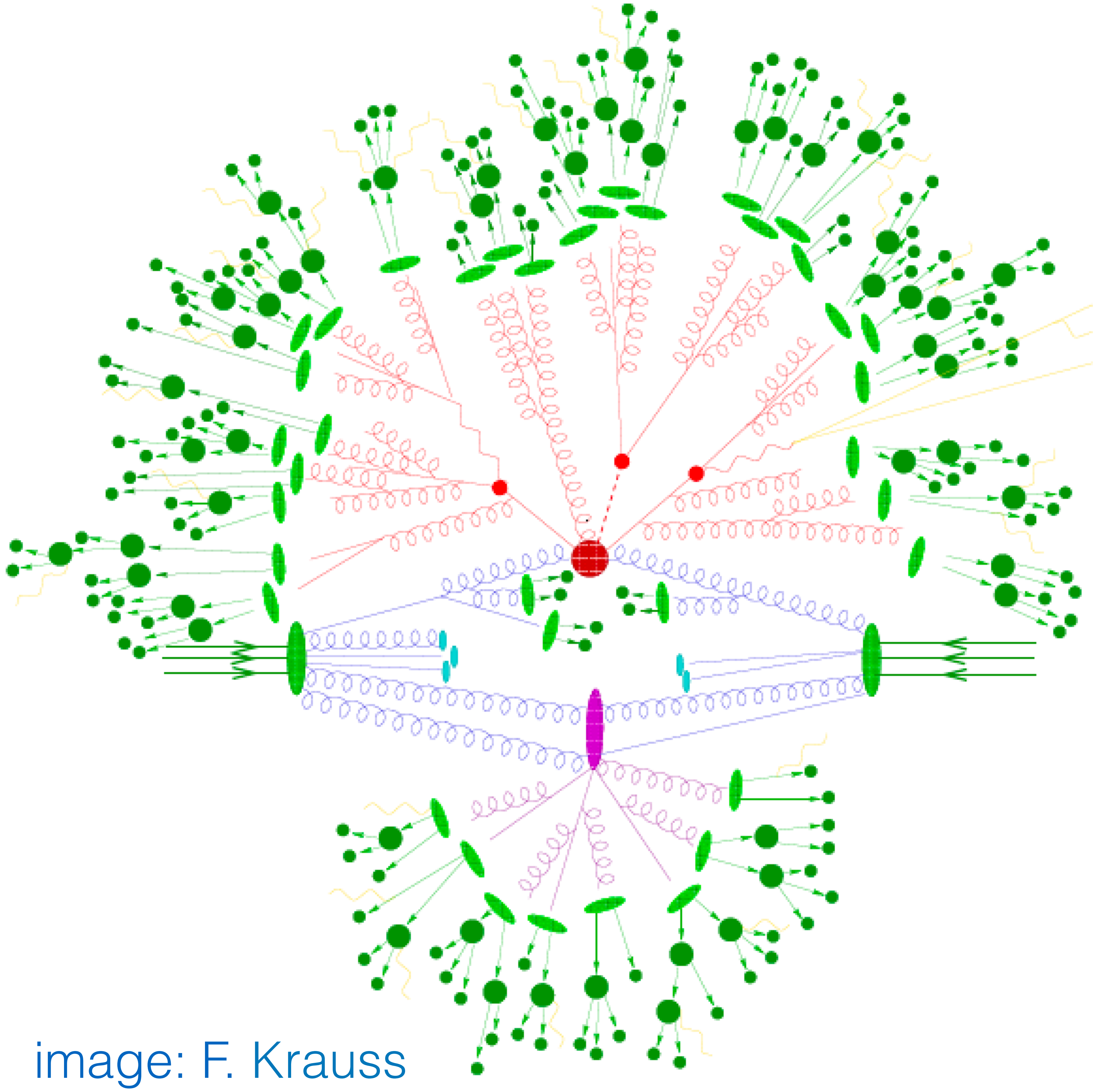

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perturbative contributions:
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non-perturbative contributions:
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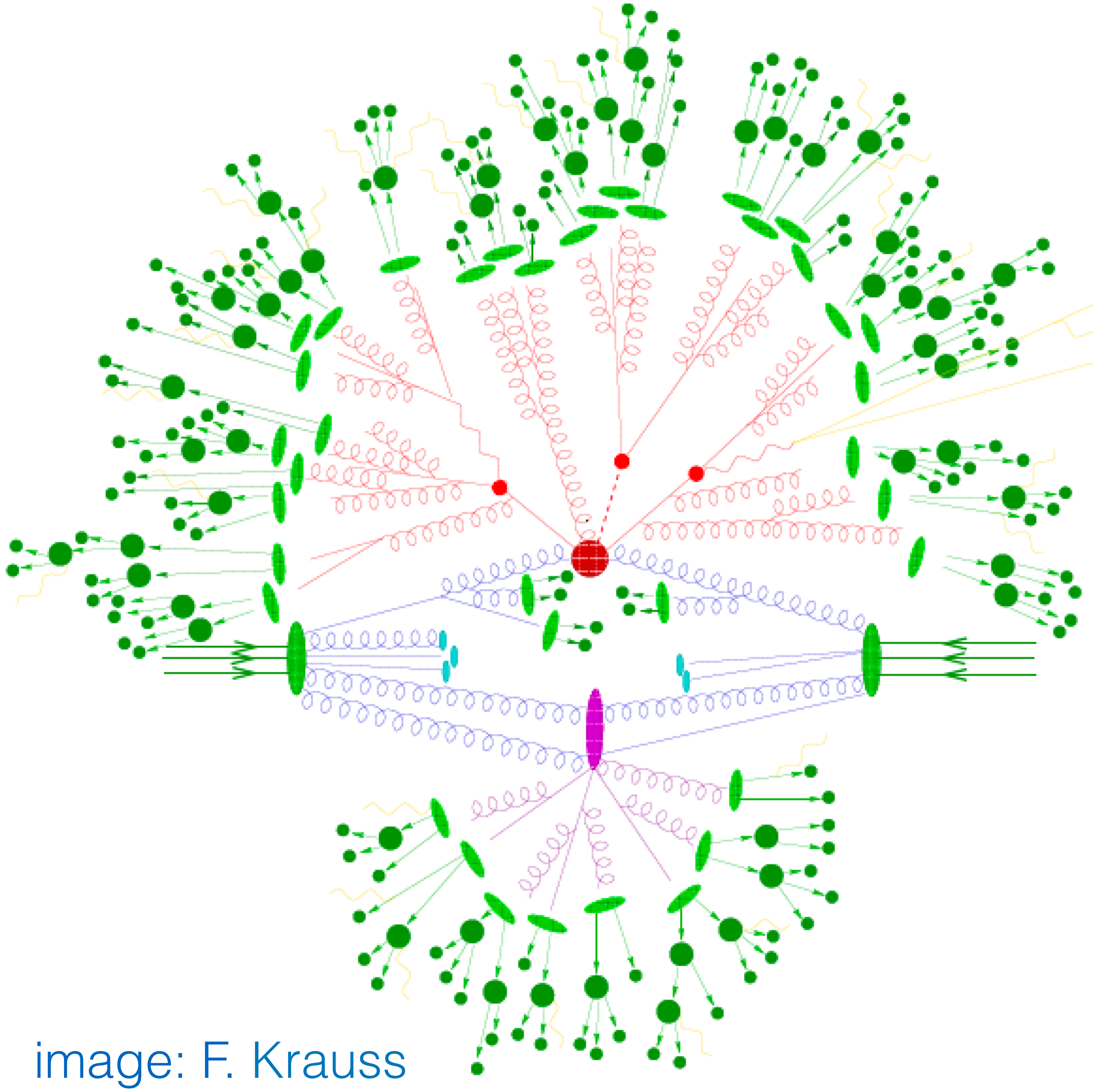
lattice?

Motivation



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lattice?

PDFs defined via light-cone correlators,
lattice formulated in Euclidean space

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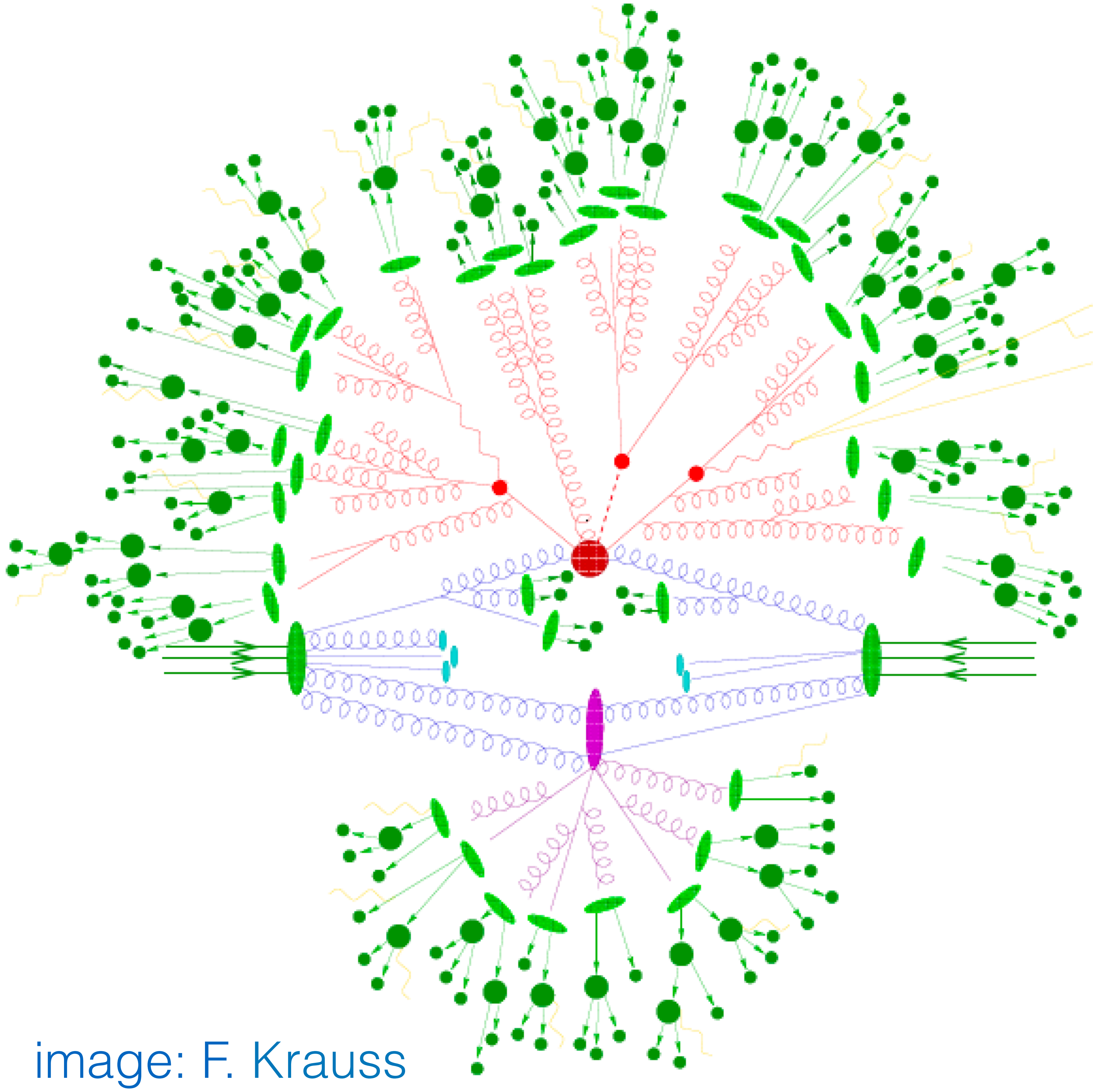

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perturbative contributions:
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PDFs defined via light-cone correlators,
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alternative: moments

$$\langle x^n \rangle_q = \int_0^1 dx\, x^n\, q(x)$$

Moments of parton densities

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad \sim \quad p_{\mu_1}\cdots\; p_{\mu_n}\langle x^{n-1}\rangle$$

Moments of parton densities

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$$\mathcal{O}_{\mu_1\cdots\mu_n} \sim \int d^4x\,\bar{\psi}(x)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\,\psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

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problem: Lattice breaks Lorentz symmetry

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problem: Lattice breaks Lorentz symmetry

→ mixing with Lorentz violating operators

→ highly complicated $a \to 0$ limit

→ only results up to $\langle x^3\rangle$ available

Martinelli, Sachrajda '87, '88

Alexandrou et al. '20, '21

Gradient flow as cutoff

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

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$$D_\mu = \partial_\mu + igA_\mu$$

In $\mathcal{O}_{\mu_1\cdots\mu_n}$, remove high-momentum modes $p^2 \gtrsim 1/t$

$$\psi(x) \to \chi(x,t)$$

$$A_\mu(x) \to B_\mu(x,t)$$

in a gauge and Lorentz invariant way

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→ continuum limit can be taken, result depends on UV cutoff t

Gradient flow as cutoff

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$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$

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$$\partial_t B_\mu(x, t) = \Box\, B_\mu(x, t)$$

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$$\partial_t B_\mu(x, t) = \Box\, B_\mu(x, t) \quad \rightarrow \quad \partial_t B_\mu(x, t) = D_\nu(t)\, G_{\nu\mu}(x, t)$$

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$$D_\mu(t) = \partial_\mu + ig[B_\mu(x,t), \cdot\,]$$

$$G_{\mu\nu}(x,t) \sim [D_\mu(t), D_\nu(t)]$$

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$$\partial_t B_\mu(x, t) = \square\, B_\mu(x, t) \quad \rightarrow \quad \partial_t B_\mu(x, t) = D_\nu(t)\, G_{\nu\mu}(x, t)$$

$$B_\mu(x, 0) = A_\mu(x)$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x, t), \cdot\,]$$

$$G_{\mu\nu}(x, t) \sim [D_\mu(t), D_\nu(t)]$$

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$\Updownarrow$

$$\partial_t B_\mu(x,t) = \Box\, B_\mu(x,t) \quad \rightarrow \quad \boxed{\begin{aligned} \partial_t B_\mu(x,t) &= D_\nu(t)\, G_{\nu\mu}(x,t) \\ B_\mu(x,0) &= A_\mu(x) \end{aligned}}$$

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Gradient flow as cutoff

$$\xcancel{B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k) \qquad \to \text{not Lorentz invariant!}}$$

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$$\partial_t B_\mu(x,t) = \Box B_\mu(x,t) \quad \to \quad \boxed{\begin{aligned} \partial_t B_\mu(x,t) &= D_\nu(t)\, G_{\nu\mu}(x,t) \\ B_\mu(x,0) &= A_\mu(x) \end{aligned}}$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x,t), \cdot\,]$$

$$G_{\mu\nu}(x,t) \sim [D_\mu(t), D_\nu(t)]$$

analogously: $$\boxed{\begin{aligned} \partial_t \chi(x,t) &= D^2 \chi(x,t) \\ \chi(x,0) &= \psi(x) \end{aligned}}$$

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad \sim \quad p_{\mu_1}\cdots\ p_{\mu_n}\langle x^{n-1}\rangle$$

Moments of parton densities

$$\begin{tikzcd}
\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad \sim \quad p_{\mu_1}\cdots\; p_{\mu_n}\langle x^{n-1}\rangle \arrow[d, bend left, "\text{Shindler ‘24}"] \\
\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle
\end{tikzcd}$$

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$\downarrow$ Shindler ‘24

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n}(t) \sim \int d^4x\,\bar{\psi}(x,t)\,\gamma_{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}(t)\cdots\overset{\leftrightarrow}{D}_{\mu_n}(t)\,\psi(x,t)$$

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

$$\Big\downarrow$$

Shindler '24

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Continuum limit exists.

Moments of parton densities

$$\langle h(p) \,|\, \mathcal{O}_{\{\mu_1\cdots\mu_n\}} \,|\, h(p) \rangle \quad \sim \quad p_{\mu_1} \cdots \; p_{\mu_n} \langle x^{n-1} \rangle$$

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Continuum limit exists.

Lorentz and gauge invariance preserved.

Moments of parton densities

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Continuum limit exists.

Lorentz and gauge invariance preserved.

No mixing with other operators.

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$t \to 0$?

Short flow time expansion

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$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \; + \; t\,(\;\cdots\;)$$

Lüscher ‘14

Suzuki ‘15

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \;+\; t\,(\,\cdots\,)$$

Lüscher ‘14
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higher-dimensional operators → neglect

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higher-dimensional operators → neglect

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lattice

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Lüscher '14
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perturbation theory

lattice

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lattice

perturbation theory

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Loops and Legs

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Lüscher '14
Suzuki '15

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perturbation theory

lattice

actually:

$$\zeta^{\mathrm{R}}_{nm}(t,\mu) = \sum_k \zeta_{nk}(t)\,Z^{-1}_{km}$$

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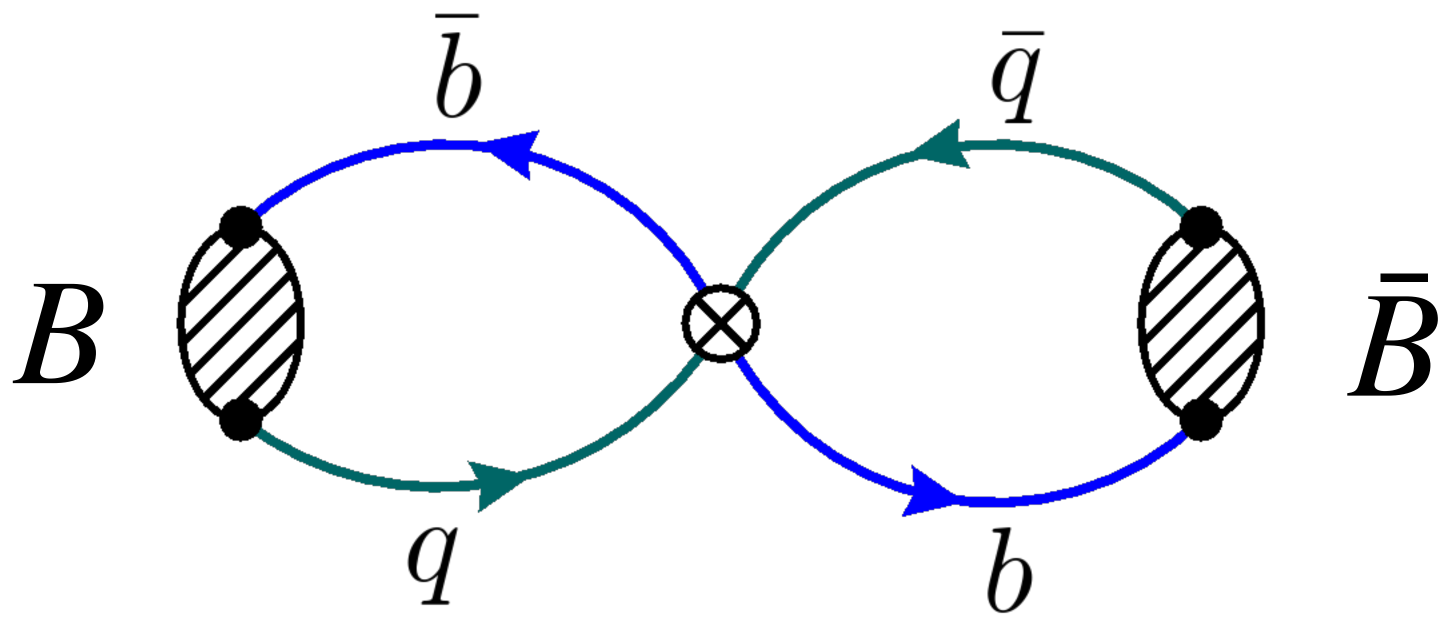
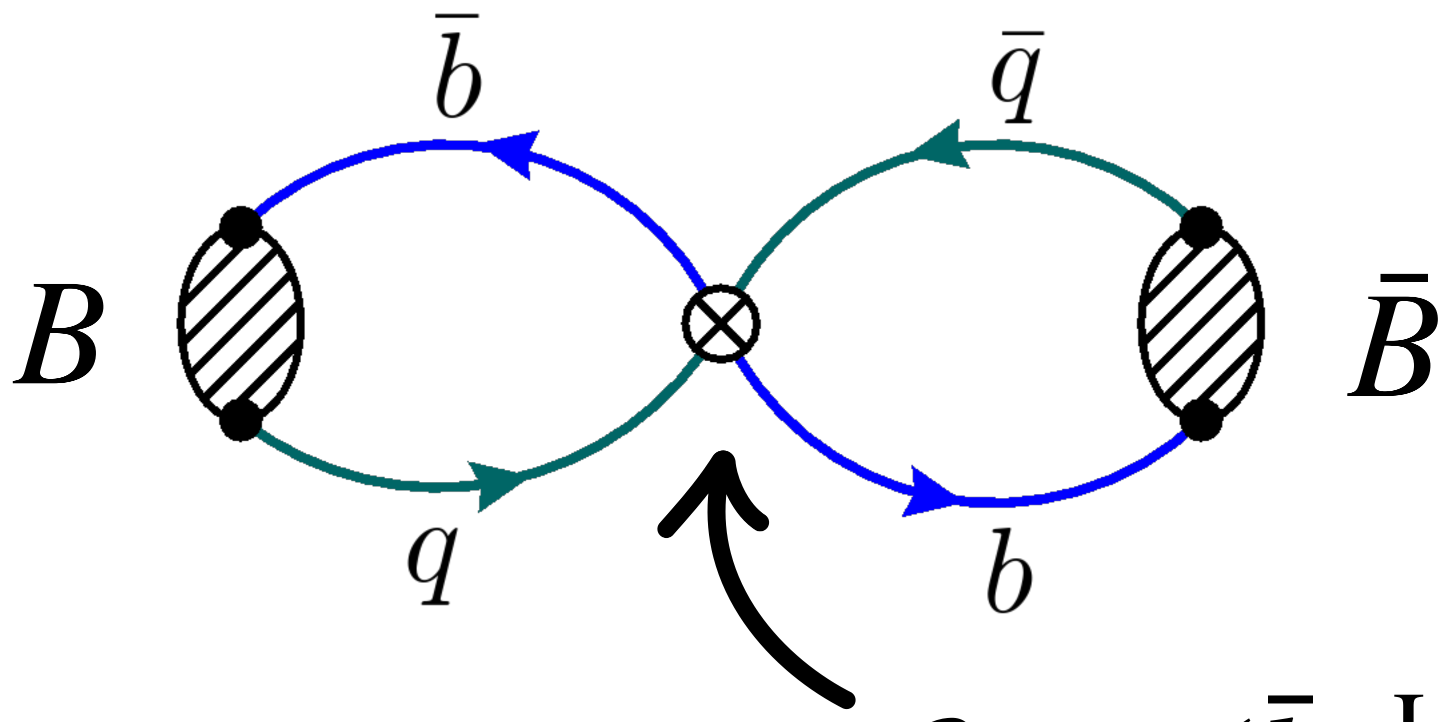
Loops and Legs

Meson mixing

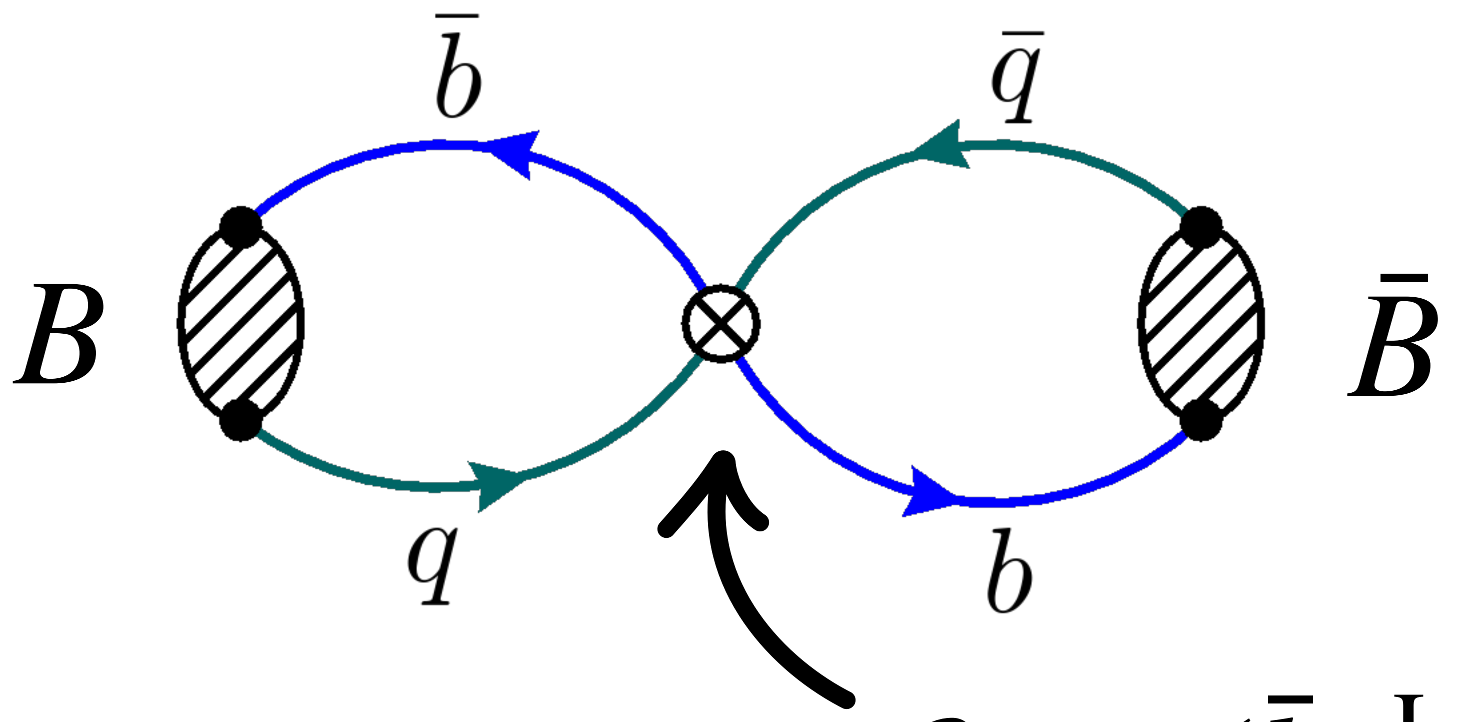


Meson mixing



$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

Meson mixing



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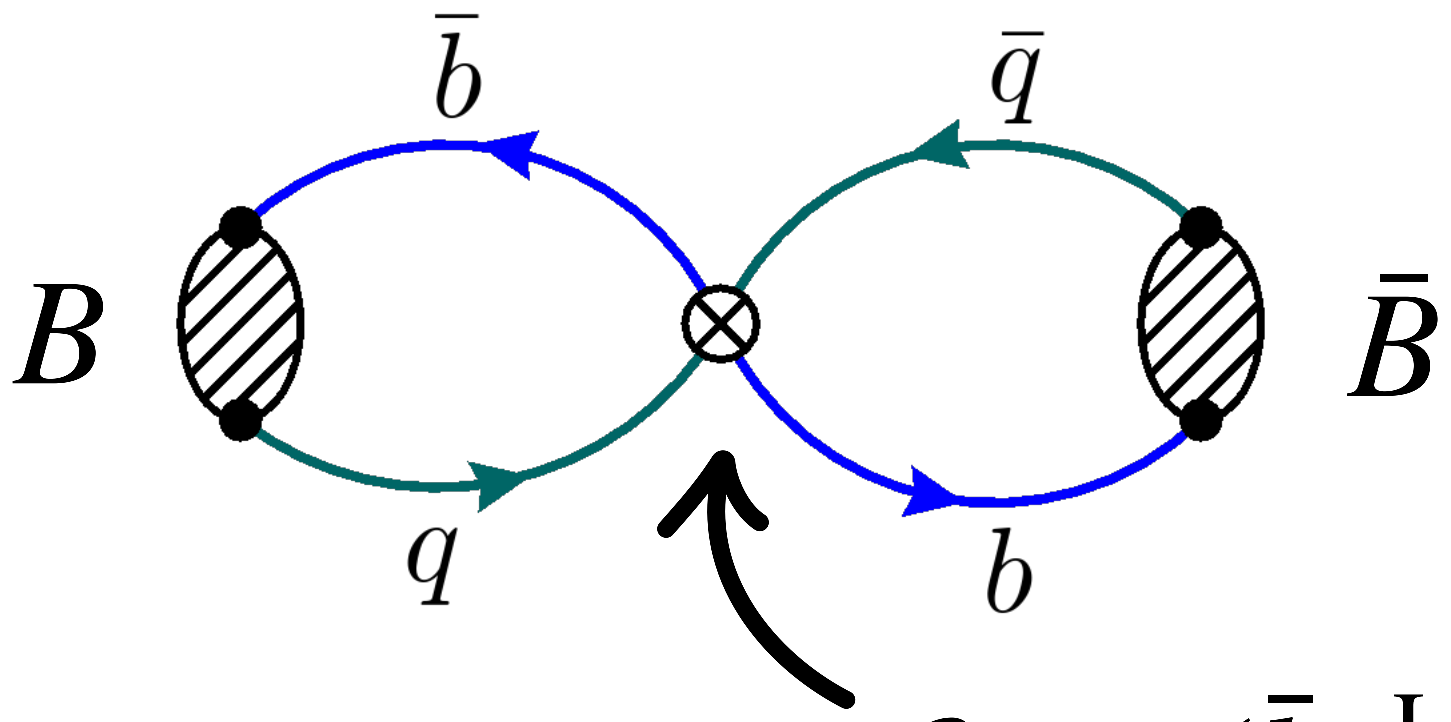
$\longrightarrow$ $B_1 \sim \langle B\,|\,\mathcal{O}_1\,|\,B\rangle$ bag parameter

Meson mixing



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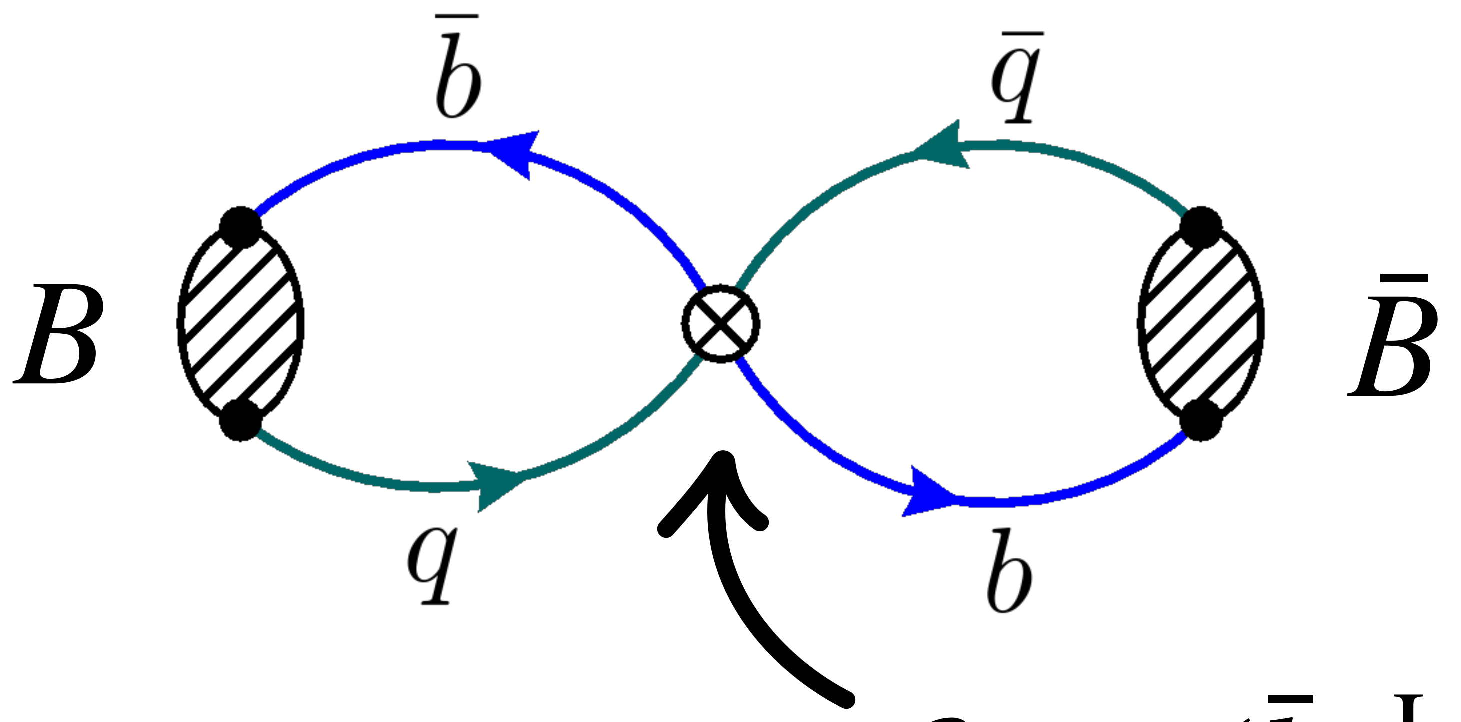
Meson mixing



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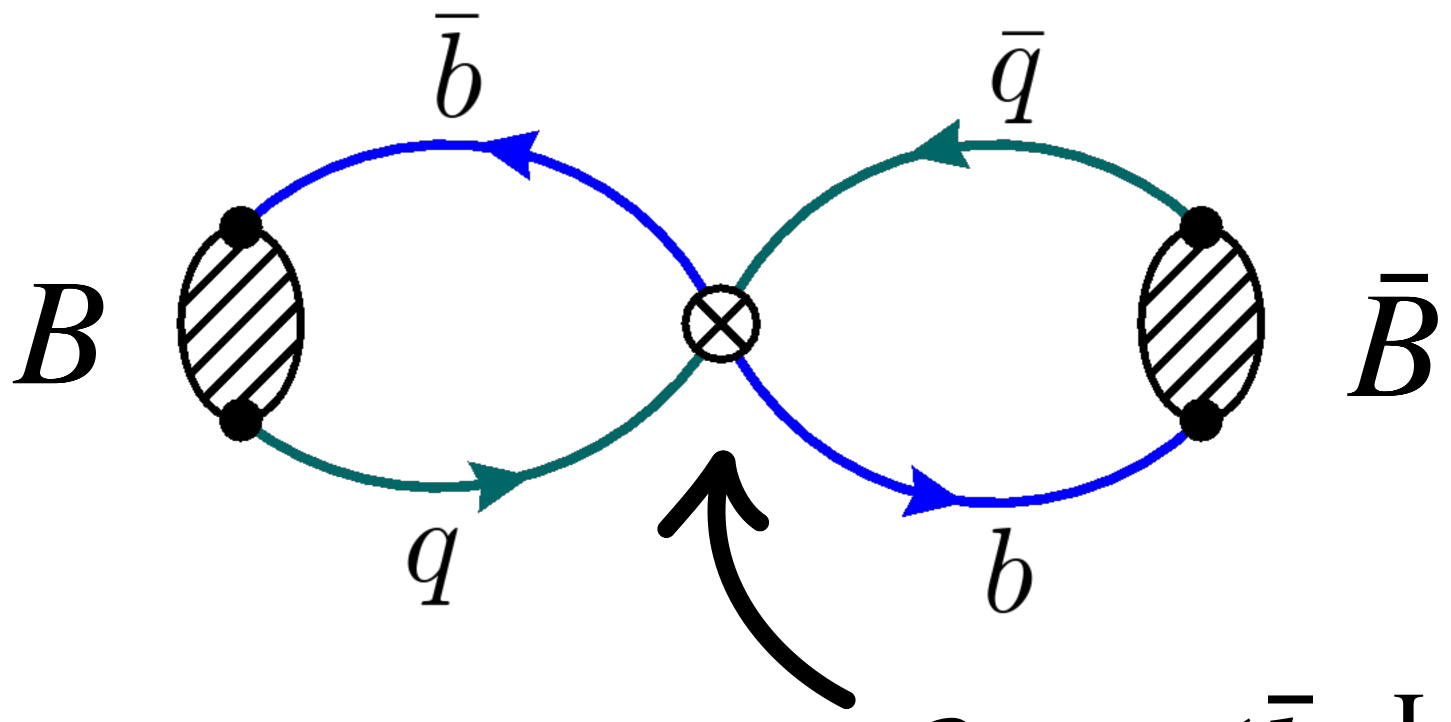
perturbative ↗

Meson mixing



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perturbative ↗ ↖ lattice

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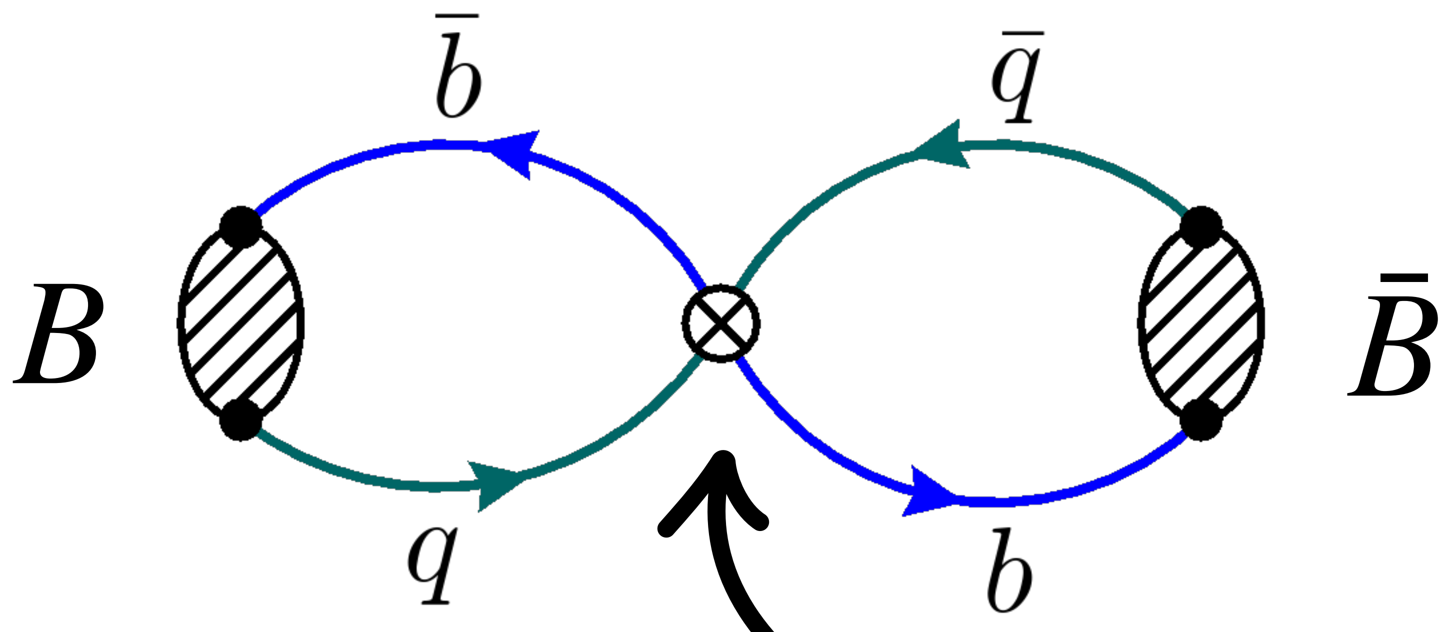

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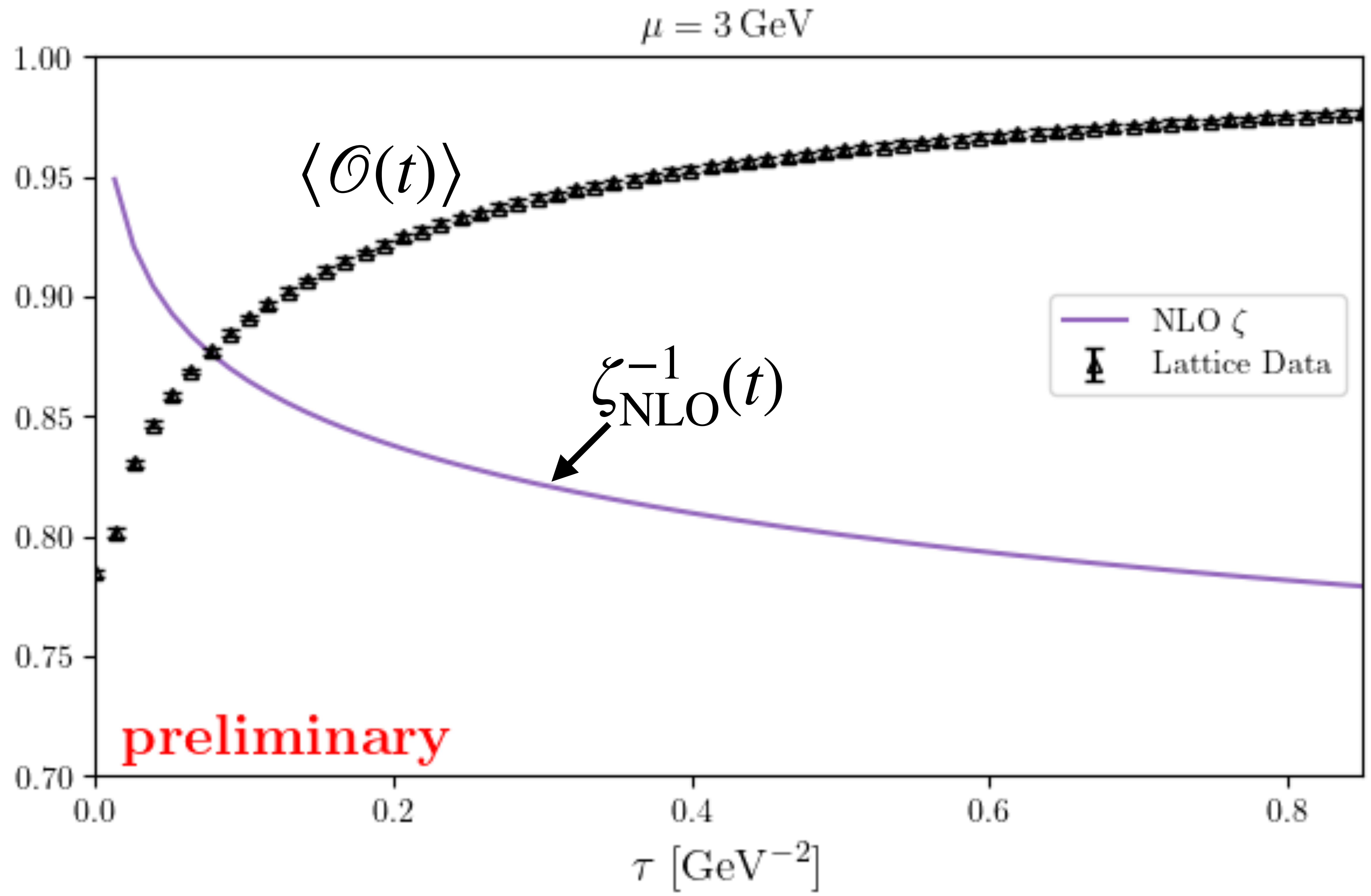
perturbative

lattice



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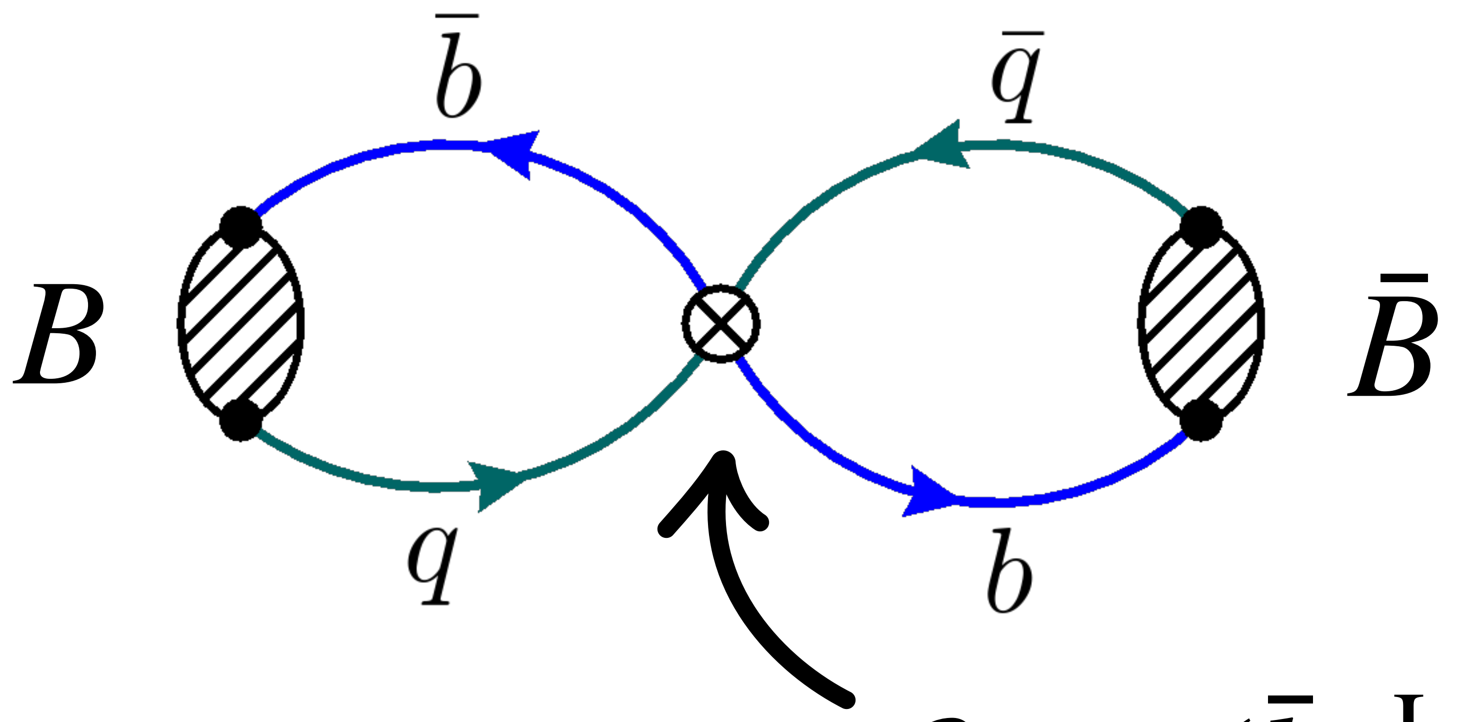

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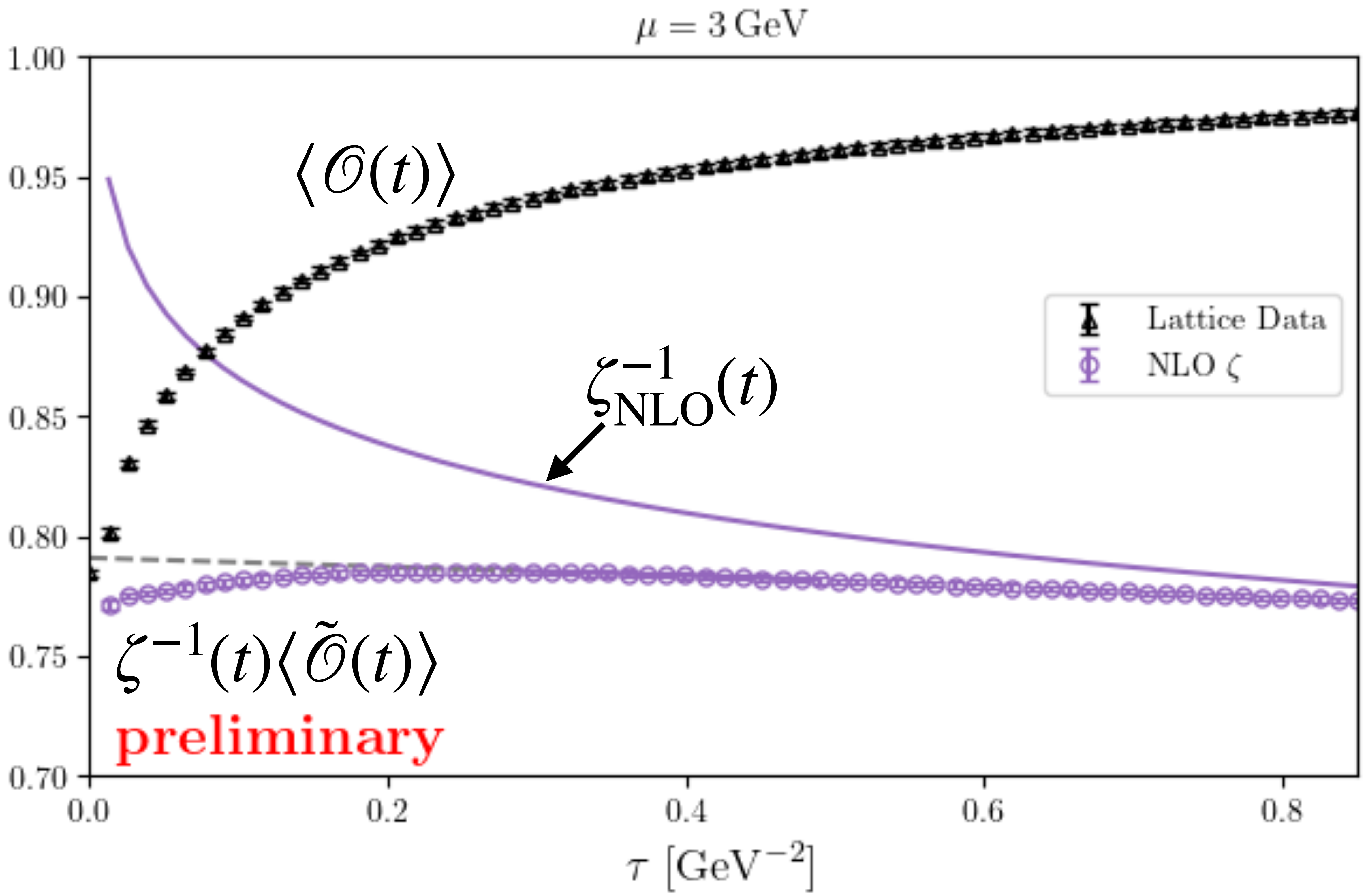
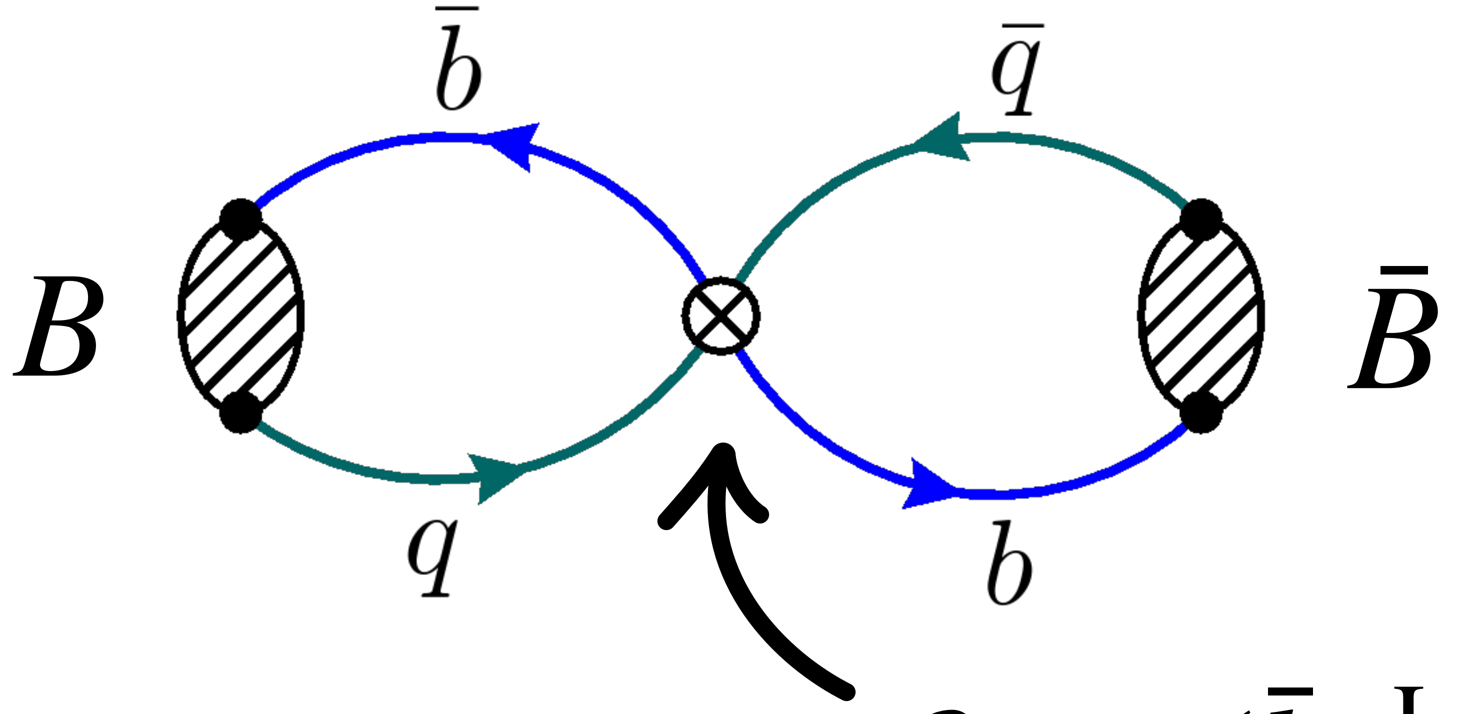


Meson mixing



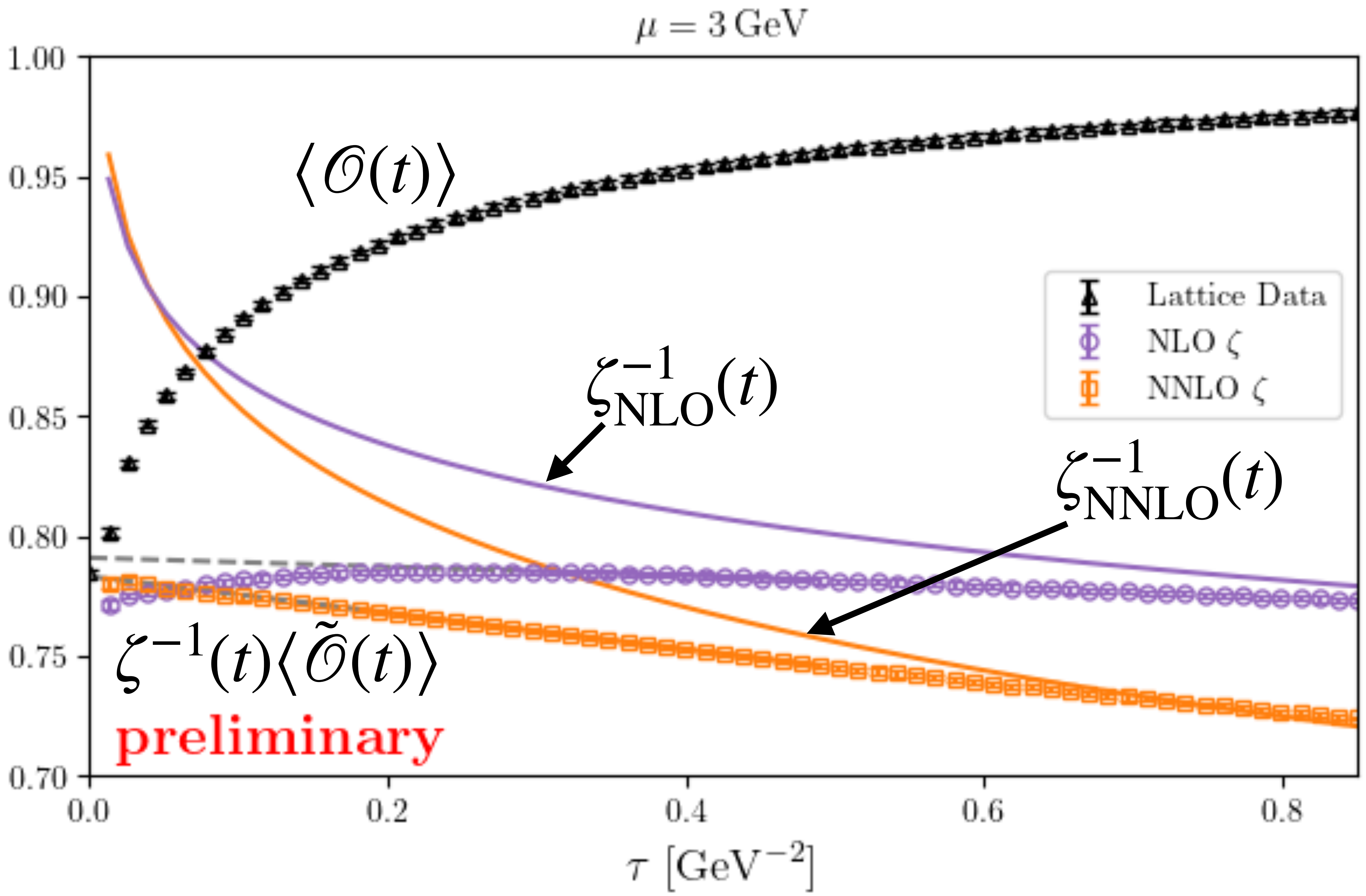
$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

$$\longrightarrow \quad B_1 \sim \langle B \,|\, \mathcal{O}_1 \,|\, B \rangle$$ bag parameter

$$= \zeta^{-1}(t)\langle B \,|\, \mathcal{O}_1(t) \,|\, B \rangle$$

perturbative

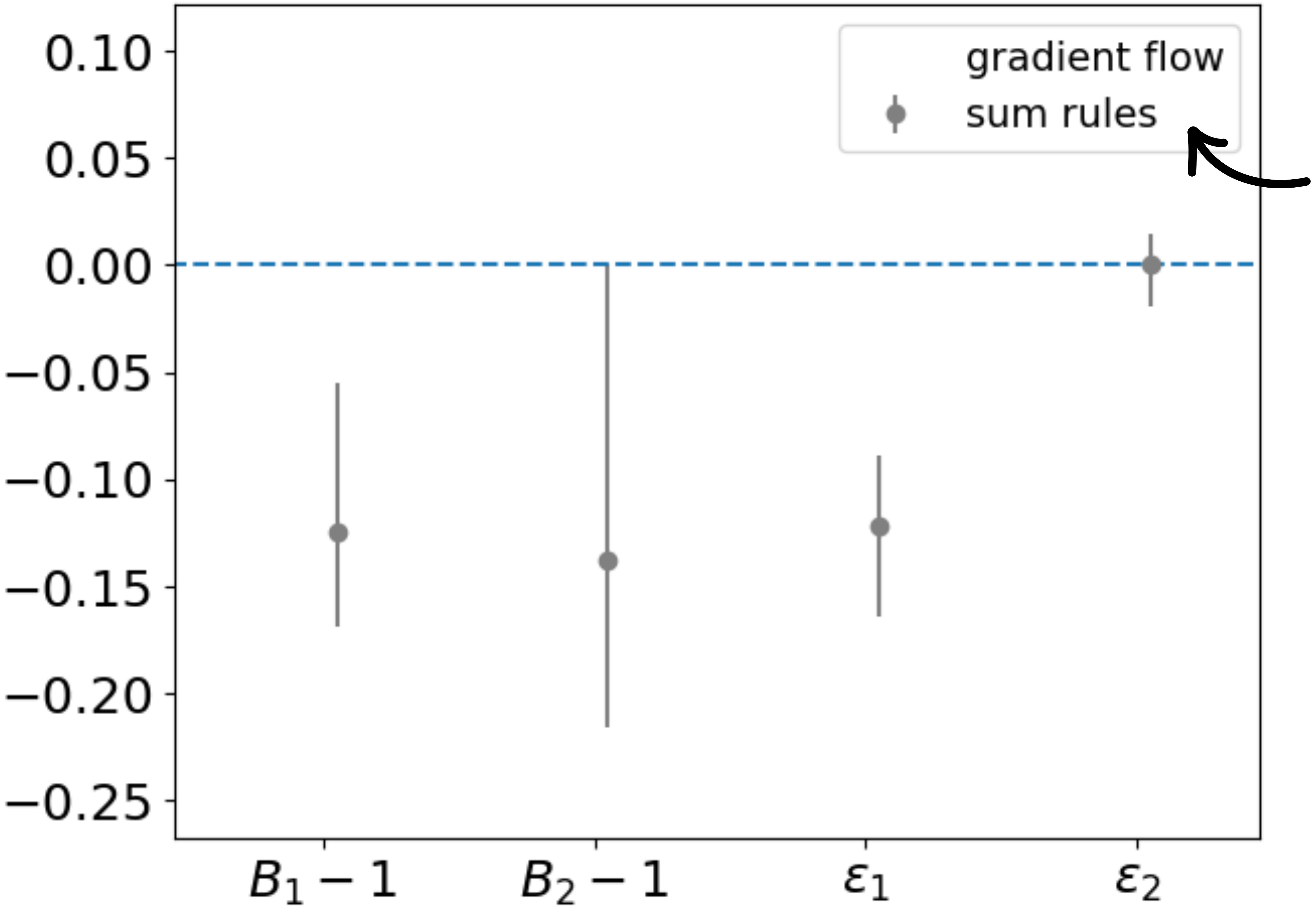
lattice



Black, RH, Lange, Rago, Shindler, Witzel ’26

CRC TRR 257

D meson lifetime bag parameters



Black, Lang, Lenz, Wüthrich ‘25

$$
\begin{aligned}
\mathcal{O}_1 &= (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)Q),\\
T_1 &= (\bar{Q}\gamma_\mu(1-\gamma_5)T^A q)(\bar{q}\gamma^\mu(1-\gamma_5)T^A Q),\\
\mathcal{O}_2 &= (\bar{Q}(1-\gamma_5)q)(\bar{q}(1+\gamma_5)Q),\\
T_2 &= (\bar{Q}(1-\gamma_5)T^A q)(\bar{q}(1+\gamma_5)T^A Q),
\end{aligned}
$$

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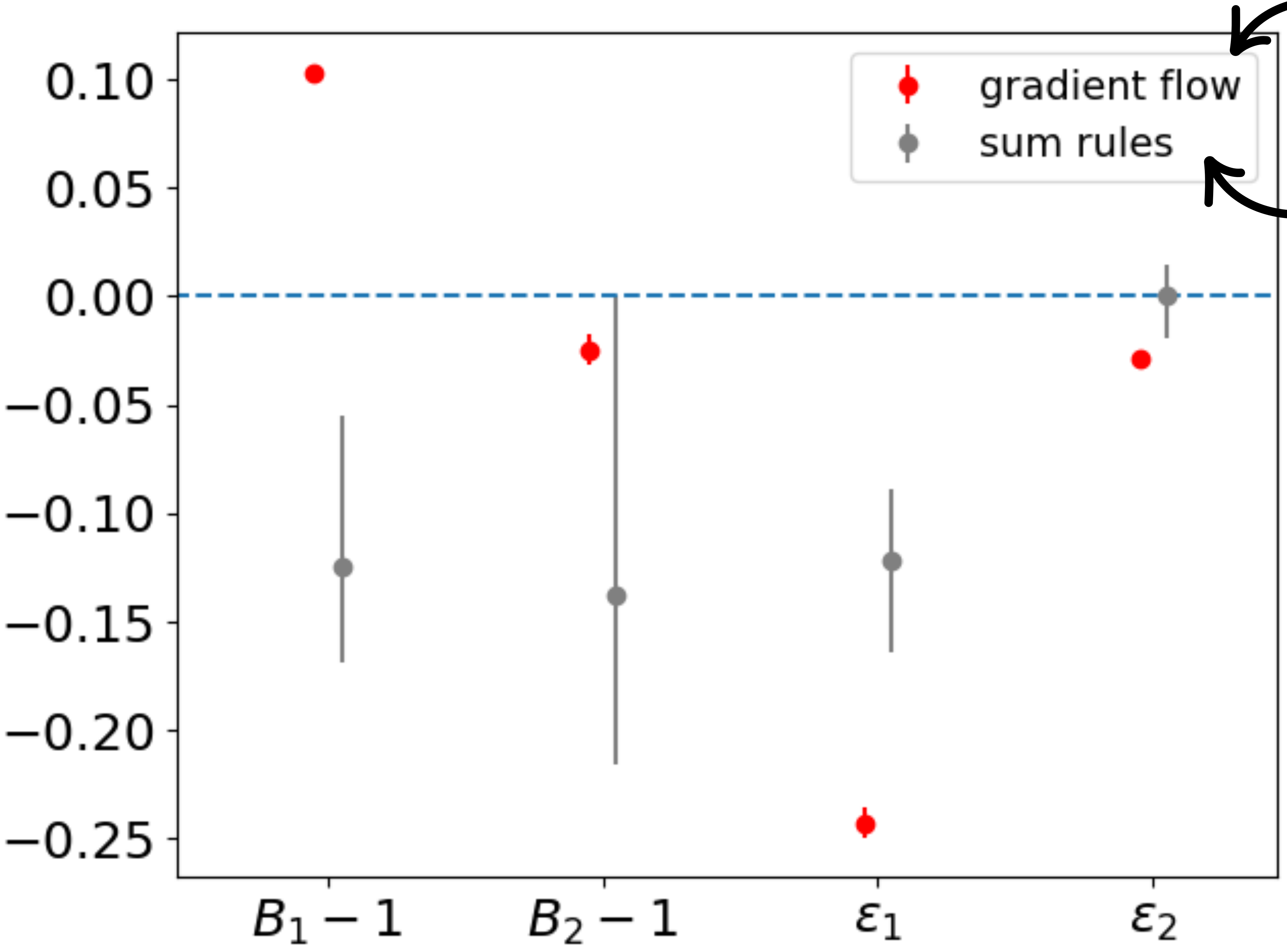

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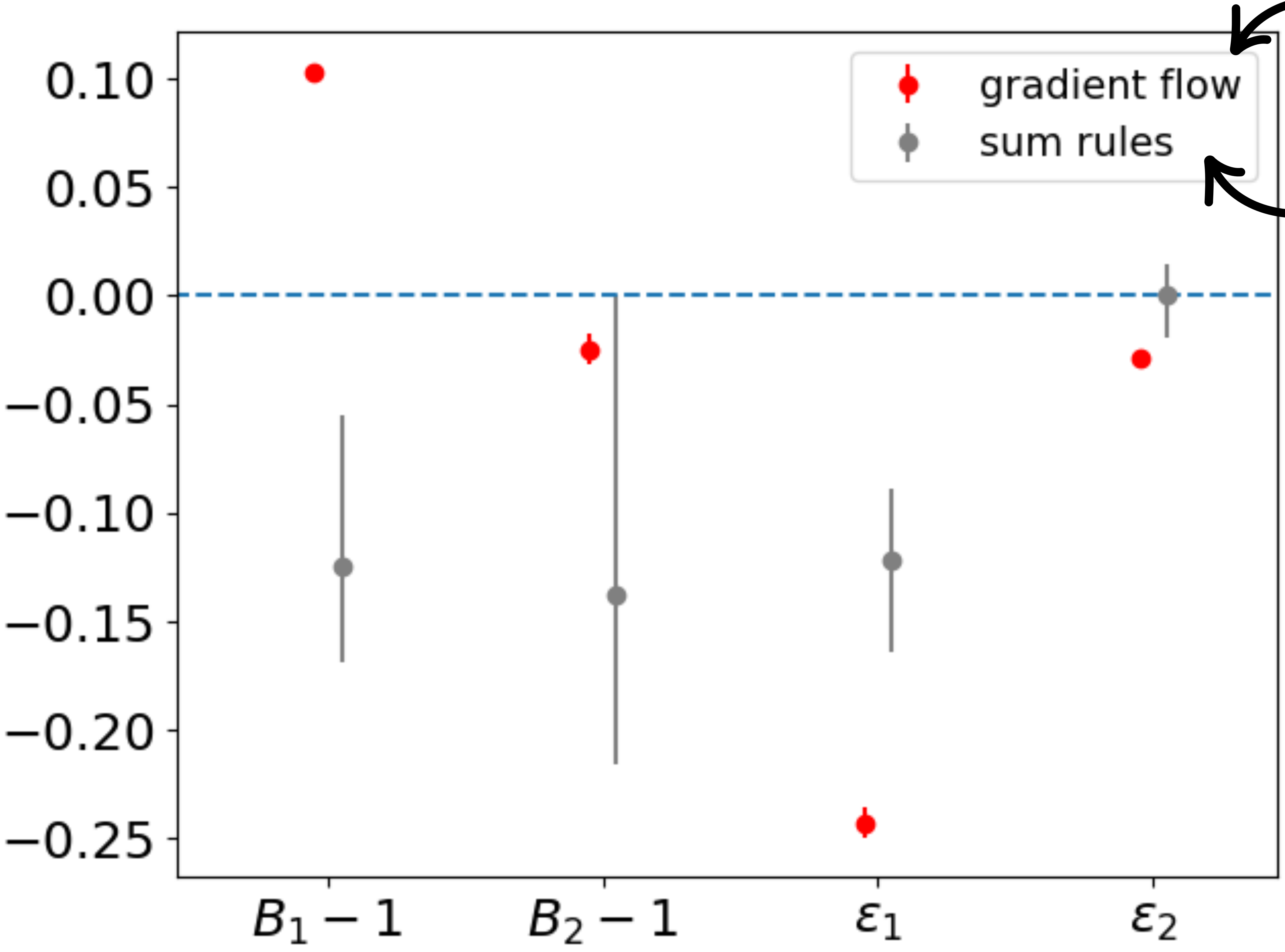

Black, RH, Lange, Rago, Shindler, Witzel '26

Black, Lang, Lenz, Wüthrich '25

$$\mathcal{O}_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)Q),$$
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$$\mathcal{O}_2 = (\bar{Q}(1-\gamma_5)q)(\bar{q}(1+\gamma_5)Q),$$
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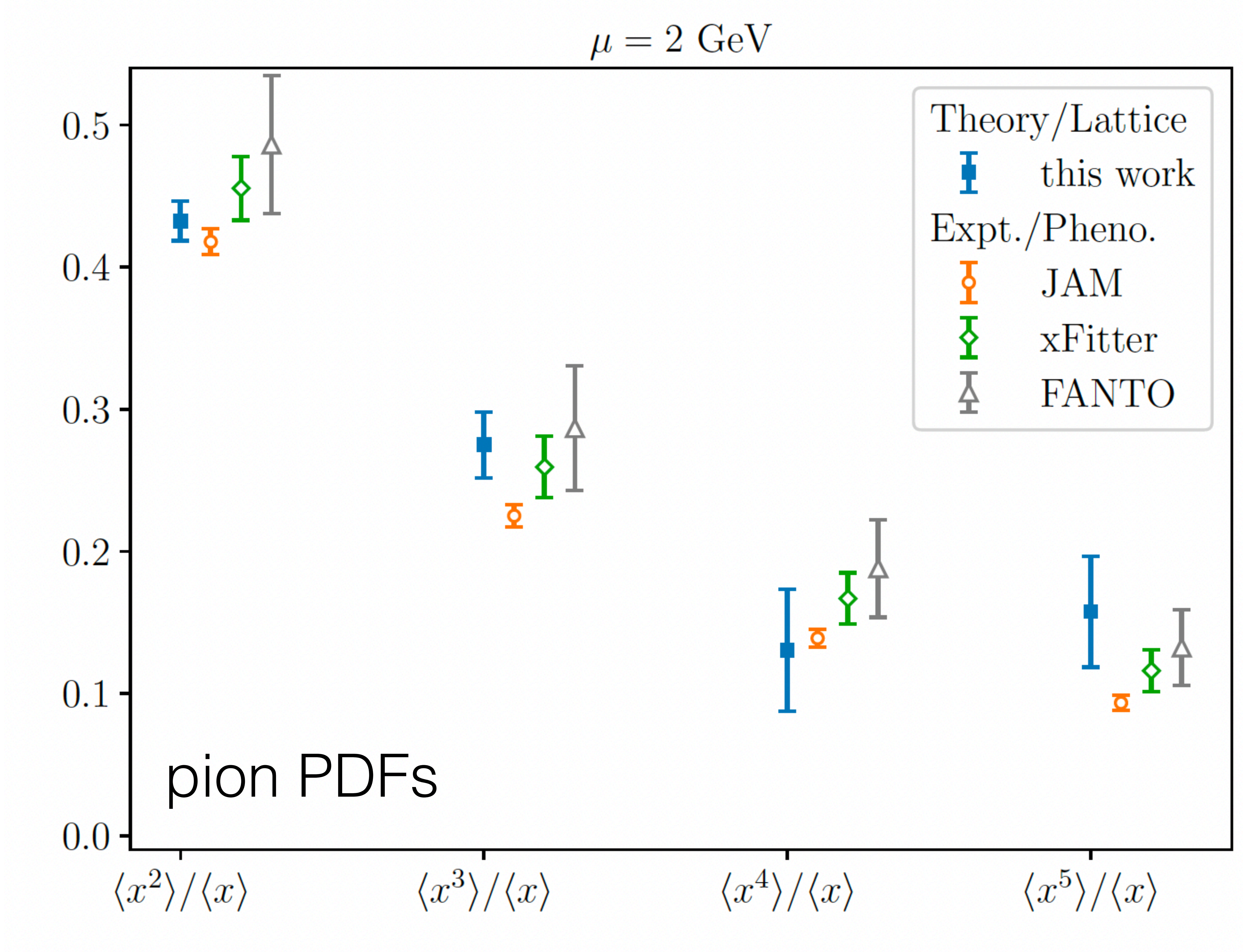
but: comparison not fully consistent

Moments of parton densities



$$\langle h(p) | \mathcal{O}_{\{\mu_1\cdots\mu_n\}} | h(p)\rangle =$$

$$= \sum_m \zeta^{-1}_{mn}(t)\, \langle h(p) | \mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t) | h(p)\rangle$$

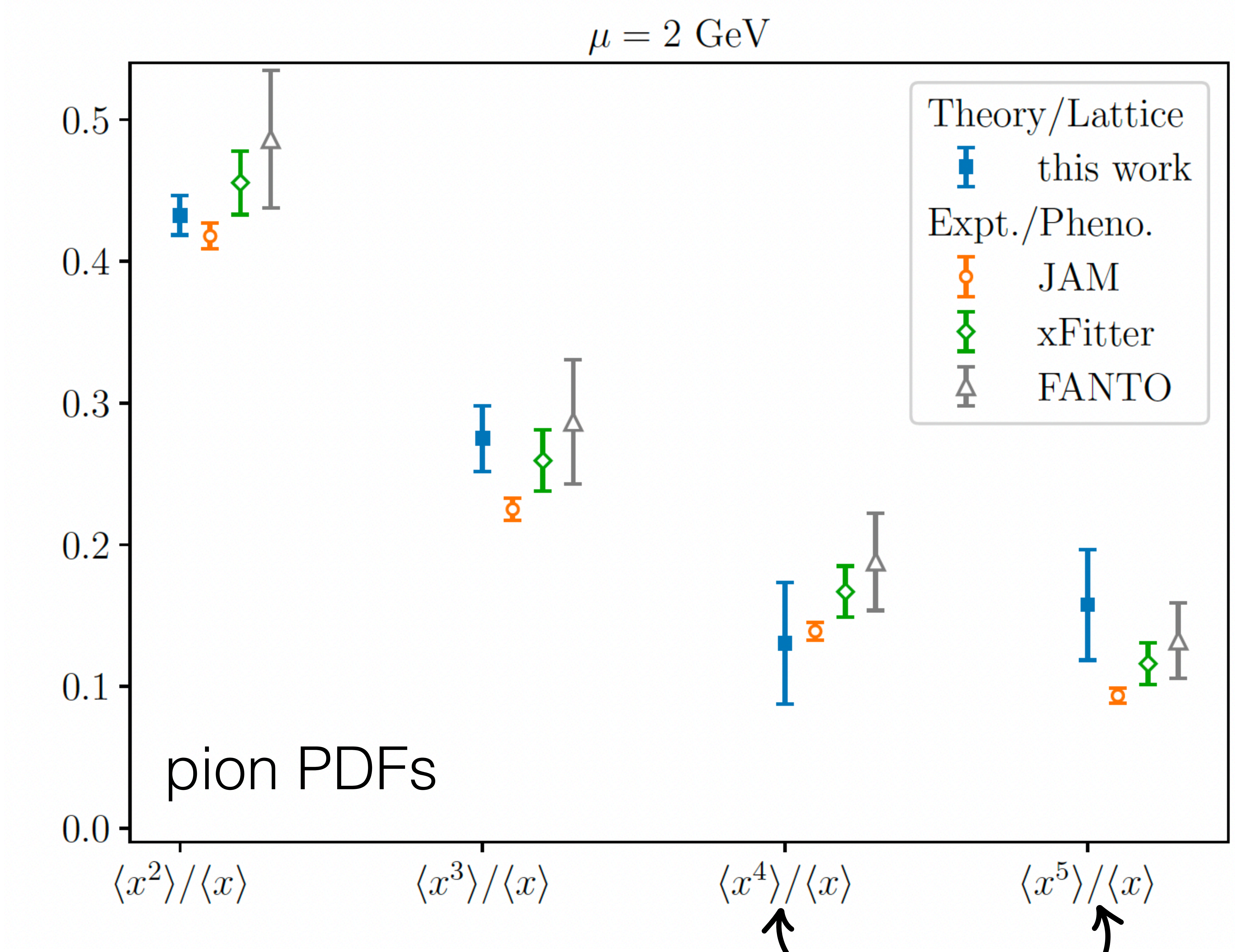
Francis, …, Shindler, … ‘25

Francis, …, RH, Kohnen, Shindler, … ‘25

RH, Kohnen, Shindler ‘25

see also Edwards et al. ‘26

Moments of parton densities



unreachable w/o gradient flow

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle =$$

$$= \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

Francis, …, Shindler, … ‘25

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see also Edwards et al. ‘26

Quark masses

Quark masses

Black, RH, Hasenfratz, Rago, Witzel ‘25

CRC TRR 257

Quark masses

Black, RH, Hasenfratz, Rago, Witzel ‘25

CRC TRR 257

PCAC:

$$R(t, \tau) = -\frac{\langle j_A^0(t, \tau) j_P(t=0)\rangle}{\langle j_P(t, \tau) j_P(t=0)\rangle}$$

Quark masses

Black, RH, Hasenfratz, Rago, Witzel '25

CRC TRR 257

PCAC:

$$R(t,\tau) = -\frac{\langle j_A^0(t,\tau) j_P(t=0)\rangle}{\langle j_P(t,\tau) j_P(t=0)\rangle}$$

$$m(\mu) = \lim_{\tau\to 0}\lim_{t\to\infty} \zeta^{-1}(\tau,\mu)\, R(t,\tau)$$

Quark masses

Black, RH, Hasenfratz, Rago, Witzel '25

CRC TRR 257

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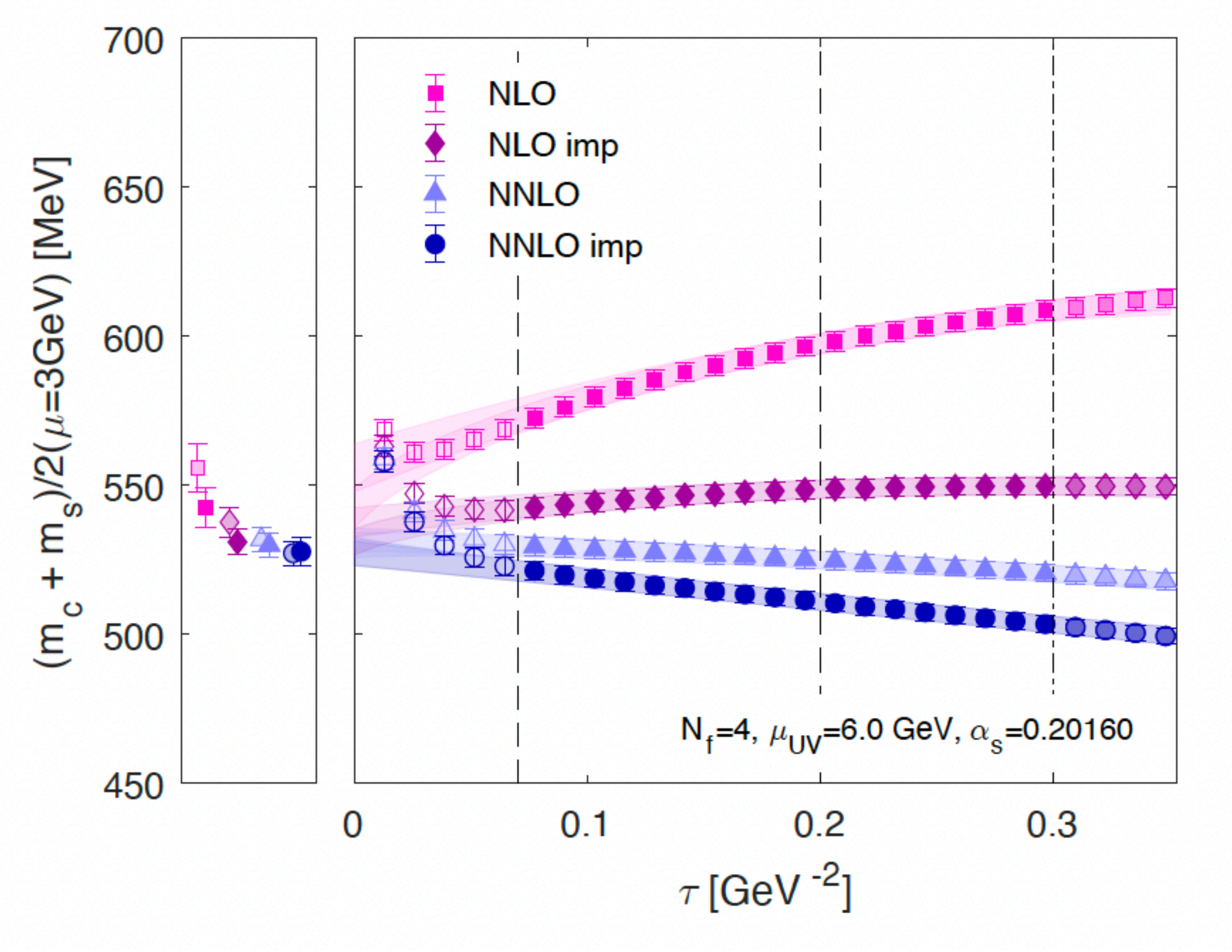

Quark masses

Black, RH, Hasenfratz, Rago, Witzel ‘25

CRC TRR 257

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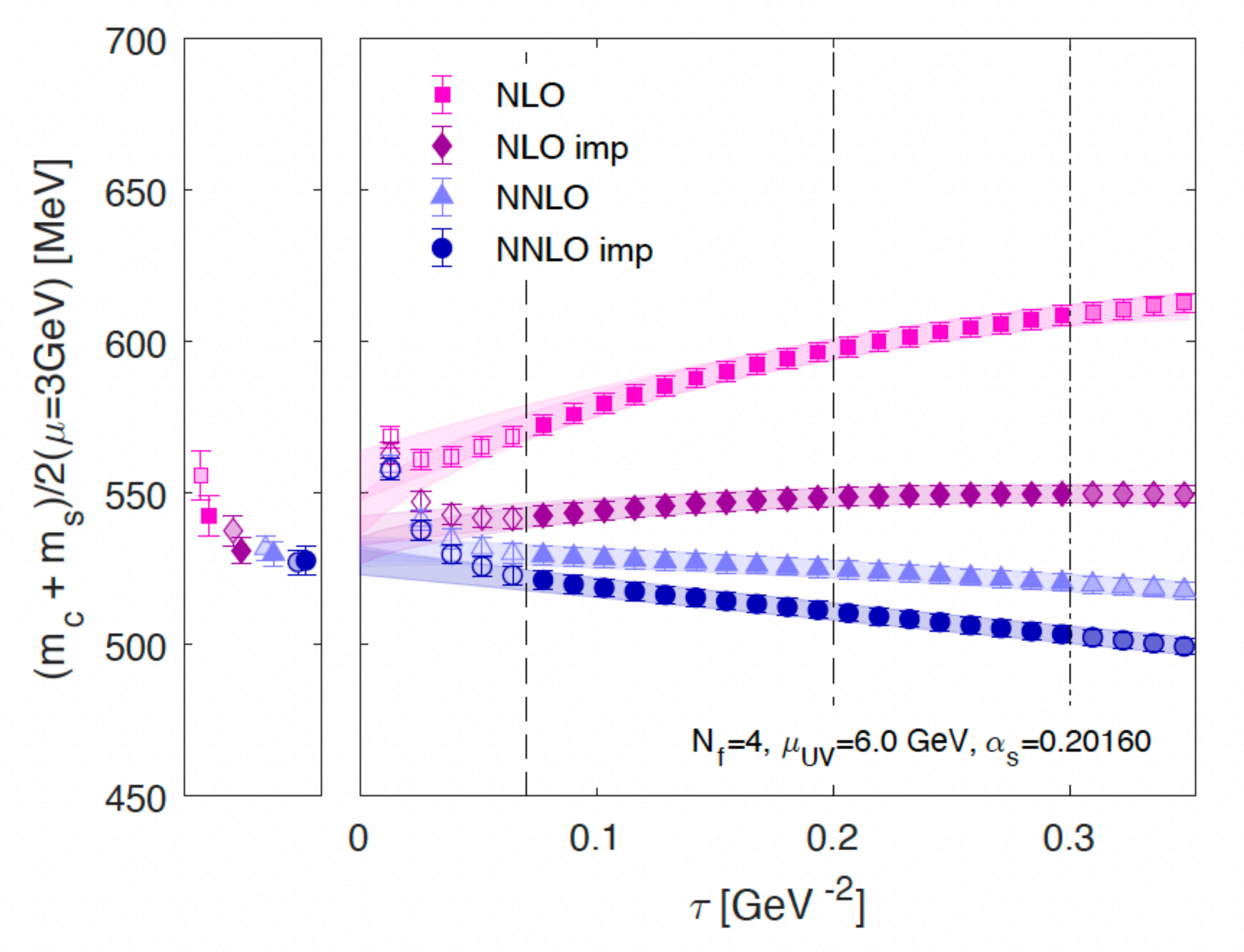

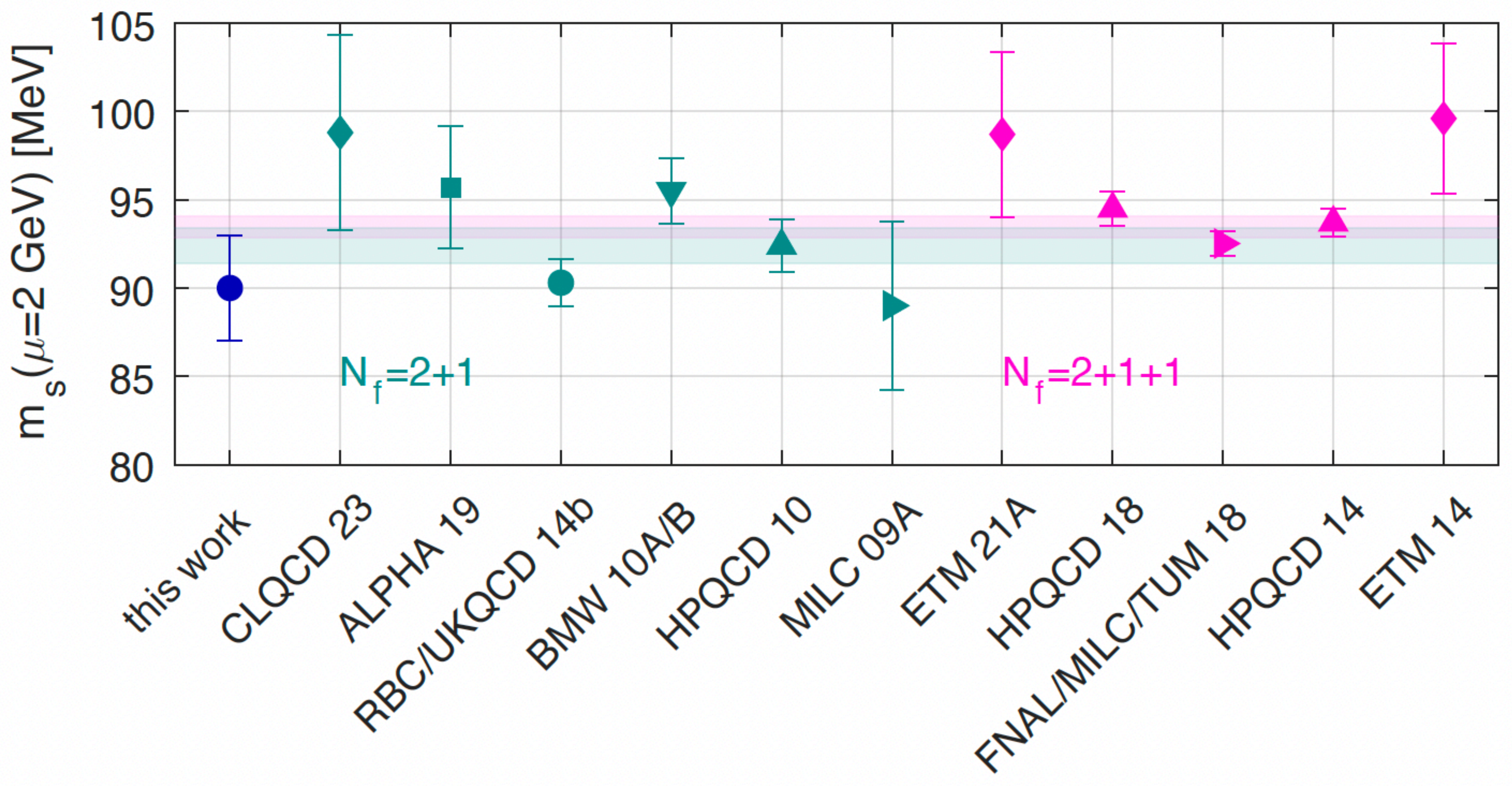

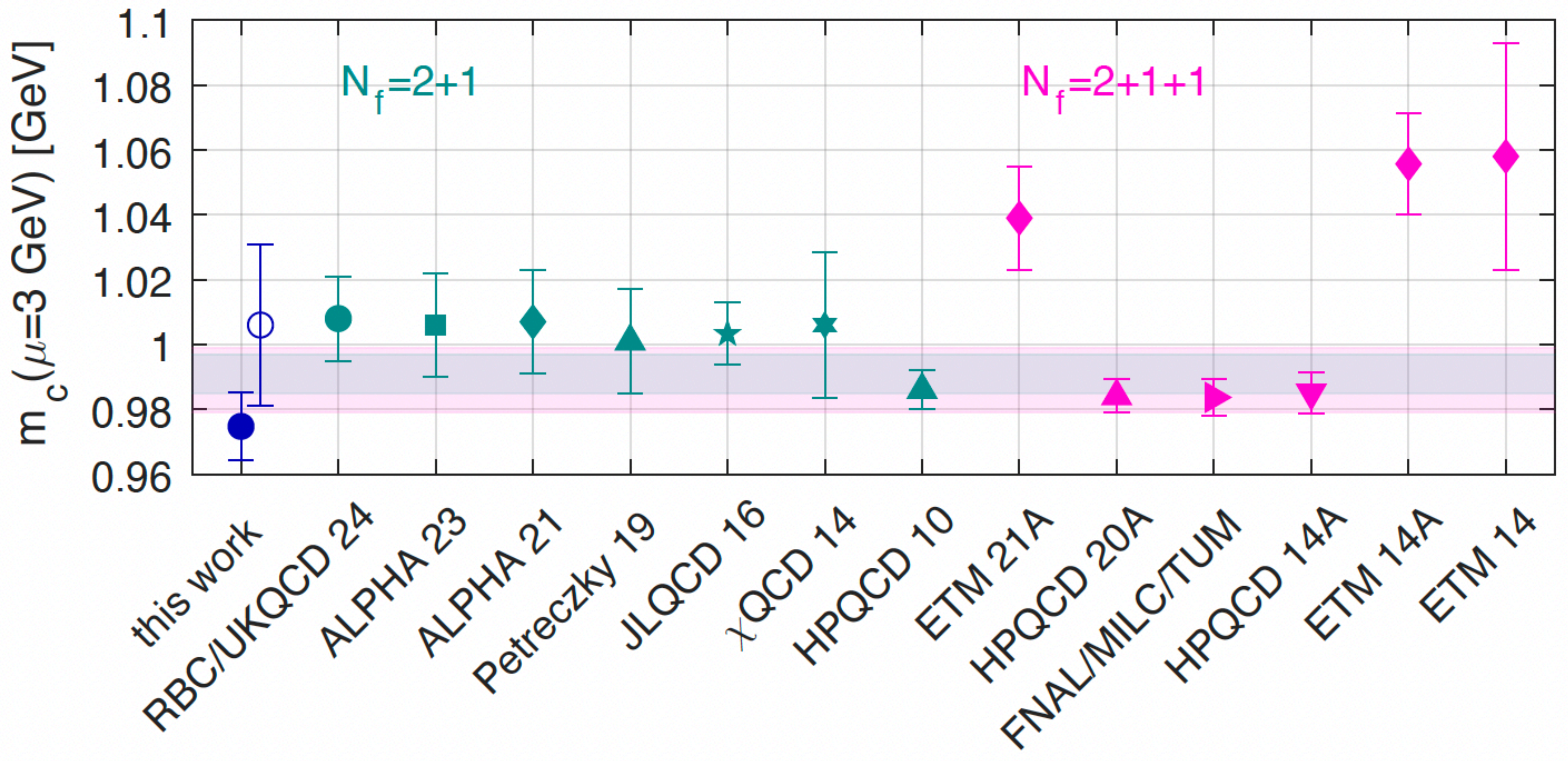

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Gradient flow Lagrangian

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t,x) = \mathscr{D}^2 \chi(t,x)$$

$$\chi(t=0,x) = \psi(x)$$

Gradient flow Lagrangian

flowed gauge field:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}B_\mu(t,x) &= \mathscr{D}_\nu G_{\nu\mu}(t,x)\\ B_\mu(t=0,x) &= A_\mu(x)\end{aligned}}$$

flowed quark field:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\chi(t,x) &= \mathscr{D}^2\chi(t,x)\\ \chi(t=0,x) &= \psi(x)\end{aligned}}$$

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda}\left(\partial_t - \mathscr{D}^2\right)\chi + \mathrm{h.c.}$$

Lüscher, Weisz 2011

Lüscher 2013

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

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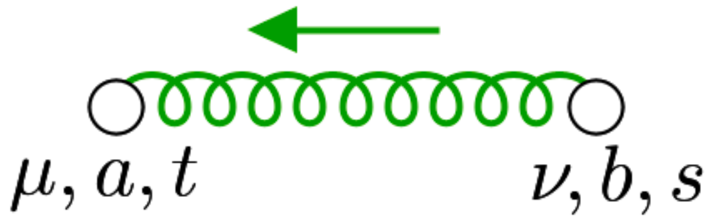


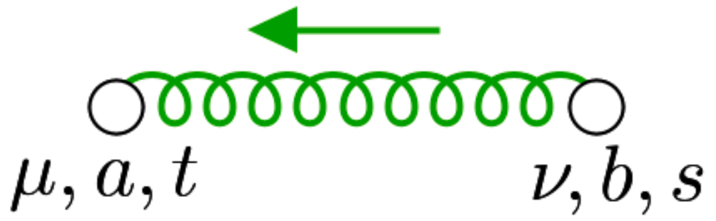
$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

“gluon flow line”

$$\sim \langle 0 | T\, L_\mu^a(t,x)\, B_\nu^b(s,0)\, | 0 \rangle$$

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

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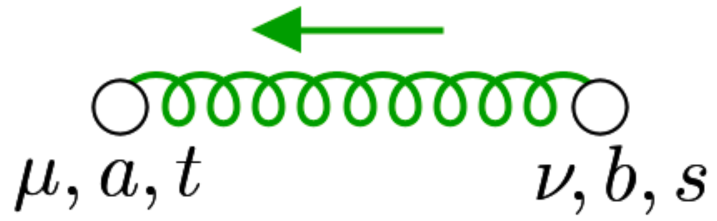
$$\sim \langle 0|\,T\,L_\mu^a(t,x)\,B_\nu^b(s,0)\,|0\rangle$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right)\,e^{-(t+s)p^2}$$

$$\sim \langle 0|\,T\,B_\mu^a(t,x)\,B_\nu^b(s,0)\,|0\rangle$$

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, {\color{green}L_\mu} \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

“gluon flow line”

$$\sim \langle 0\,|\,T\,{\color{green}L^a_\mu(t,x)}\,B^b_\nu(s,0)\,|\,0\rangle$$

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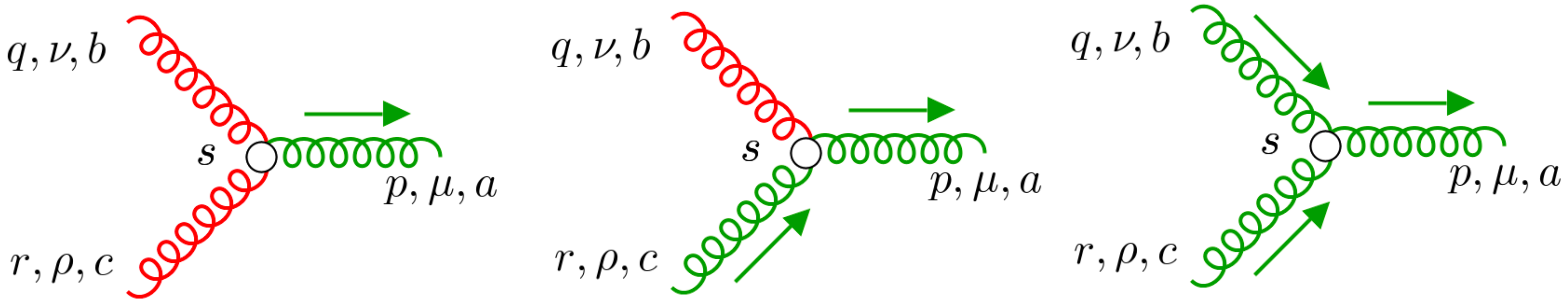

$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu$$
$$+\,(\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

+ 4-gluon vertex

Let’s calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

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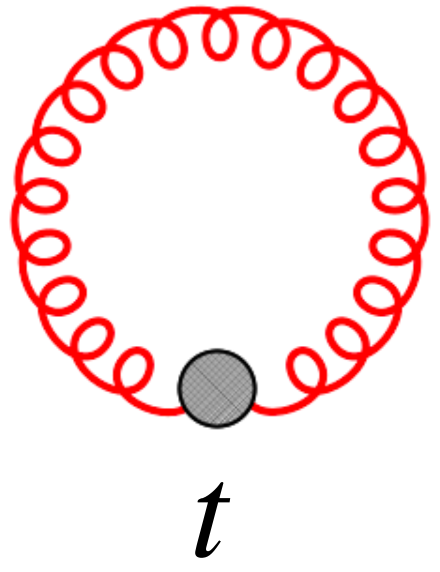
LO:



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$$E(t) \equiv \frac{1}{4}\langle G^{a}_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

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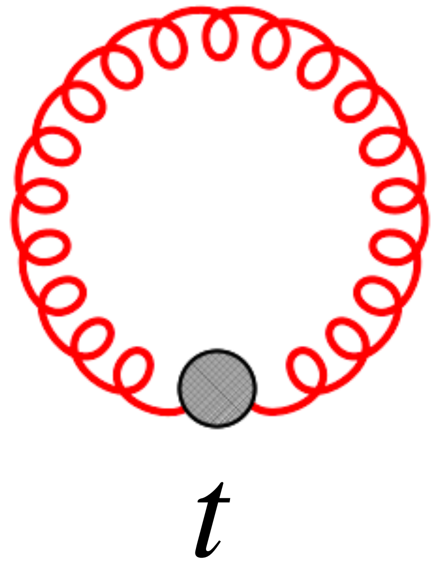

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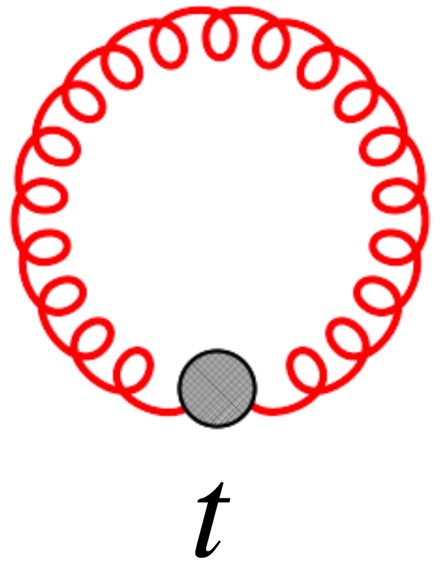

$$\sim \int d^Dp\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

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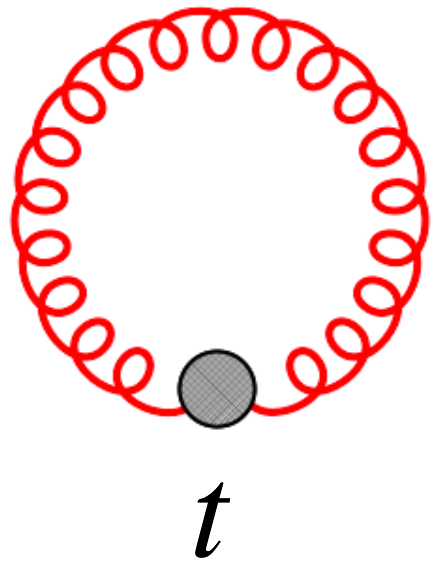

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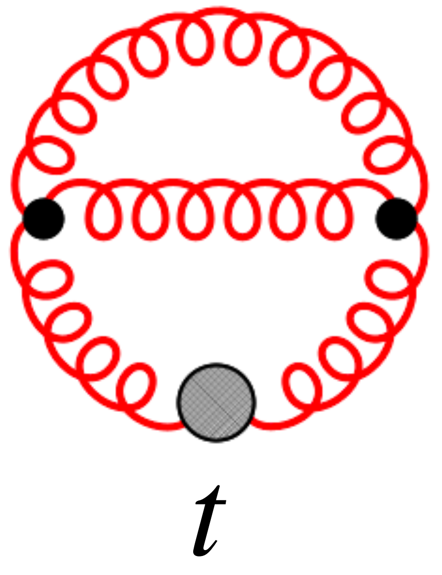
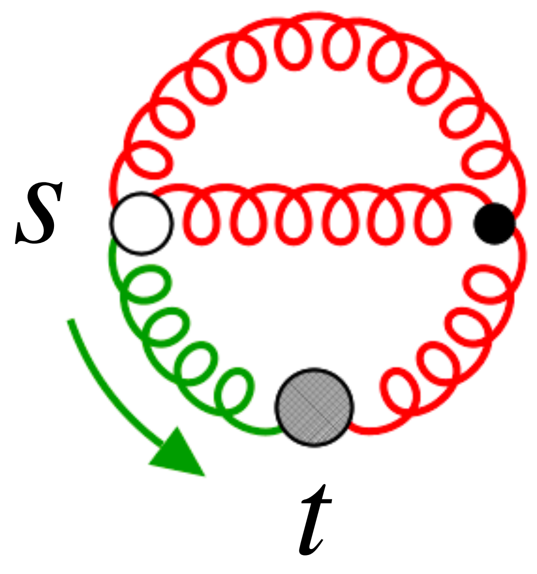
NLO:





$$\sim \int_p\int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

$$\int_0^t ds \int_p\int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

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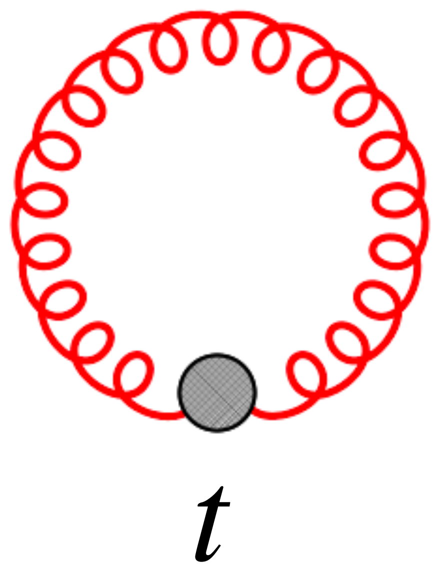

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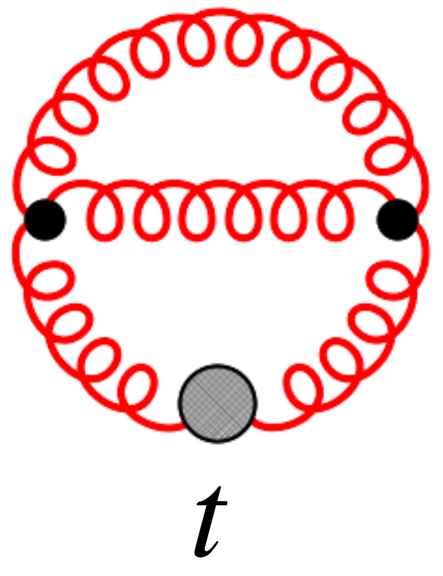

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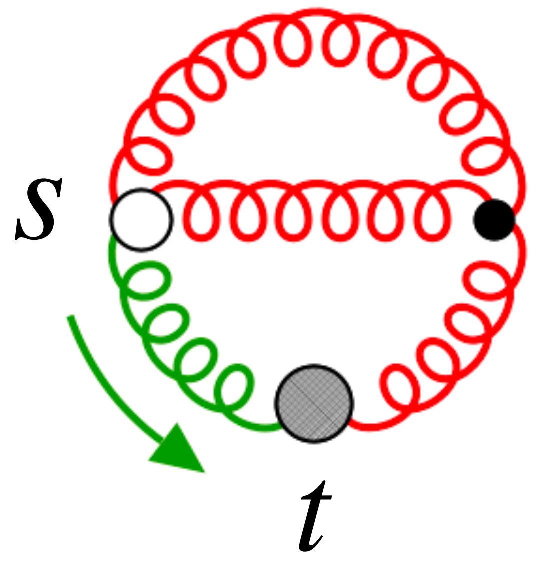

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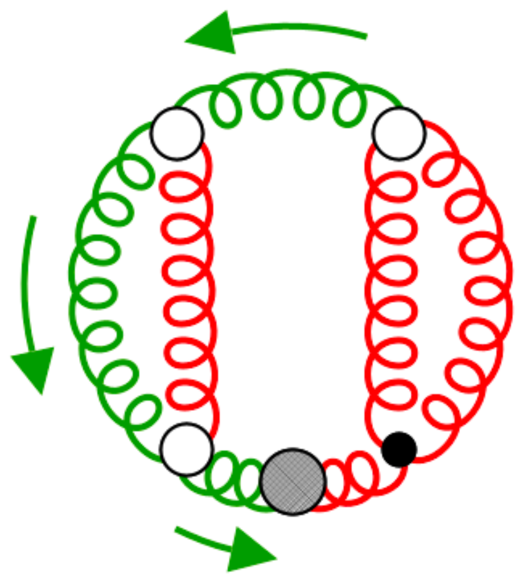
NNLO:



$$\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_p \int_k \int_l \cdots$$

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: `qgraf` Nogueira 1993

diagram analyzation: `q2e/exp` RH, Seidensticker, Steinhauser 1997

→ `tapir/exp` Gerlach, Herren, Lang 2022

algebraic manipulations: `FORM` Vermaseren 2000, …

reduction to masters: `Kira` $\otimes$ `FireFly` Usovitsch, Uwer, Maierhöfer 2017 $\otimes$ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981
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sector decomposition:

Binoth, Heinrich 2002

$$\int d^Dk \int d^Dp \int_0^t ds \frac{e^{-tp^2-s(k-p)^2}}{k^2p^2(k-p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \dots$$

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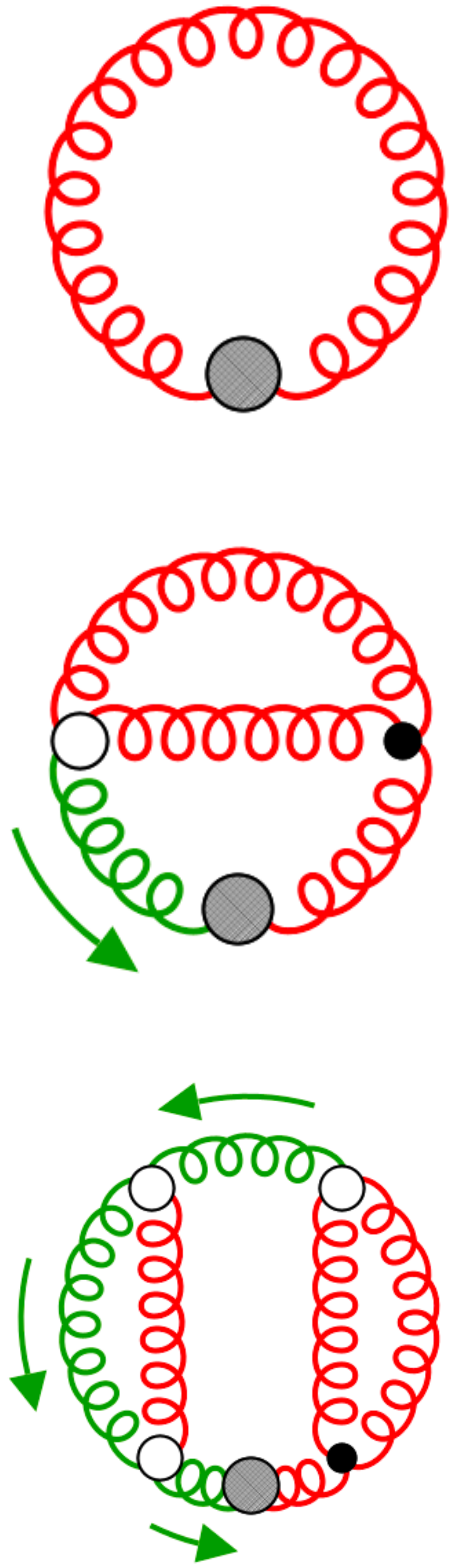
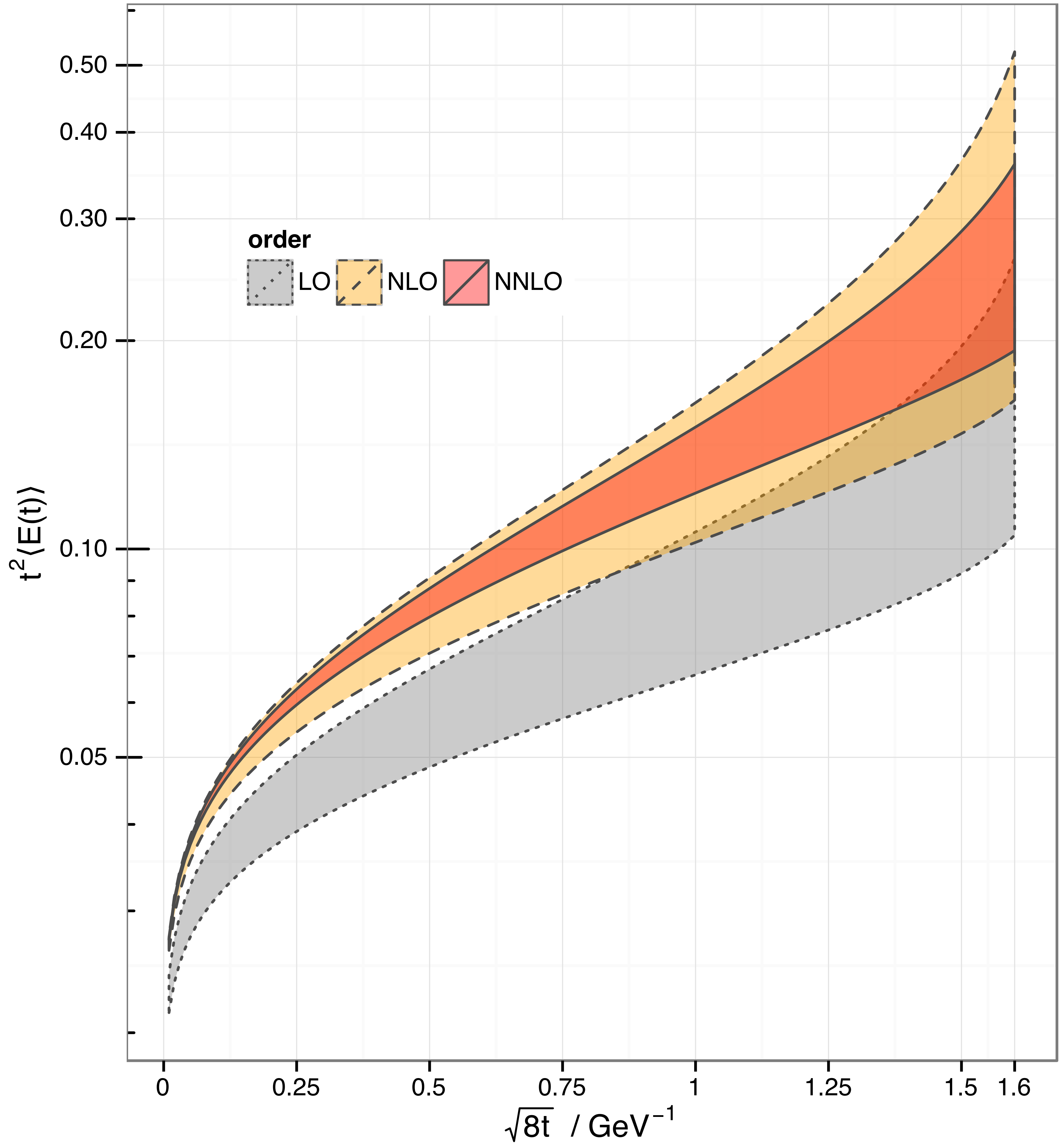
→ `ftint` RH, Nellopoulos, Olsson, Wesle '25

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

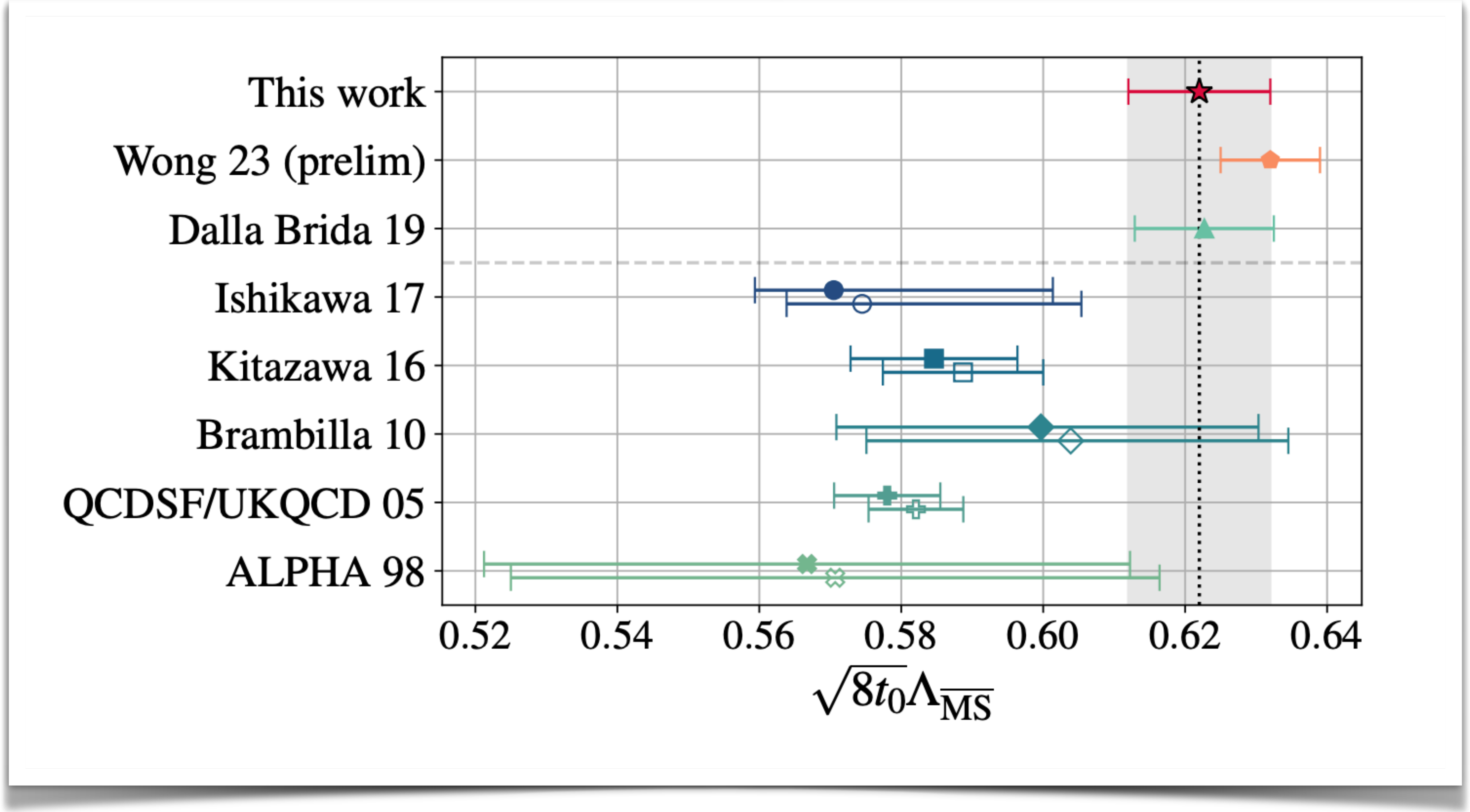
RH, Neumann 2016





Renormalization scheme for α_s compatible with lattice and PT!

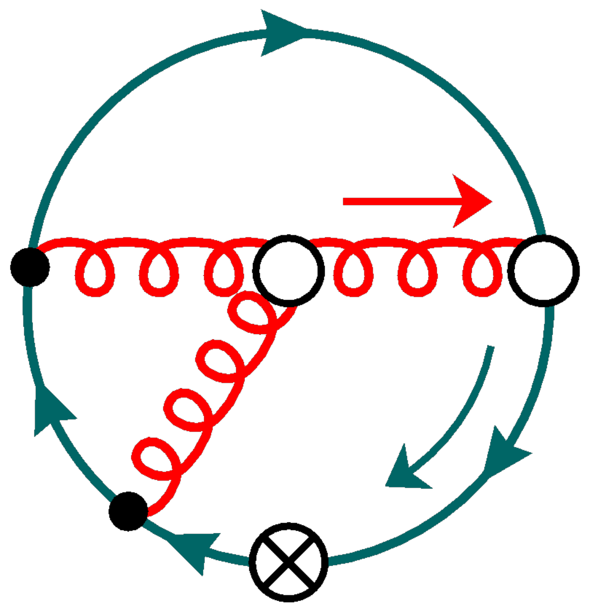
Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel (2023)

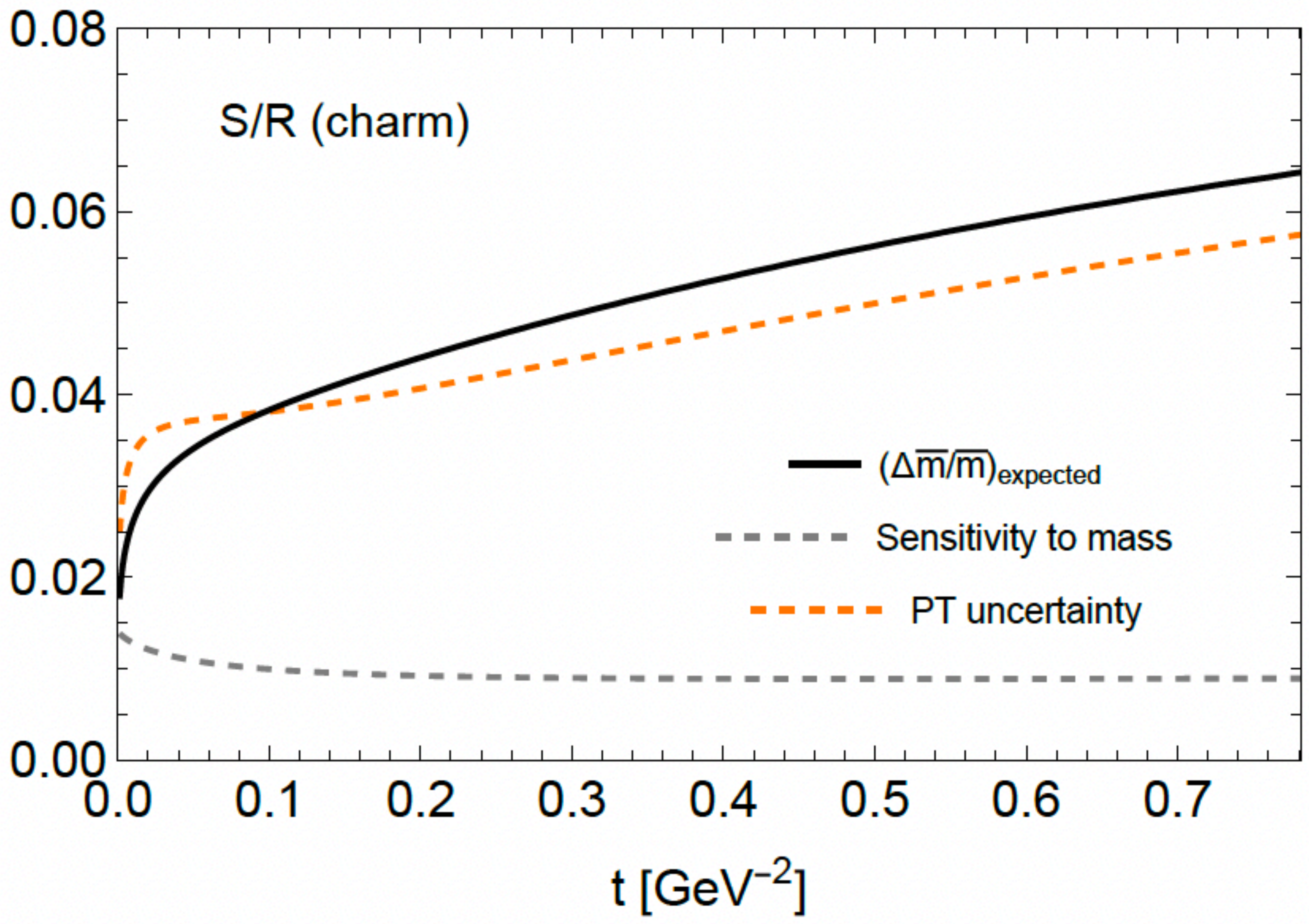
see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

Quark masses



$$\equiv \quad \langle\bar{\chi}(t)\chi(t)\rangle = -\frac{3}{8\pi^2 t}m(\mu) \quad + \text{higher orders}$$

Artz, RH, Lange, Neumann, Prausa ‘19
Georg, RH, Mason ‘26



Takaura, RH, Lange ‘25

Renormalon subtraction

Beneke, Takaura ‘25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 \,|\, T j(x) j(0) \,|\, 0 \rangle$$

Adler function: $$D(Q^2) = -Q^2 \frac{d}{dQ^2}\Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Baikov, Chetyrkin, Kühn ’08

RH ‘98

perturbative

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Baikov, Chetyrkin, Kühn ’08

RH ‘98

perturbative

$$\langle G_{\mu\nu}(t)G^{\mu\nu}(t)\rangle = t^2\tilde{C}_1(t) + \tilde{C}_F(t)\langle F_{\mu\nu}F^{\mu\nu}\rangle + \cdots$$

Renormalon subtraction

Beneke, Takaura ‘25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 \,|\, T\, j(x) j(0) \,|\, 0 \rangle$$

Adler function: $$D(Q^2) = -\,Q^2 \frac{d}{dQ^2}\Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu}F^{\mu\nu} \rangle + \cdots$$

Baikov, Chetyrkin, Kühn ’08

RH ‘98

perturbative

RH, Neumann ‘16

RH, Lange, Neumann ‘20

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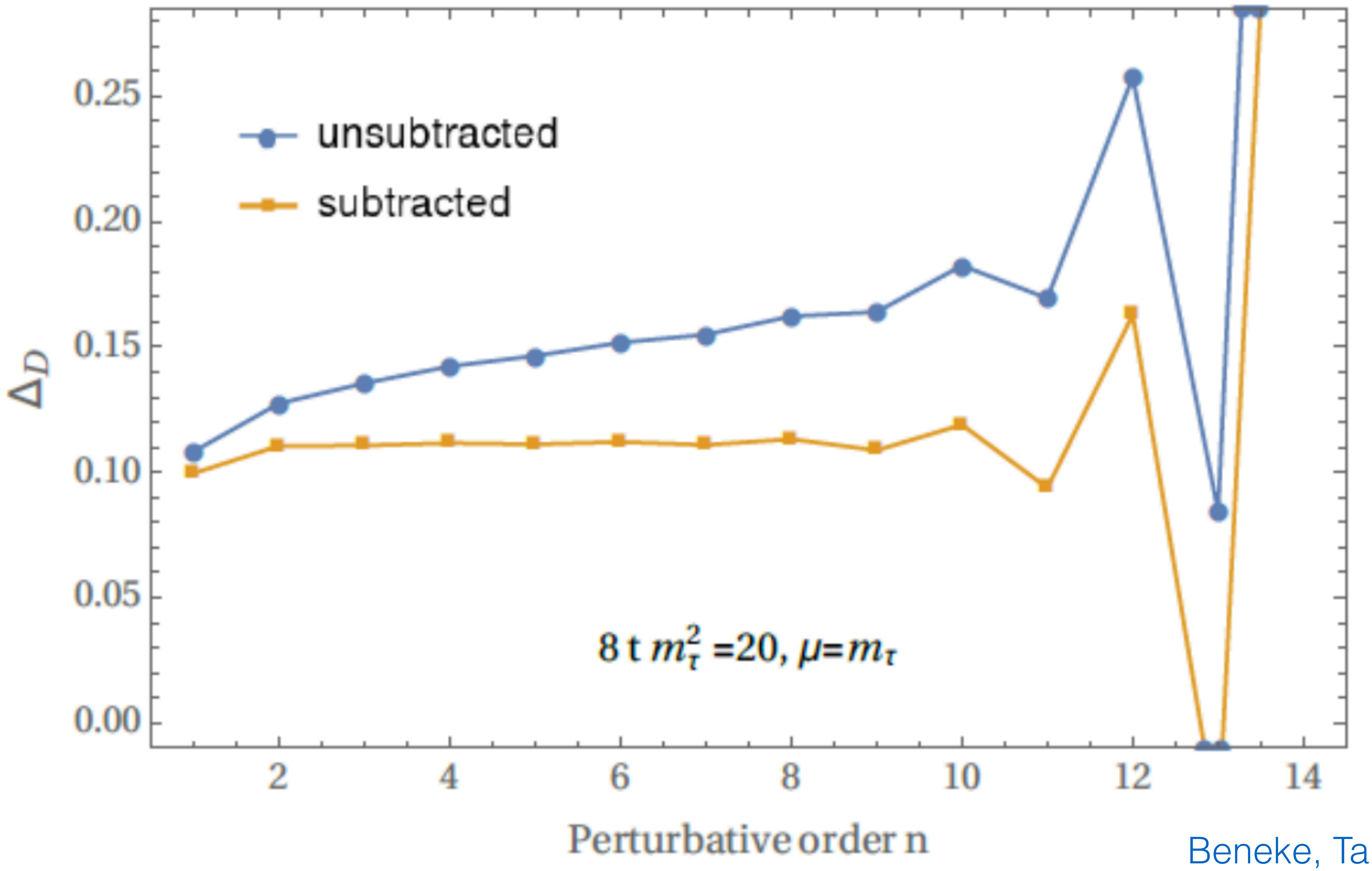
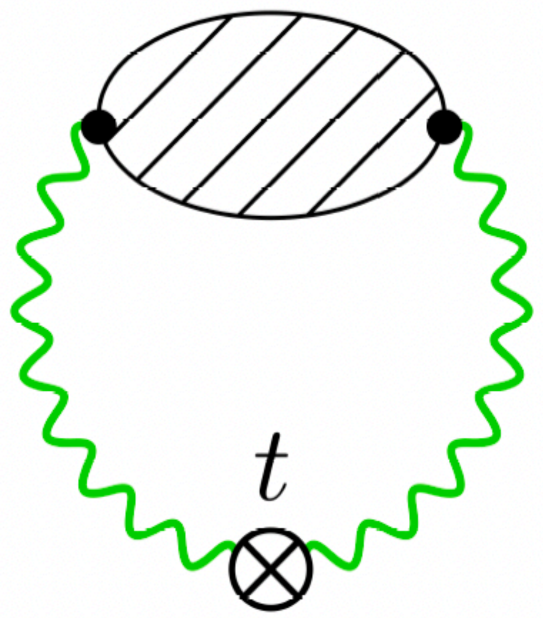
renormalon cancels



Beneke, Takaura ‘25

Application to QED

$$E(t) =$$



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Application to QED

in D dimensions:

Georg, RH, Mason '26

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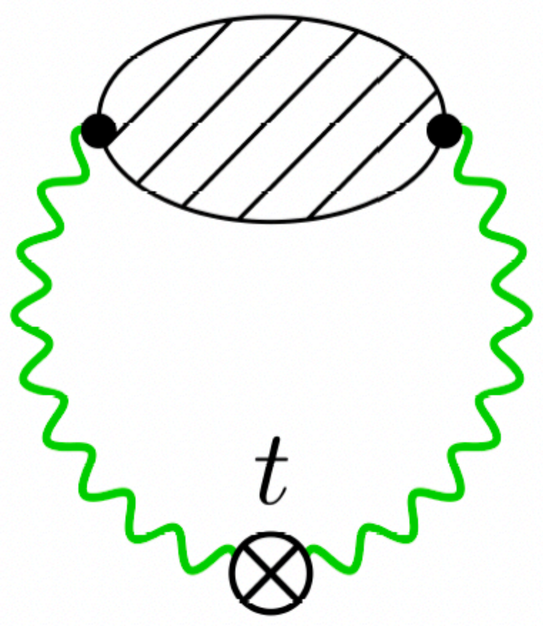

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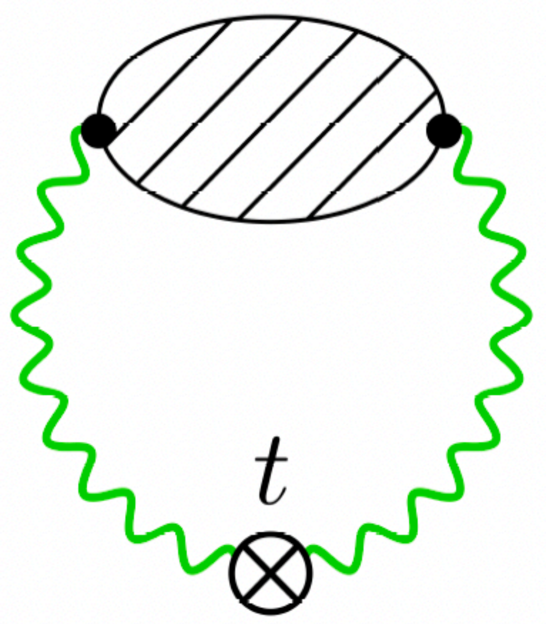

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regular photon
polarization function

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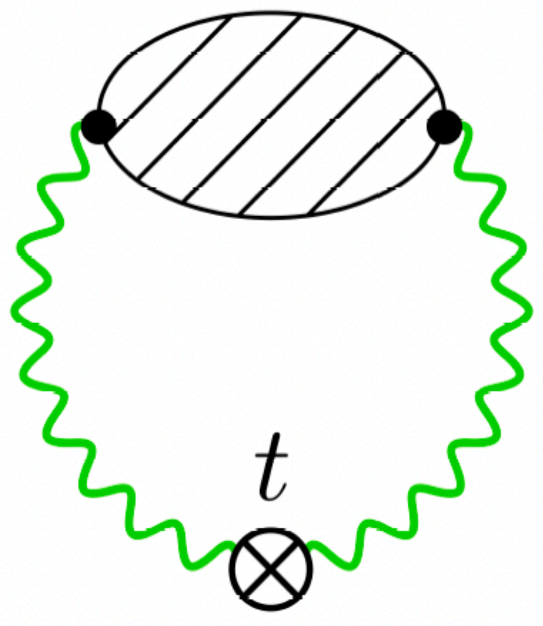

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Application to QED

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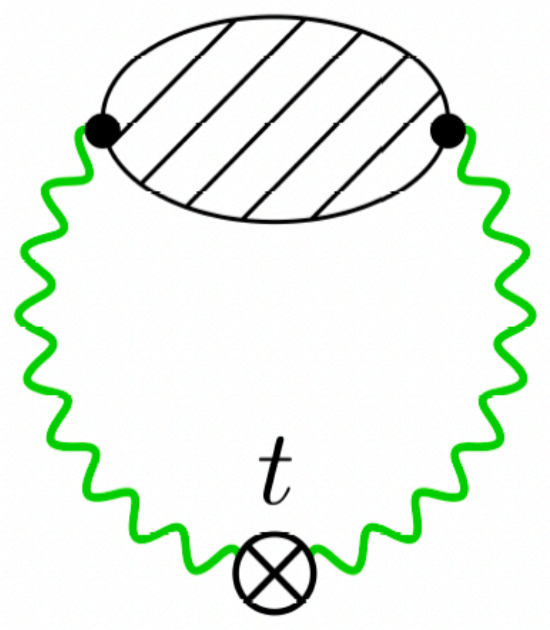
Georg, RH, Mason ‘26

$E(t) =$ 

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D=3: α mass dimension = 1

$\hat{\alpha} \sim \alpha/p$

Appelquist, Pisarski ‘81
Pisarski ‘84

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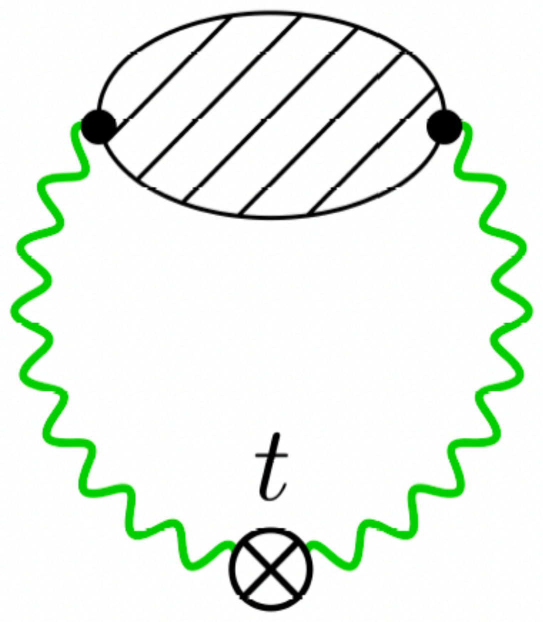

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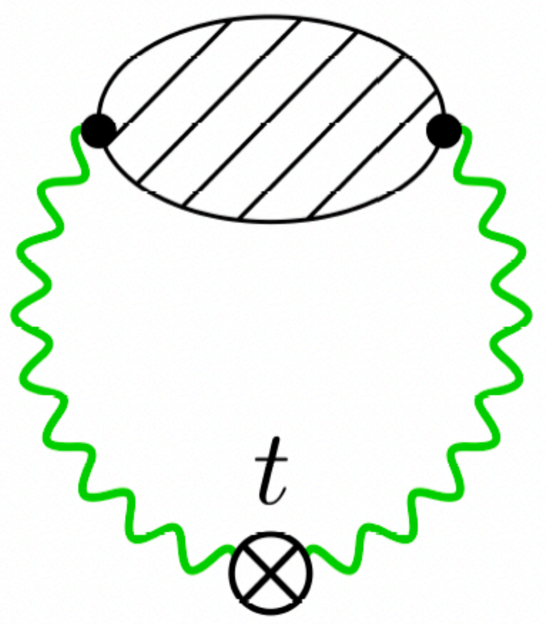

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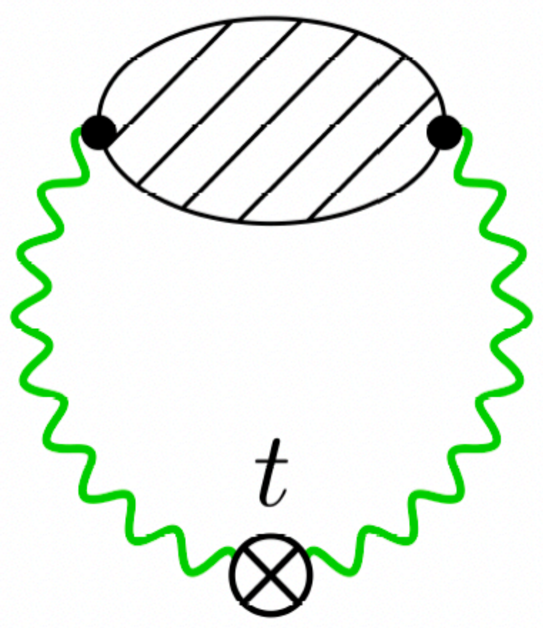

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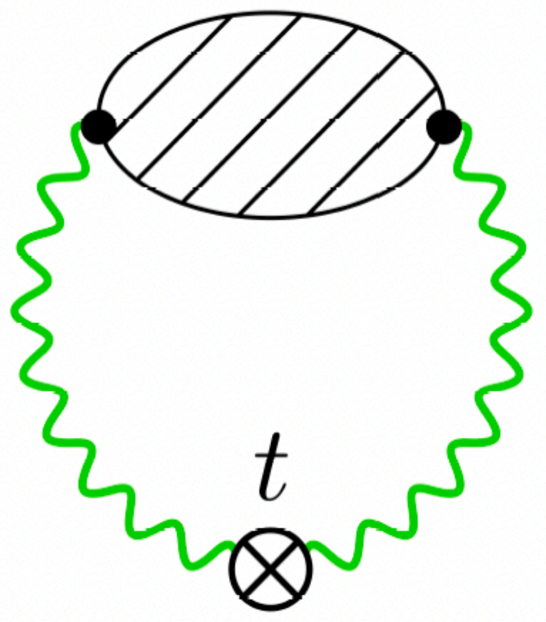

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dynamical fermion mass generation

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Application to QED$_3$

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α^{GF} IR divergent beyond NNLO

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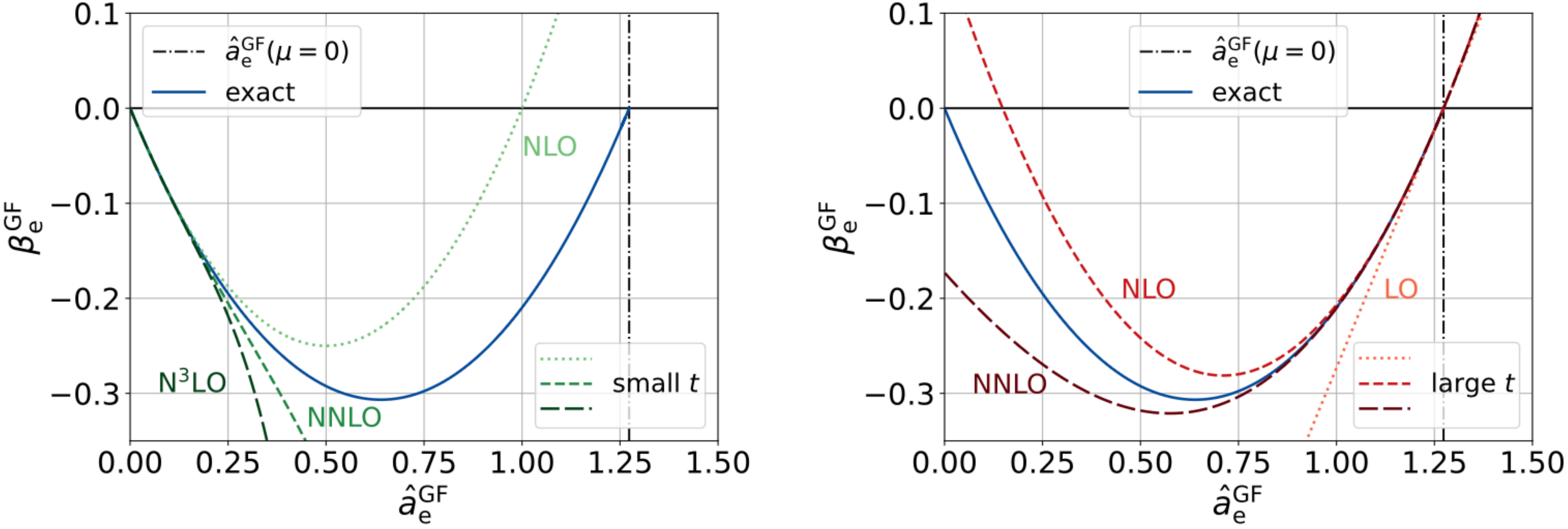
$$\equiv \sqrt{\frac{\pi}{8}}\,\hat{a}_e(p)$$

Application to QED₃



Georg, RH, Mason 2025

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathscr{O}_n \rangle$$

GF	$\overline{\text{MS}}$

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$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\text{R}}(\mu) = C^{\text{R}}(\mu)\,\gamma$

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RH, Lange, Neumann ‘20

Borgulat, Felten, RH, Kohnen ‘25

Application to gravity

flowed gauge field:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}B_\mu(t,x) &= \mathscr{D}_\nu G_{\nu\mu}(t,x)\\ B_\mu(t=0,x) &= A_\mu(x)\end{aligned}}$$

$$\xrightarrow{\text{gravity}}$$

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

… in preparation …

RH, Kluth, Kohnen, Werthenbach

Conclusions

Gradient flow: interesting field theoretical concept

Established tool in **lattice calculations**

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Lattice/PT interplay picks up momentum:

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Lattice/PT interplay picks up momentum:
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Many things not even discussed:
Static QCD potential, EDMs, …