Gradient flow and the short flow-time expansion

Robert Harlander

RWTH Aachen University

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Lattice Meets Continuum
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The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t,x) = \mathscr{D}^2 \chi(t,x)$$

$$\chi(t=0,x) = \psi(x)$$

Narayanan, Neuberger 2006
Lüscher 2009
Lüscher 2010
Lüscher, Weisz 2011
Lüscher 2013

Schematically…

$$\frac{\partial}{\partial t} B_\mu(t) = \mathscr{D}_\nu G_{\nu\mu}(t)$$

$$\mathscr{D}_\mu = \partial_\mu - iT^a g_0 B^a_\mu(t)$$

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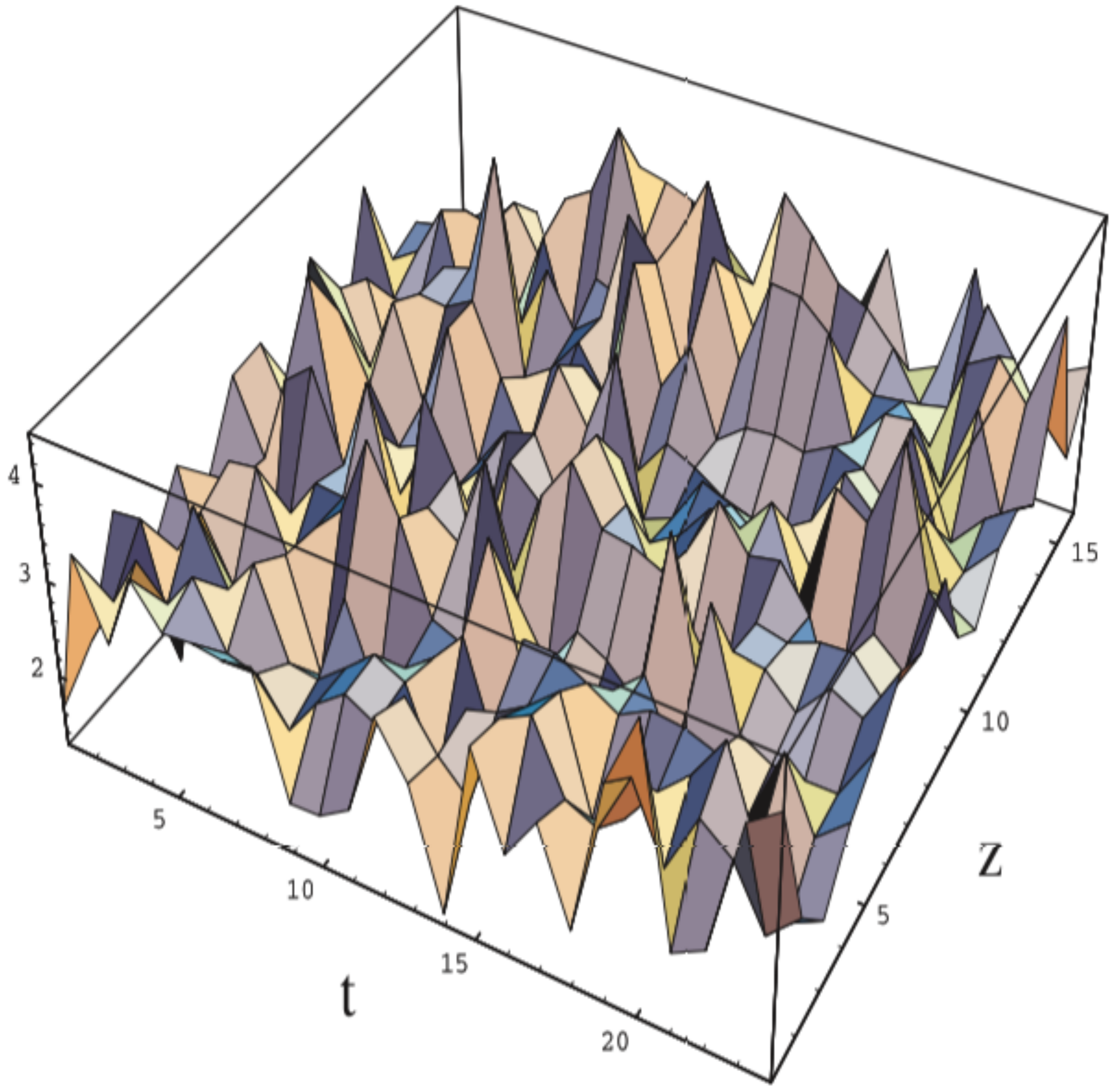
cf. heat equation: $$\partial_t u(t, \mathbf{x}) = \Delta u(t, \mathbf{x})$$

Lattice QCD

quantum fluctuations:



Engel 2009

Lattice QCD

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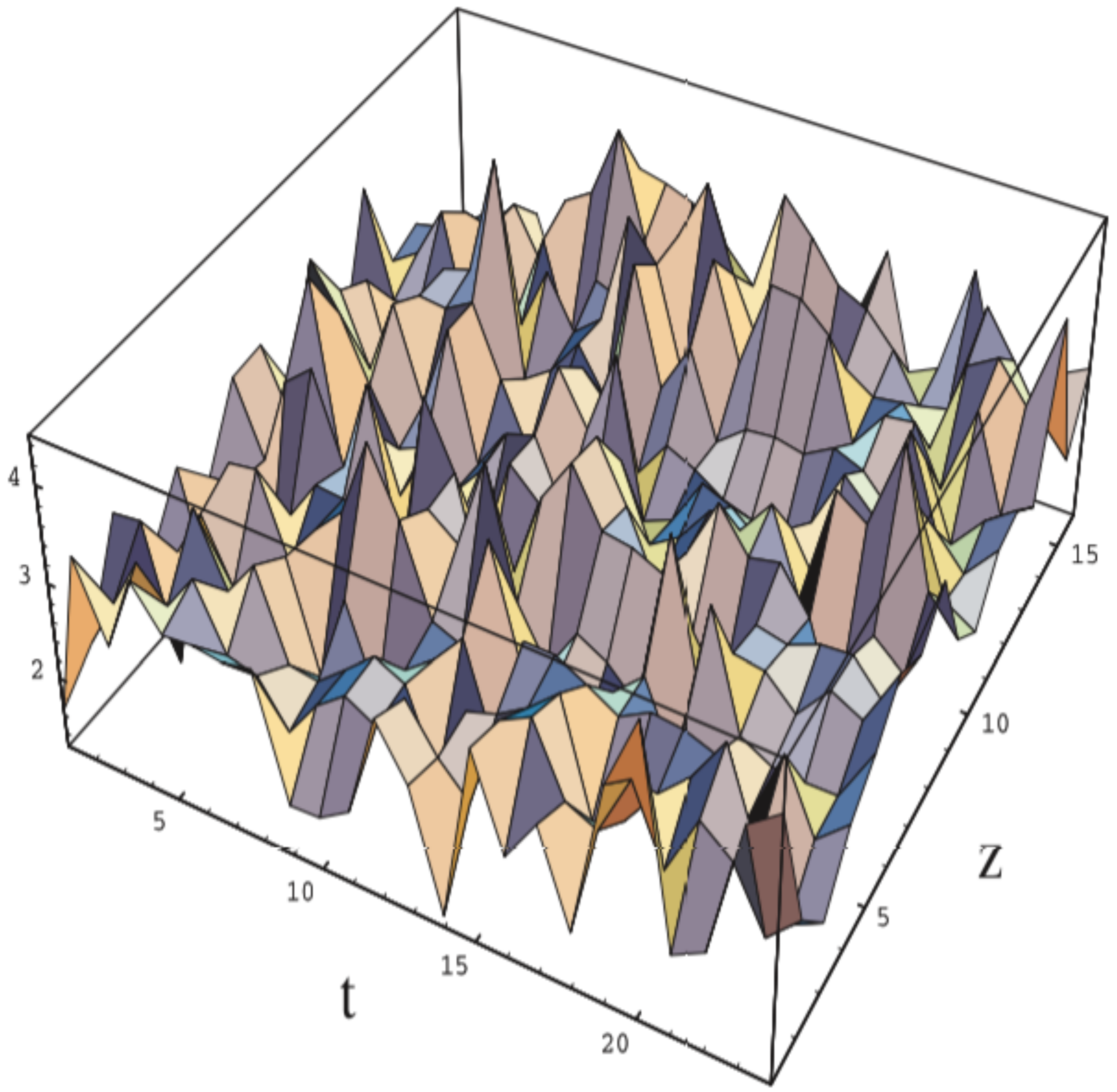
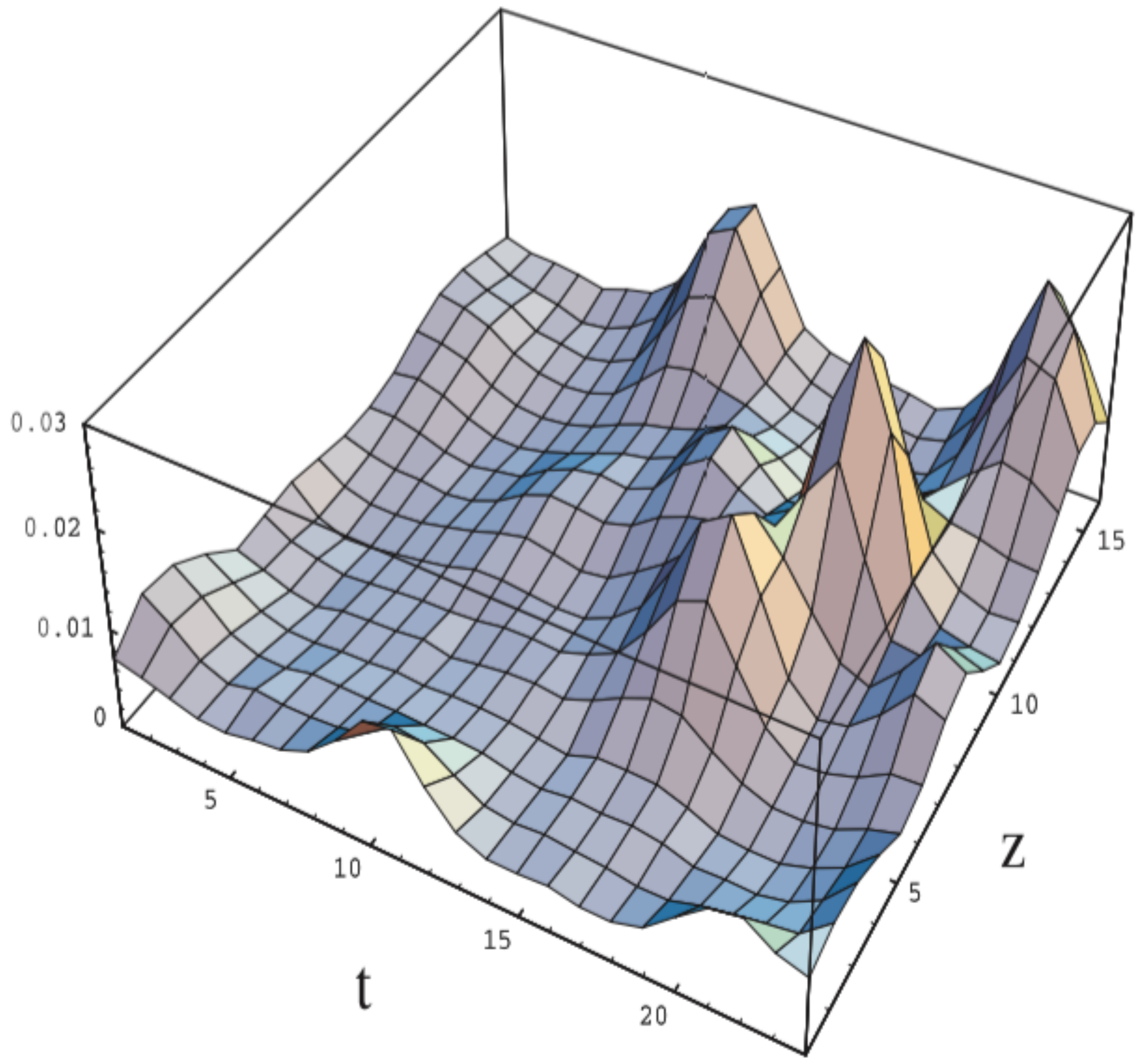


"smearing":



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Perturbative solution

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Perturbative solution

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perturbative ansatz: $$B = g_0 B_1 + g_0^2 B_2 + \ldots$$

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$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

$$K(t,s,p,q) \sim \exp\big[-tp^2 - 2sq(q-p)\big]$$

etc.

Exponential damping in momentum integrals!

5-dimensional field theory

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$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$
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Perturbative approach

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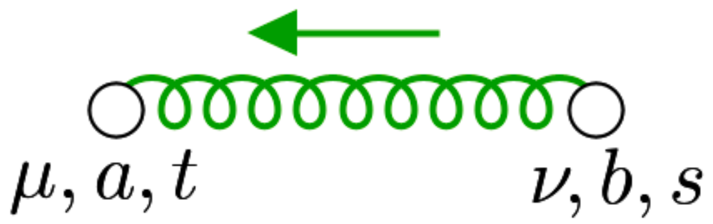
$$\sim \langle 0 \,|\, T\, B_\mu^a(t,x)\, B_\nu^b(s,0) \,|\, 0 \rangle$$



$$\delta_{ab}\delta_{\mu\nu}\, \theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

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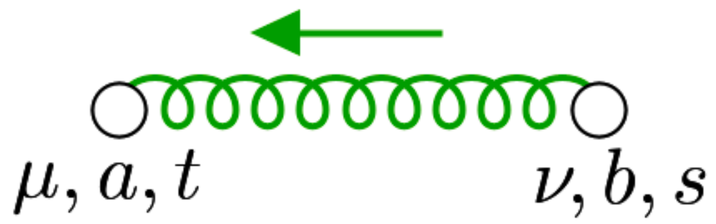

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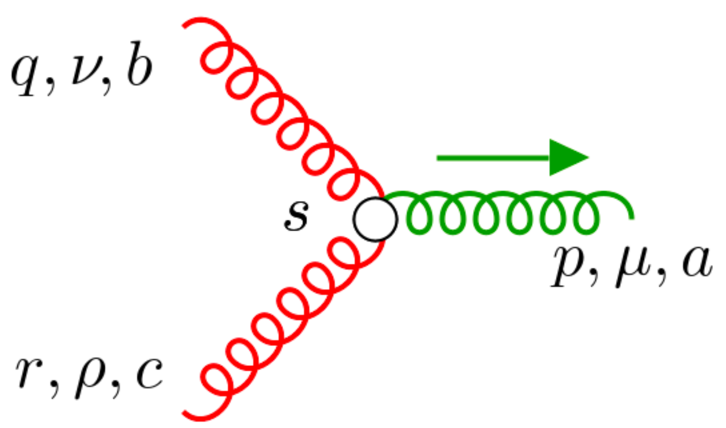
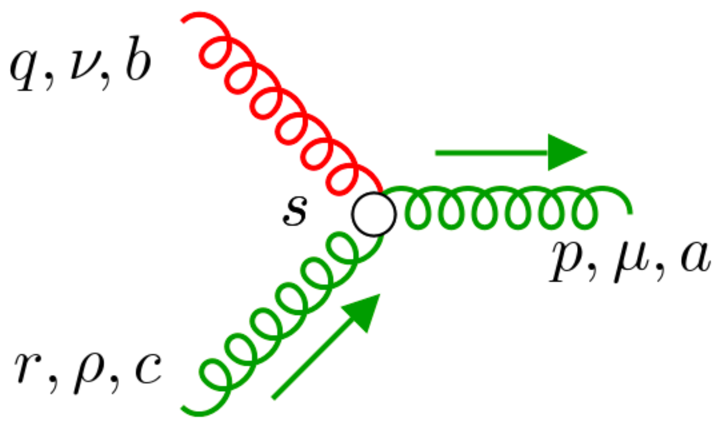
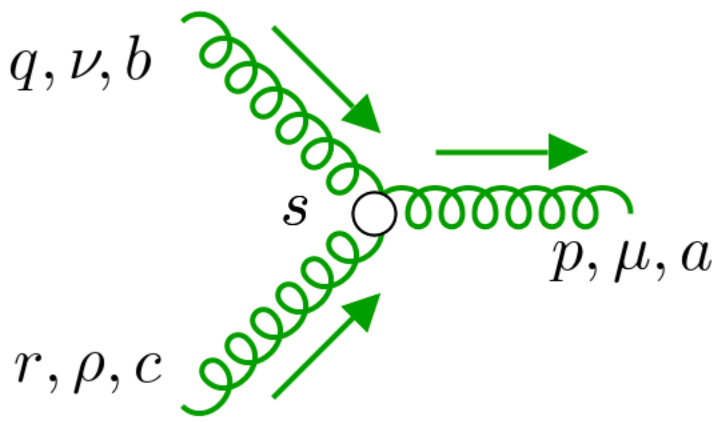


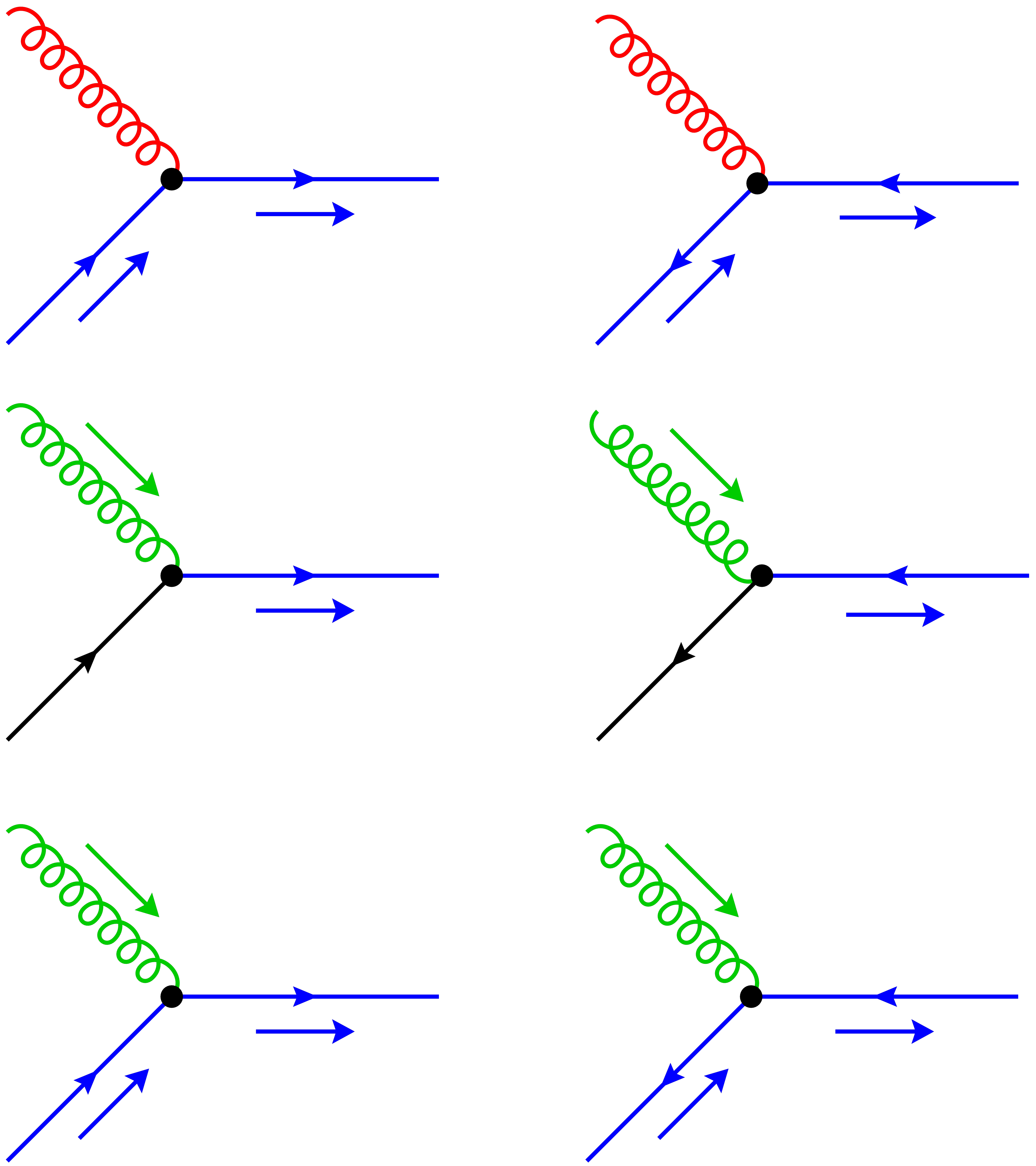




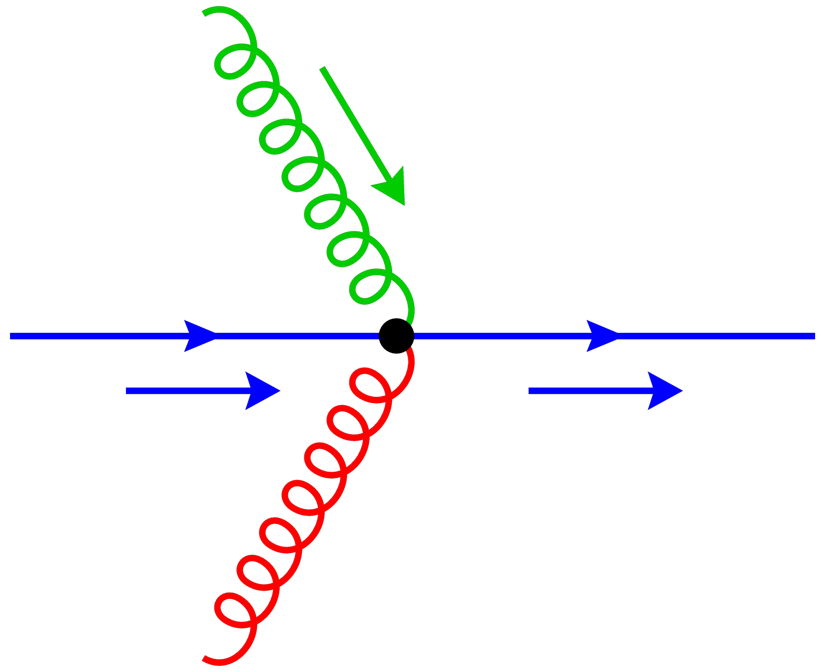
$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu$$
$$+ (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

+ 4-gluon vertex

Perturbative approach



$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \mathscr{D}^2\right) \chi + \mathrm{h.c.}$$



etc.

Renormalization

$$\mathcal{L} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

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⇒ renormalization of

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flowed field renormalization:

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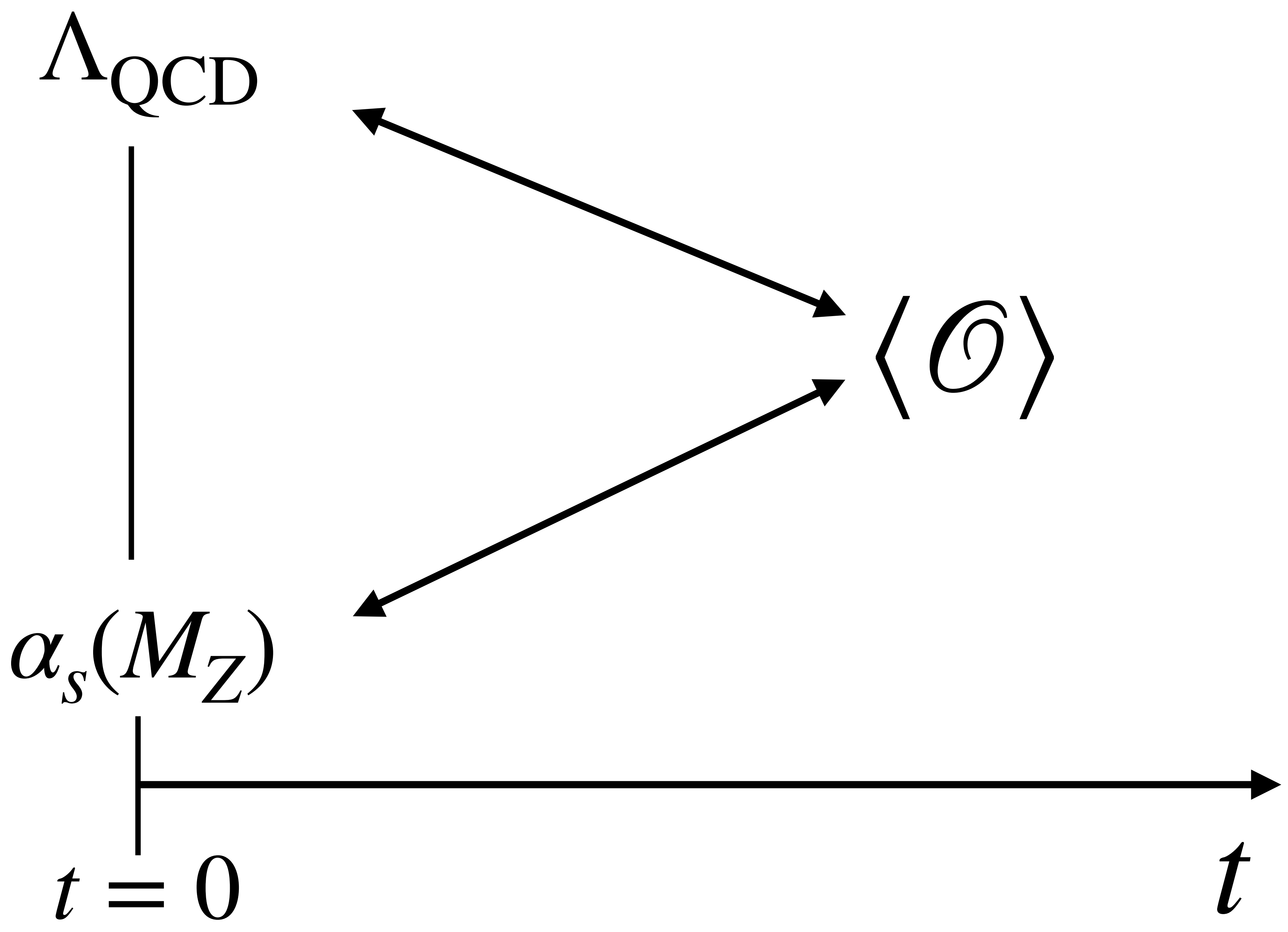
$$\mathring{Z}_\chi \langle \bar{\chi}(t) \not{D} \chi(t) \rangle \Big|_{m=0} \equiv \text{LO}$$

NLO: Suzuki, Makino (2014)
NNLO: Artz, RH, Lange, Neumann, Prausa (2019)


Λ_{QCD}
$\langle \mathcal{O} \rangle$
$\alpha_s(M_Z)$
$t = 0$
t

Let’s calculate

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

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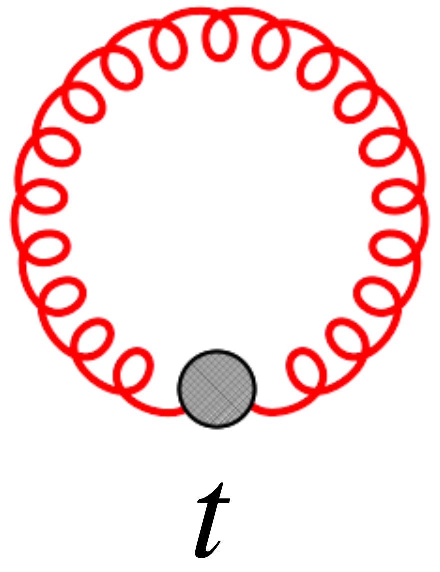

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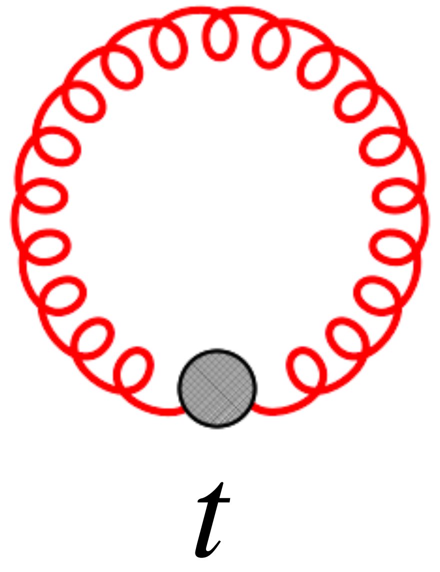

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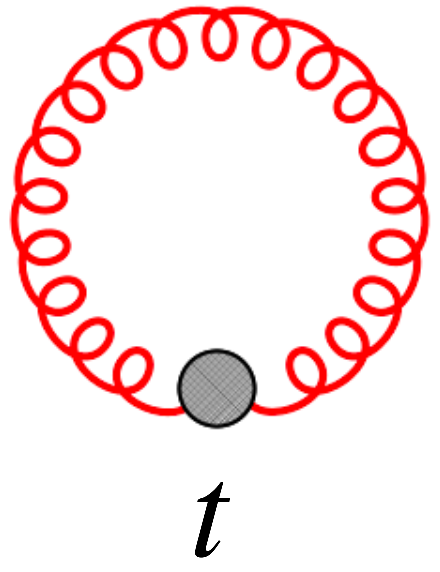

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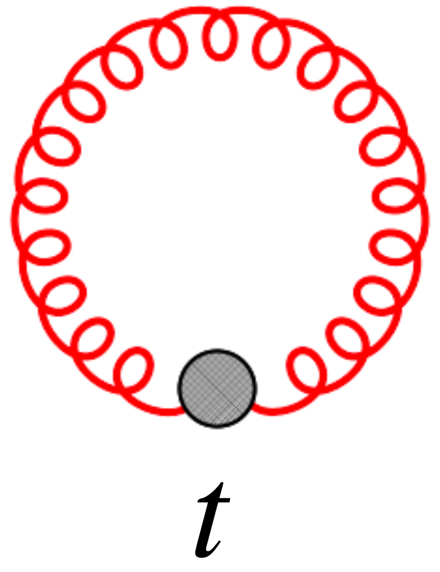
LO: 

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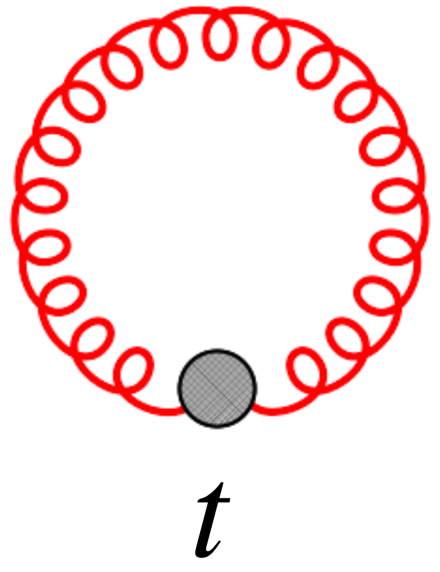

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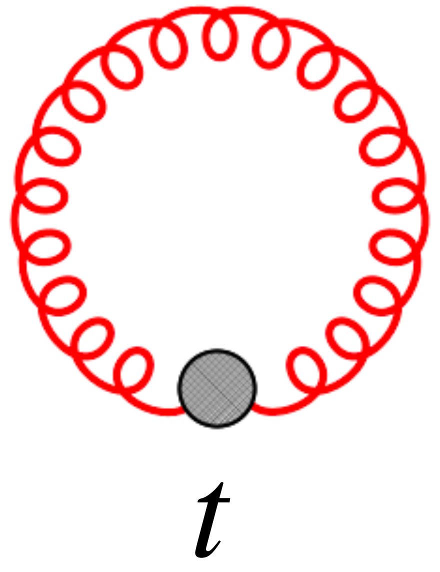
LO: 

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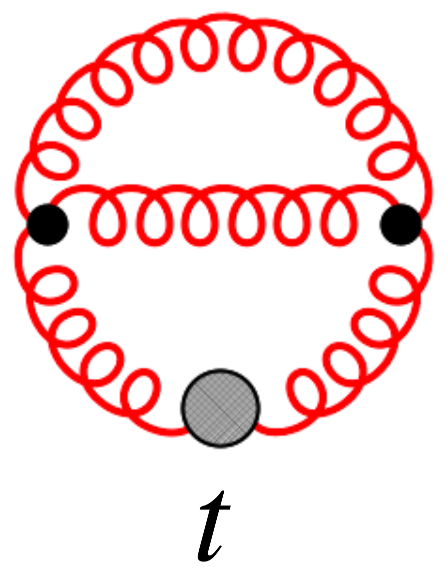
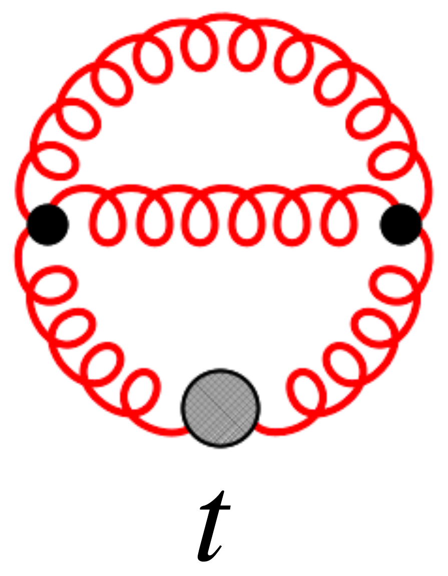
$$\alpha_s = \alpha_s(\mu)$$

Higher orders



$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

Higher orders



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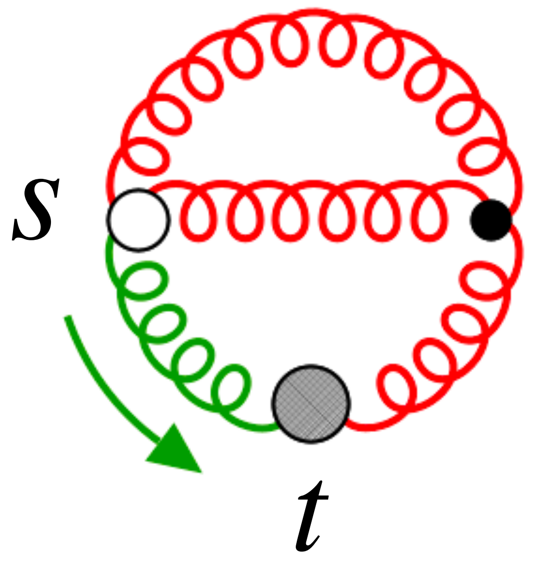

$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

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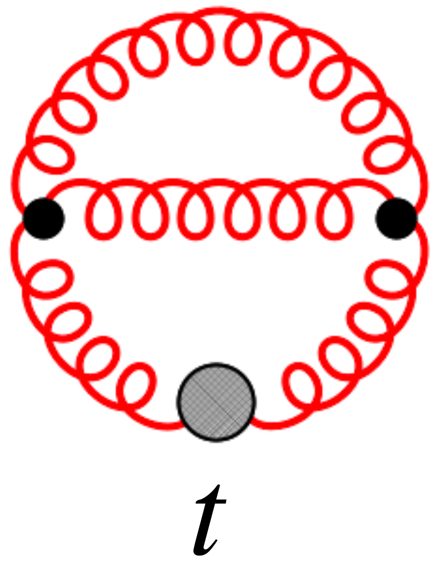

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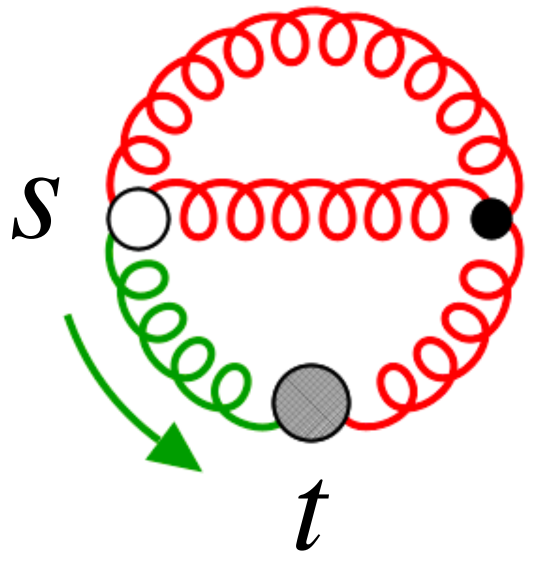

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- generalized loop integrals

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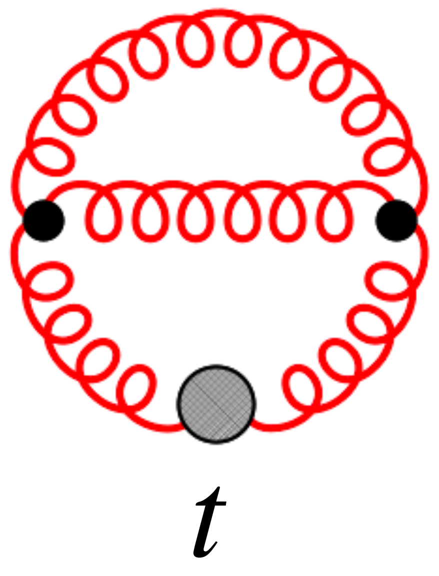

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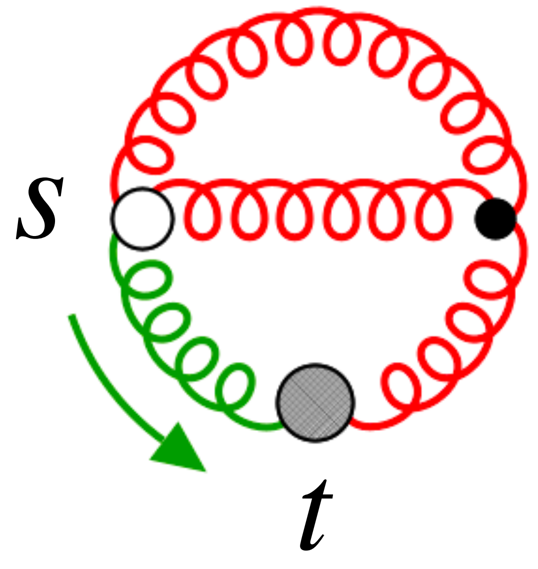

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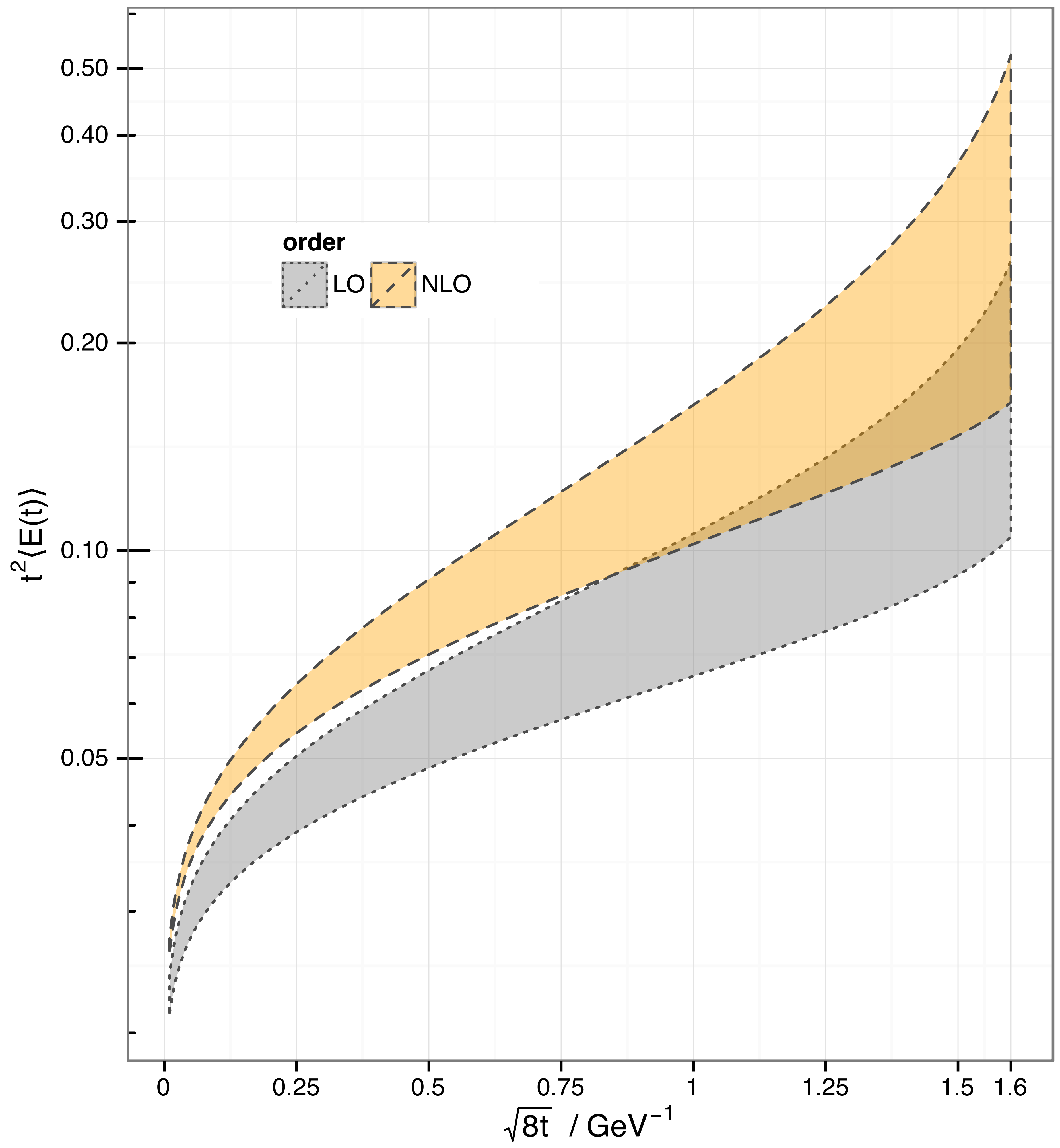
- generalized loop integrals
- integration over flow-time parameters

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

Three-loop calculation

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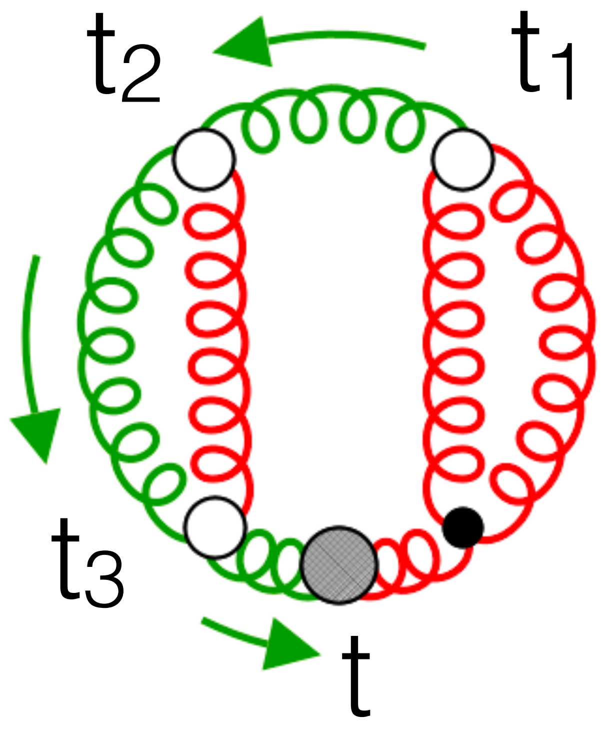
The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



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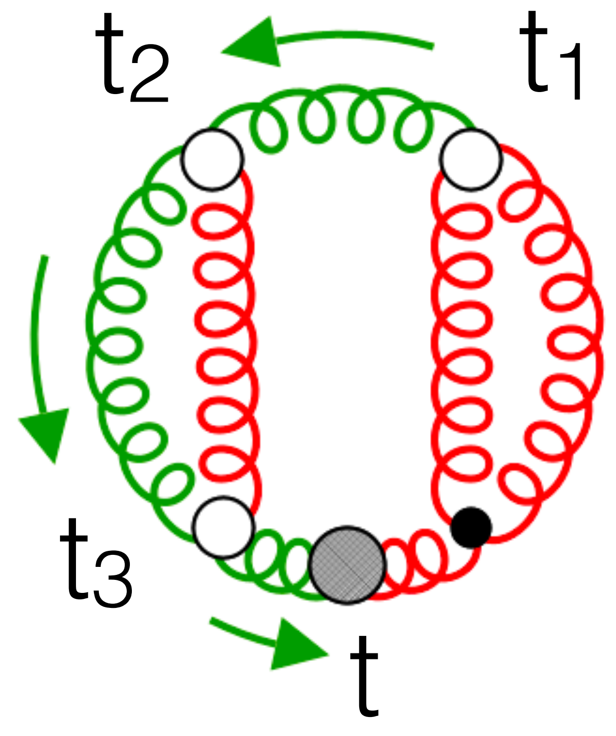


The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals

Artz, RH, Lange, Neumann, Prausa ’19

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: qgraf Nogueira 1993

diagram analyzation: q2e/exp RH, Seidensticker, Steinhauser 1997

→ tapir/exp Gerlach, Herren, Lang 2022

algebraic manipulations: FORM Vermaseren 2000, …

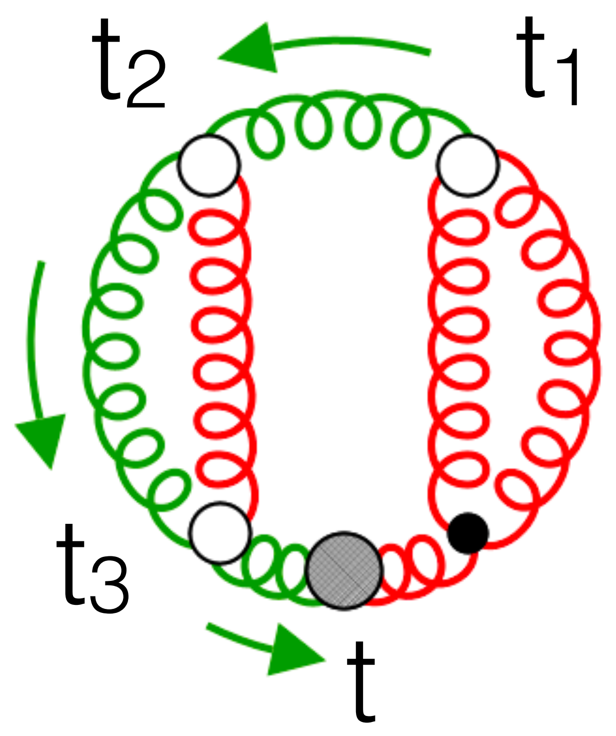
reduction to masters: Kira ⊗ FireFly Usovitsch, Uwer, Maierhöfer 2017 ⊗ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981
Laporta 2000

Three-loop calculation

$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\}) =$$

$$= \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3 \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2)^{b_1}\,\cdots\,(p_6^2)^{b_6}}$$



$u_i = t_i/t$

IbP identities:

$$\frac{\partial}{\partial p_i}\cdot p_j\, I(c,a,b) = D\,\delta_{ij}\, I(c,a,b) + \sum I(c',a,b')$$

$$\frac{\partial}{\partial u_i} I(c,a,b) = I(c',a(u=1),b') - I(c',a(u=0),b')$$

Artz, RH, Lange, Neumann, Prausa 2019]

Huge systems of linear equations, solved by “master integrals”.

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diagram generation:	`qgraf`	Nogueira 1993
diagram analyzation:	`q2e/exp`	RH, Seidensticker, Steinhauser 1997
	→ `tapir/exp`	Gerlach, Herren, Lang 2022
algebraic manipulations:	`FORM`	Vermaseren 2000, …
reduction to masters: Chetyrkin, Tkachov 1981 Laporta 2000	`Kira` ⊗ `FireFly`	Usovitsch, Uwer, Maierhöfer 2017 ⊗ Klappert, Klein, Lange 2019

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The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: qgraf — Nogueira 1993

diagram analyzation: q2e/exp — RH, Seidensticker, Steinhauser 1997

→ tapir/exp — Gerlach, Herren, Lang 2022

algebraic manipulations: FORM — Vermaseren 2000, …

reduction to masters: Kira ⊗ FireFly — Usovitsch, Uwer, Maierhöfer 2017 ⊗ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981
Laporta 2000

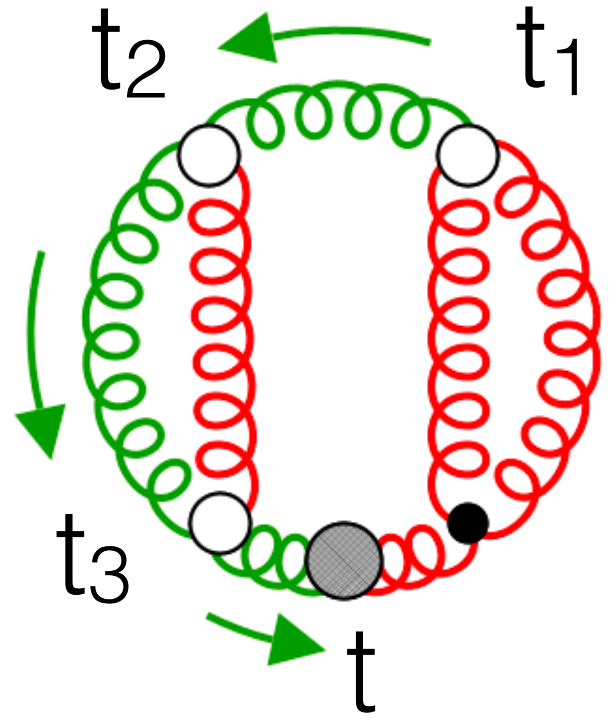
sector decomposition:

Binoth, Heinrich 2002

$$\int d^Dk \int d^Dp \int_0^t ds \frac{e^{-tp^2-s(k-p)^2}}{k^2p^2(k-p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \ldots$$

Numerical evaluation

RH, Neumann (2016)

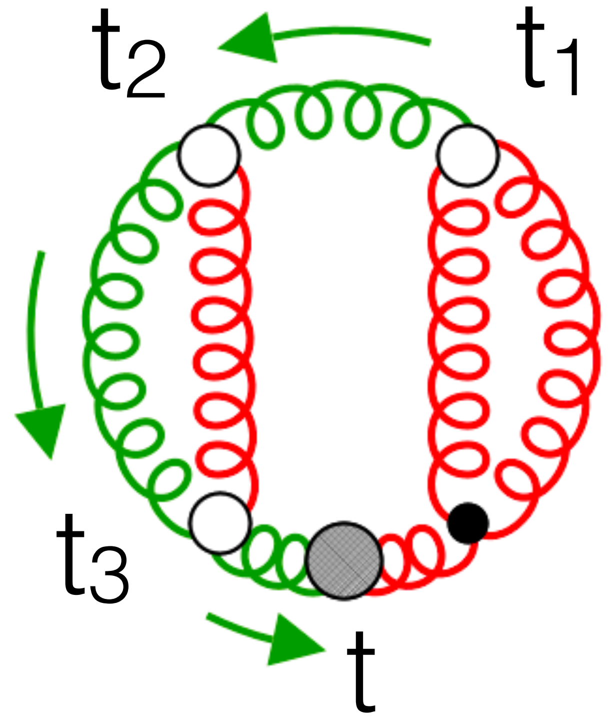


$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\}) =$$

$$= \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2)^{b_1}\,\cdots\,(p_6^2)^{b_6}}$$

Numerical evaluation

RH, Neumann (2016)



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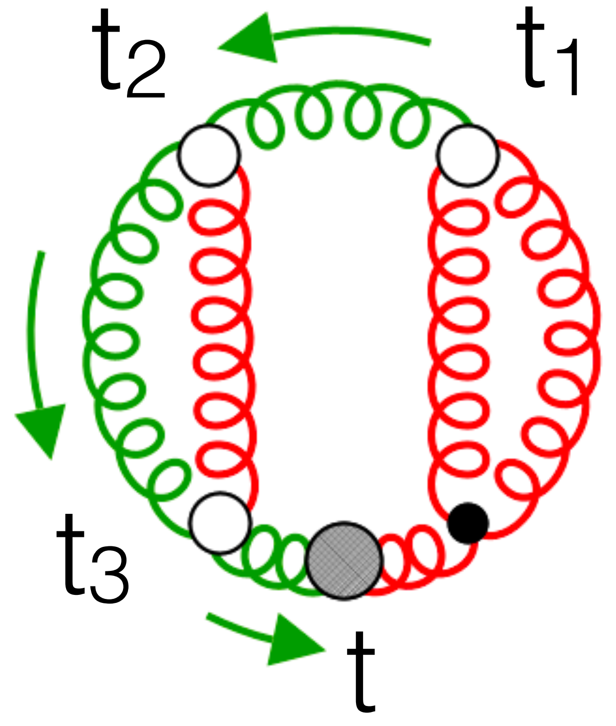
Schwinger parameters: $$\frac{1}{(p^2)^b} \sim \int_0^\infty dx\, x^{b-1}\, e^{-xp^2}$$

Numerical evaluation

RH, Neumann (2016)



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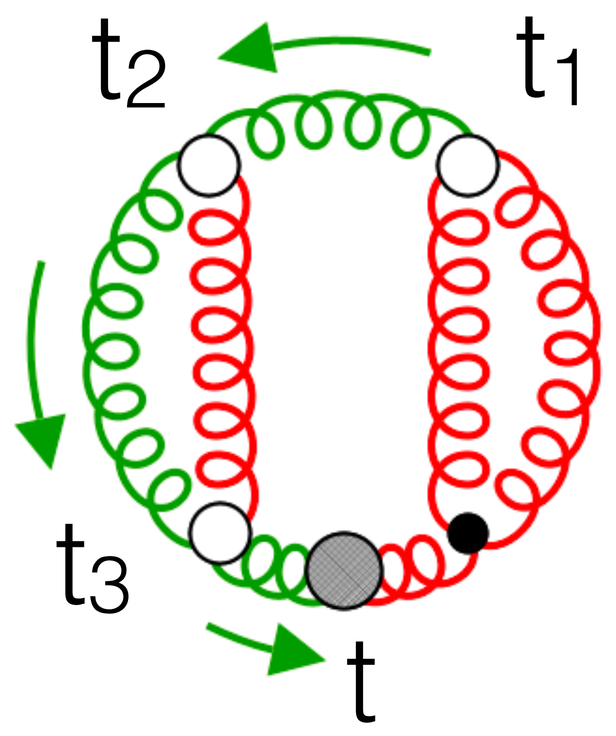
$$\frac{1}{(p^2)^b} \sim \int_0^\infty dx\, x^{b-1}\, e^{-xp^2}$$

$$\sim \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\left(\prod_{j=1}^{6}\int_0^\infty dx_j\, x_i^{b_j-1}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3 \exp\left[-t\,\mathbf{p}^T A(x,u)\,\mathbf{p}\right]$$

Numerical evaluation

RH, Neumann (2016)



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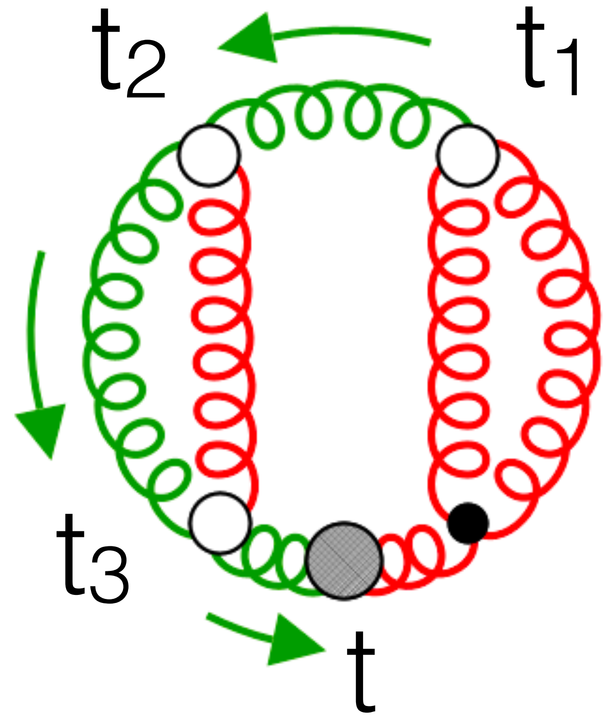
Numerical evaluation

RH, Neumann (2016)



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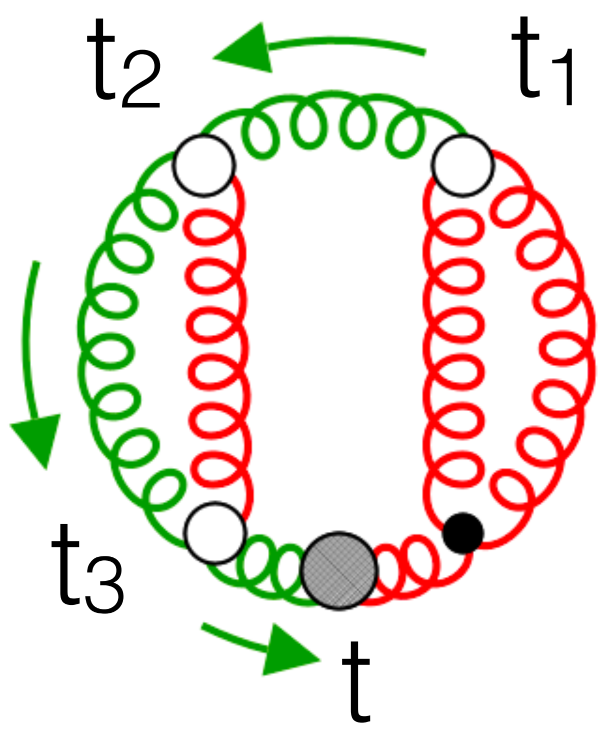
Numerical evaluation

RH, Neumann (2016)



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Schwinger parameters: $\frac{1}{(p^2)^b} \sim \int_0^\infty dx\, x^{b-1}\, e^{-xp^2}$ $\left(\overset{\text{map}}{\rightarrow} \int_0^1 dx \cdots\right)$

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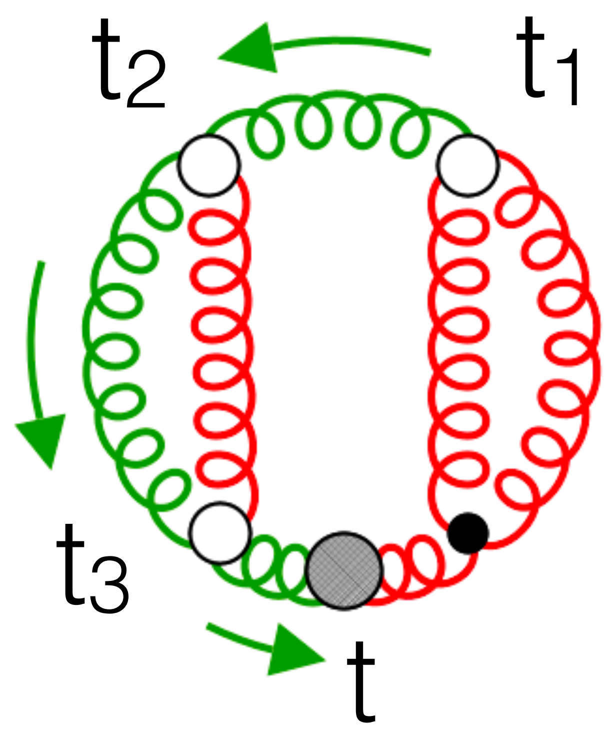
Numerical evaluation

RH, Neumann (2016)



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→ sector decomposition

Binoth, Heinrich (2000)

Implementation

$$I(\{c_1, \cdots, c_f\}, \{a_1(u), \cdots, a_6(u)\}, \{b_1, \cdots, b_6\})$$

$$c_1 = c_2 = 0$$

$$a_1 = u_1 u_2\,, \quad a_2 = u_2\,, \quad a_3 = u_2 - u_1 u_2$$

$$a_4 = 1\,, \quad a_5 = 1 + u_1 u_2\,, \quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

$$b_2 = b_3 = b_5 = b_6 = 0$$

Implementation

$$I(\{c_1, \cdots, c_f\}, \{a_1(u), \cdots, a_6(u)\}, \{b_1, \cdots, b_6\})$$

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ftint RH, Nellopoulos, Olsson, Wesle '24

(based on pySecDec)

Heinrich, Magerya, Kerner, Jones, …

Implementation

$$I(\{c_1, \cdots, c_f\}, \{a_1(u), \cdots, a_6(u)\}, \{b_1, \cdots, b_6\})$$

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ftint RH, Nellopoulos, Olsson, Wesle '24

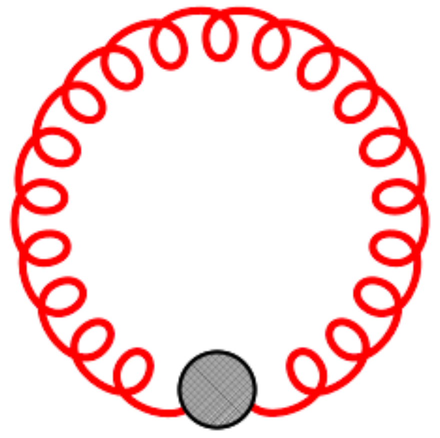
(based on pySecDec)

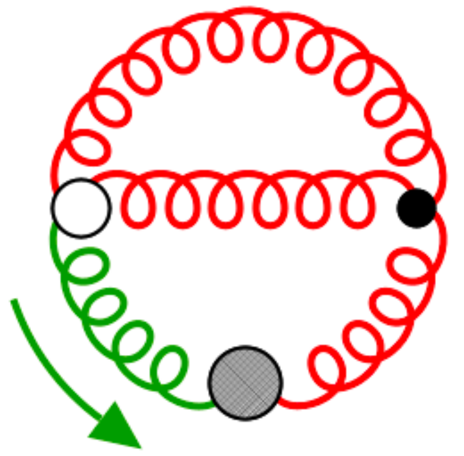
Heinrich, Magerya, Kerner, Jones, …

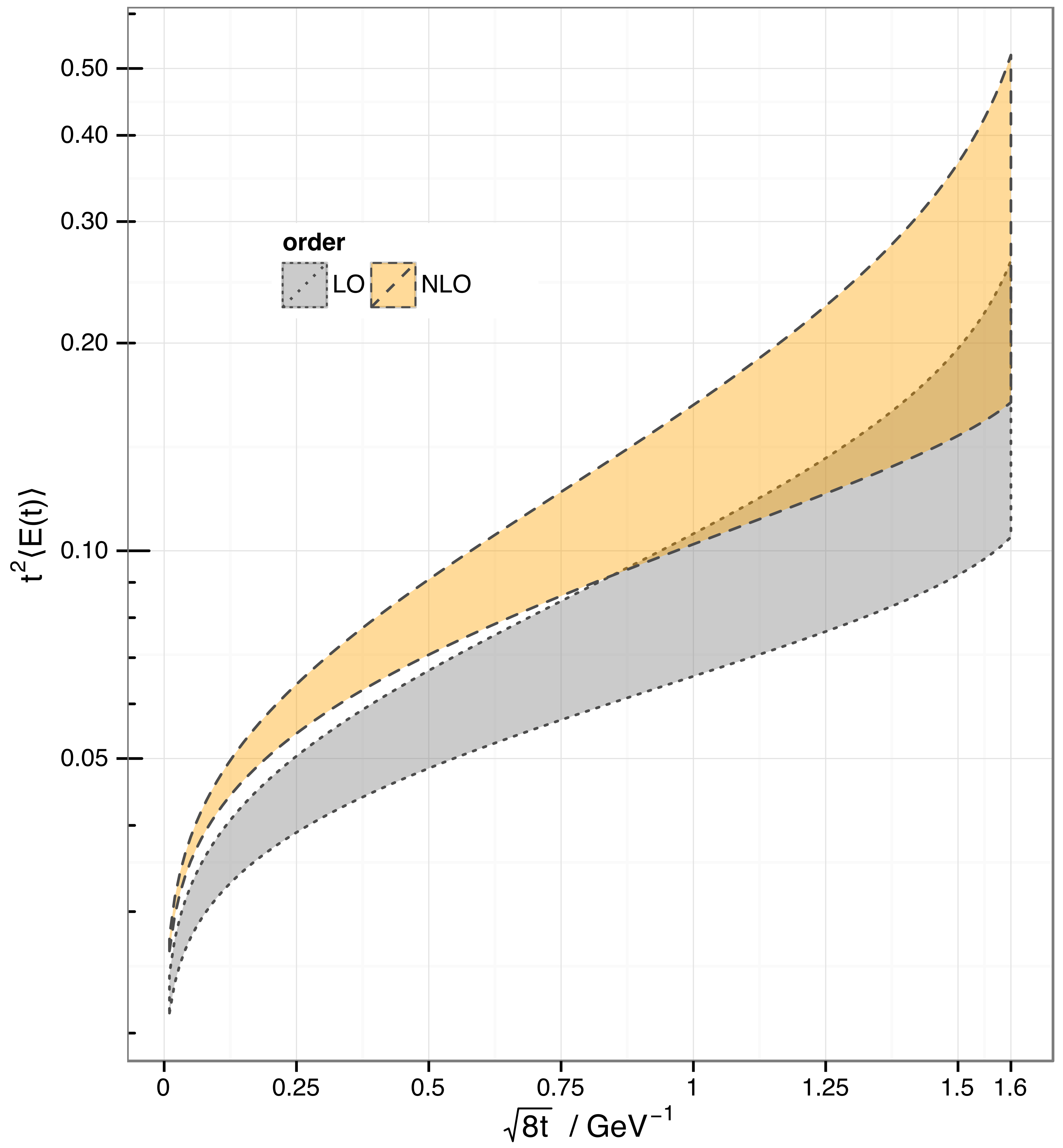
```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}},{1,0,0,1,0,0}] -> (
    +eps^-1*(+8.333333333333343*10^-02+0.0000000000000000*10^+00*I)
    +eps^-1*(+1.4433895444086145*10^-15+0.0000000000000000*10^+00*I)*plusminus
    +eps^0*(+3.0238270284562663*10^-01+0.0000000000000000*10^+00*I)
    +eps^0*(+1.6918362746499228*10^-08+0.0000000000000000*10^+00*I)*plusminus
    +eps^1*(+6.5531010458012129*10^-01+0.0000000000000000*10^+00*I)
    +eps^1*(+3.7857260802916662*10^-08+0.0000000000000000*10^+00*I)*plusminus
  ),
```

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010







$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

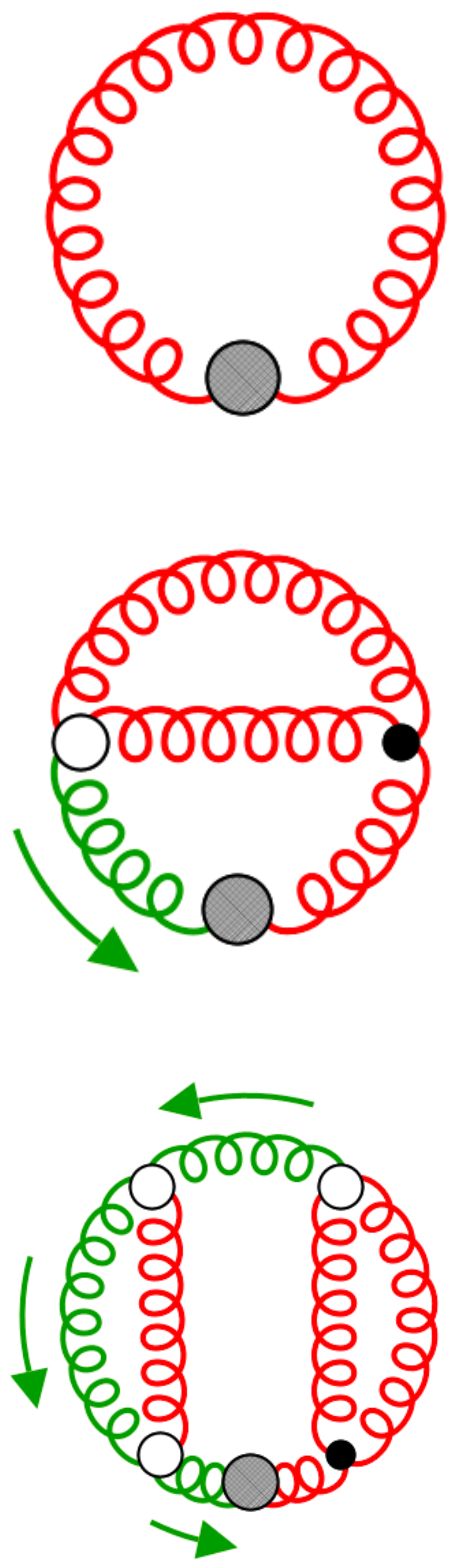
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

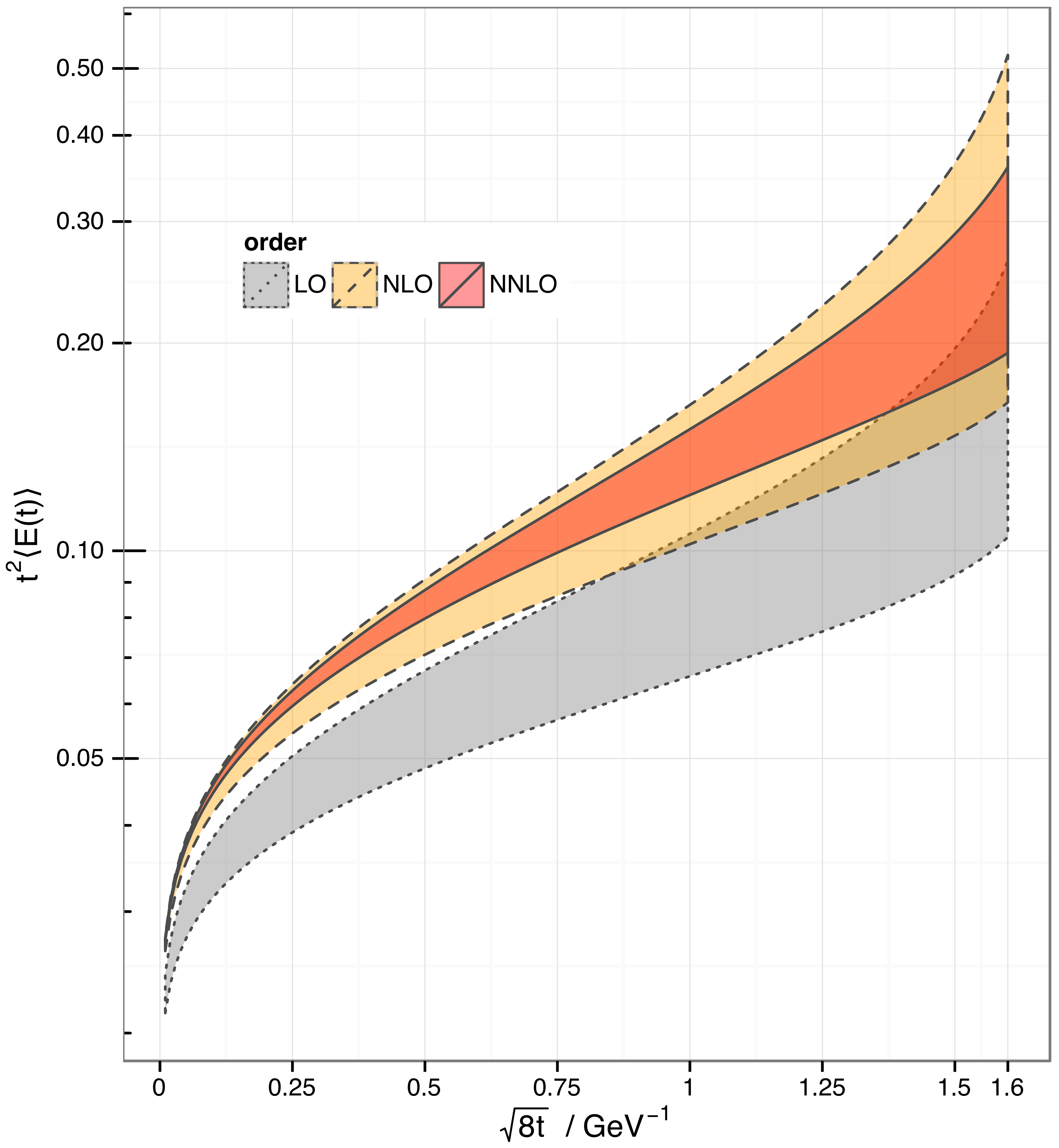
resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann 2016





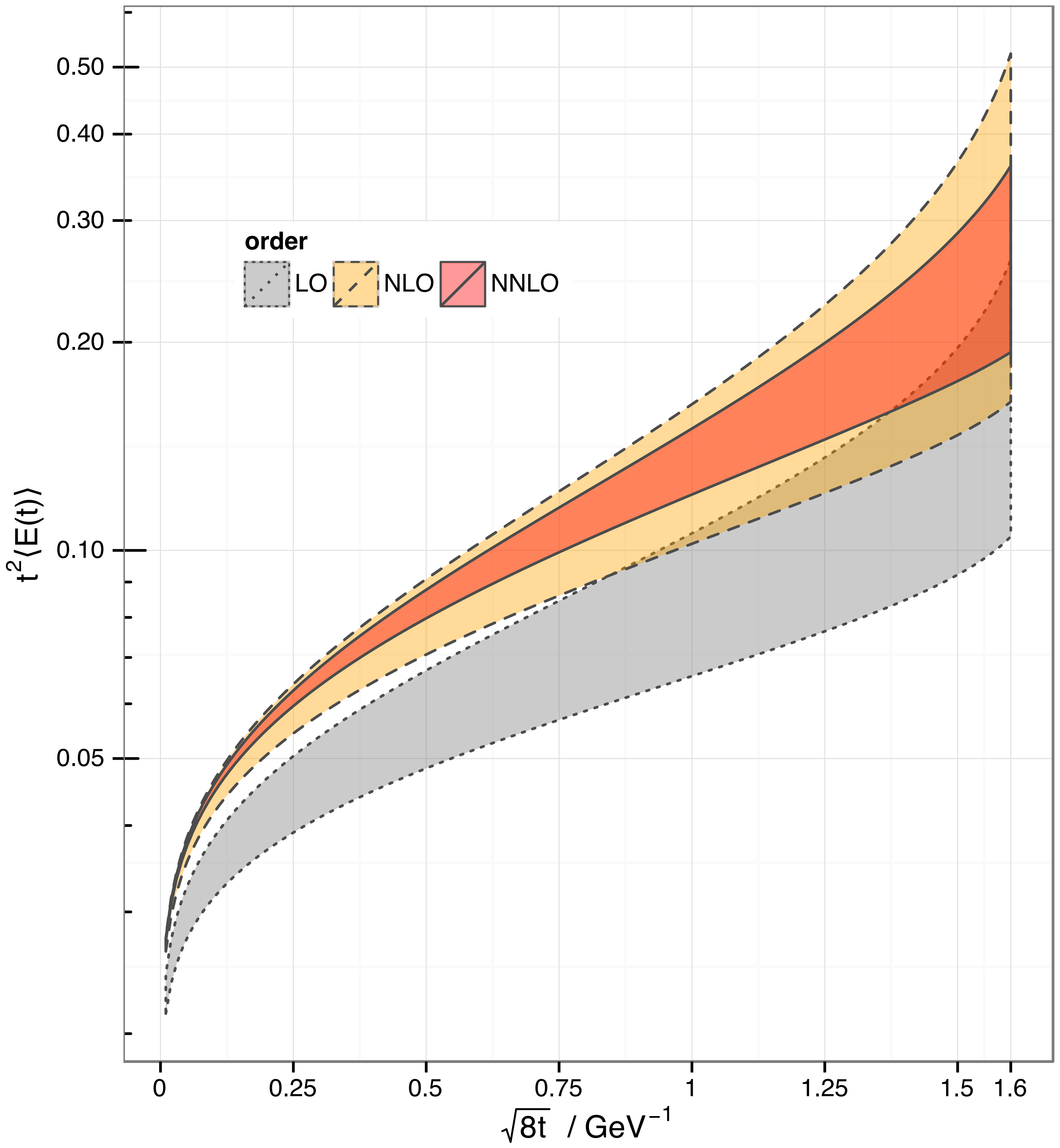
resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

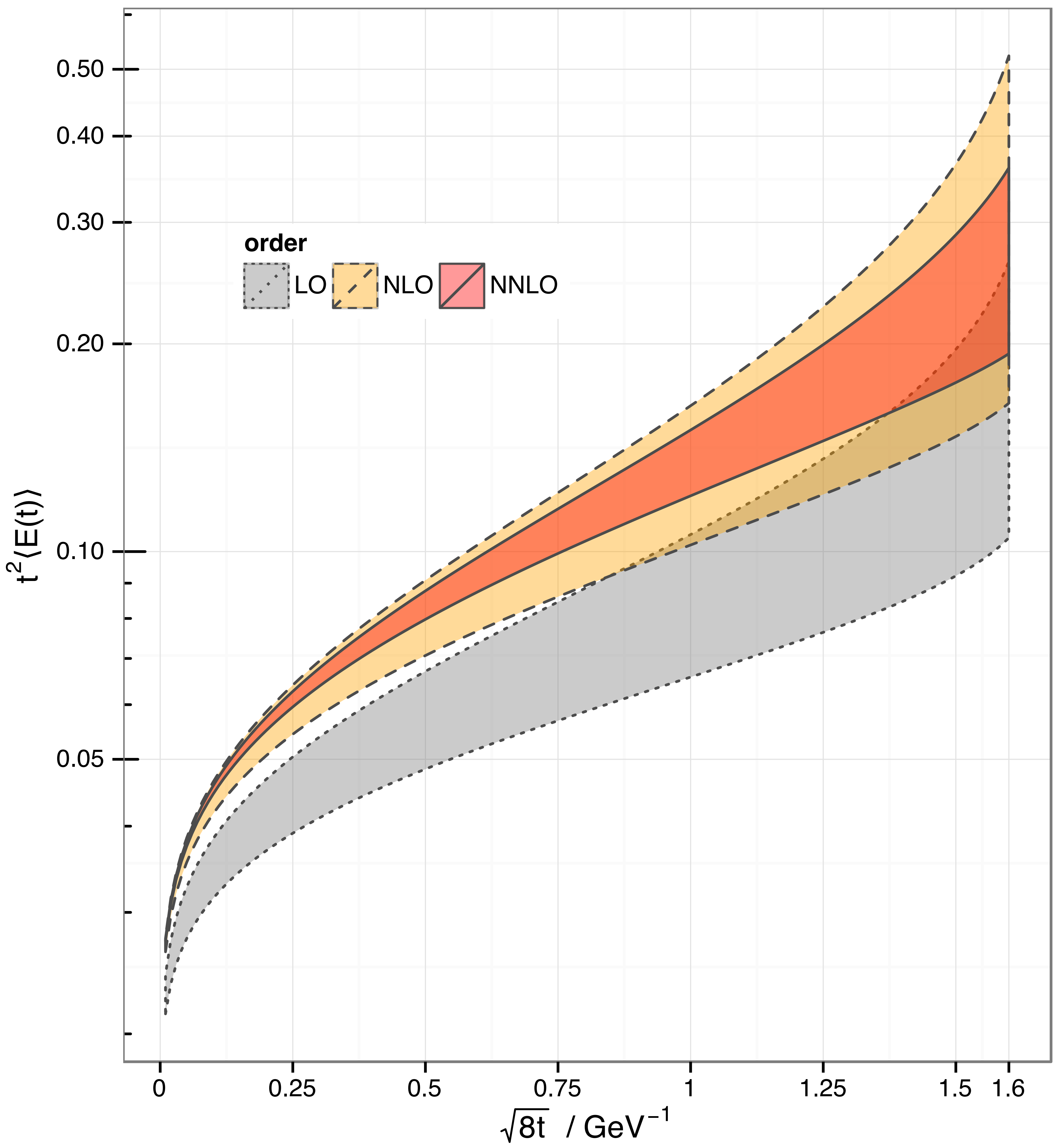
Derive $\alpha_s(m_Z)$

	$t^2\langle E(t)\rangle \cdot 10^4$							
q_8	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$



$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



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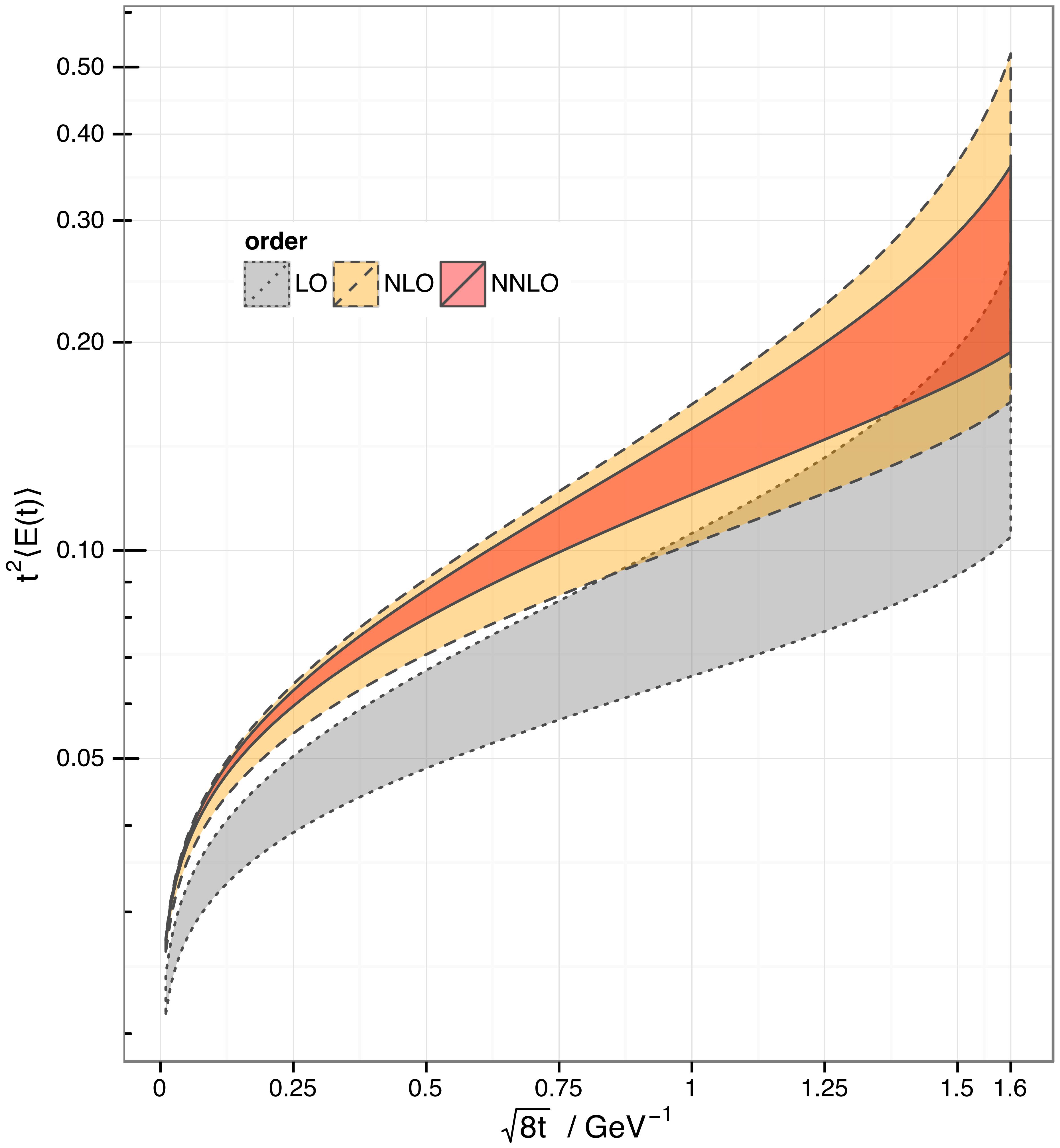
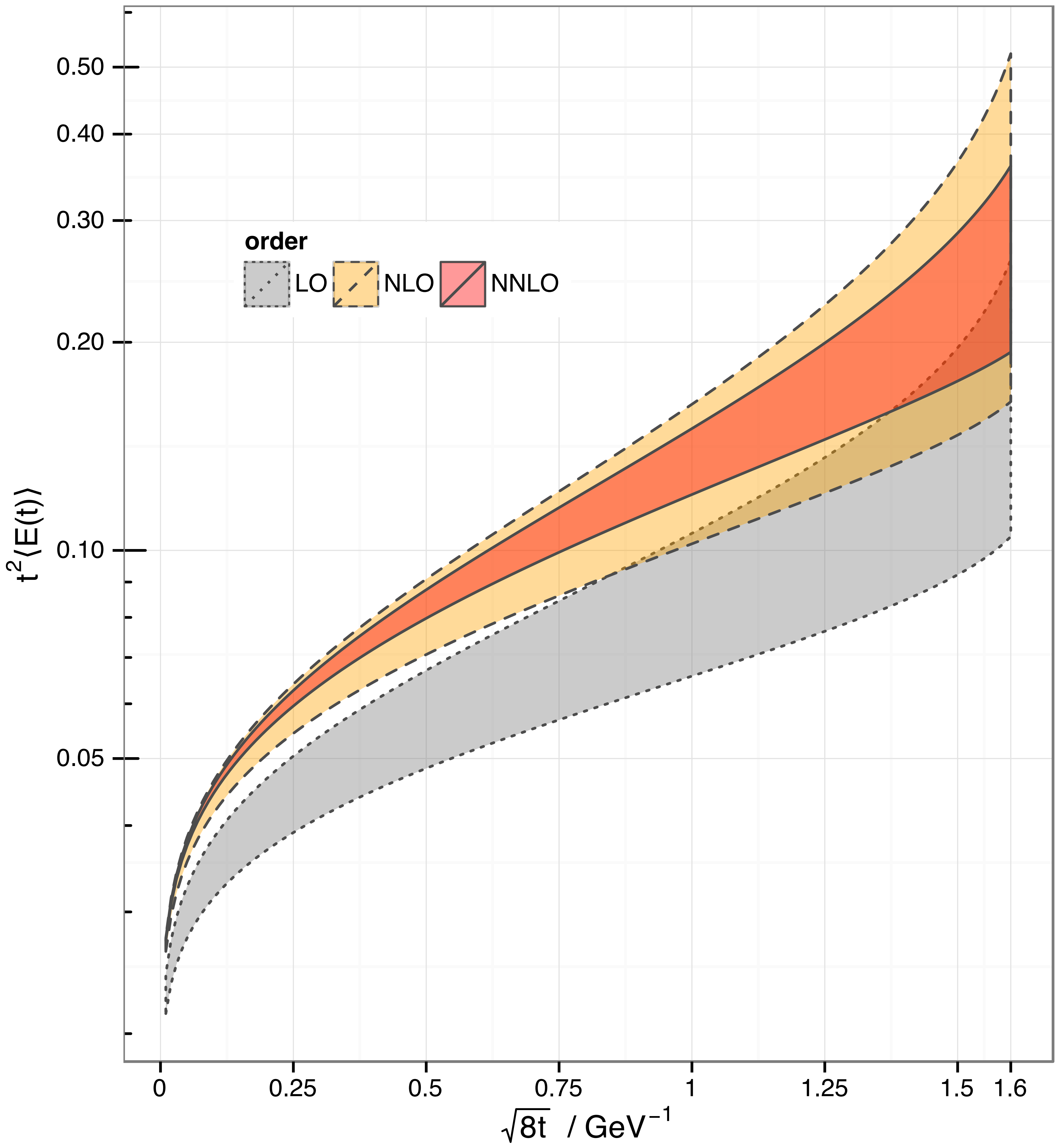


$$t\frac{d}{dt}\hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

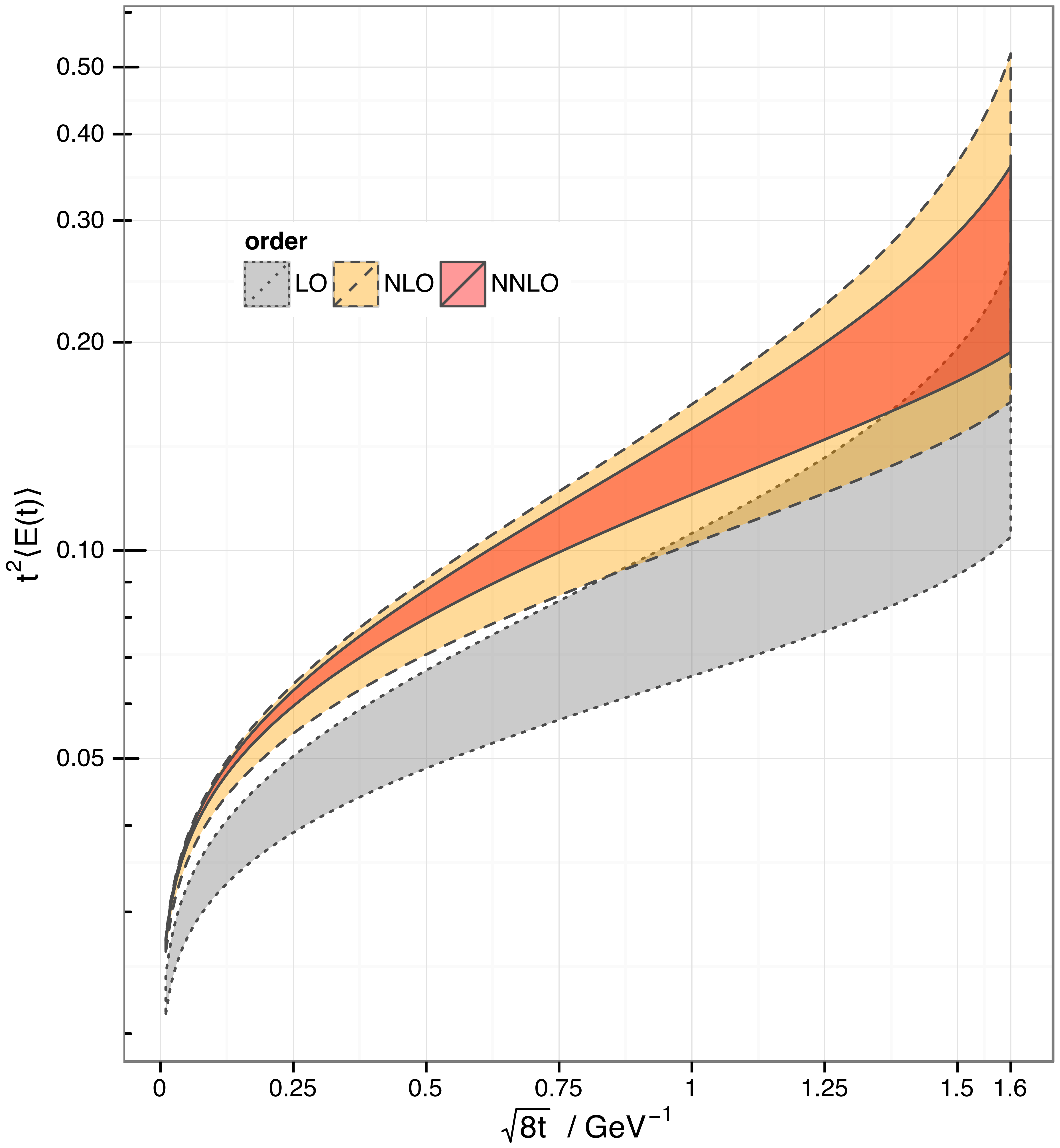
$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



$$t\frac{d}{dt}\hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

$$= \hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



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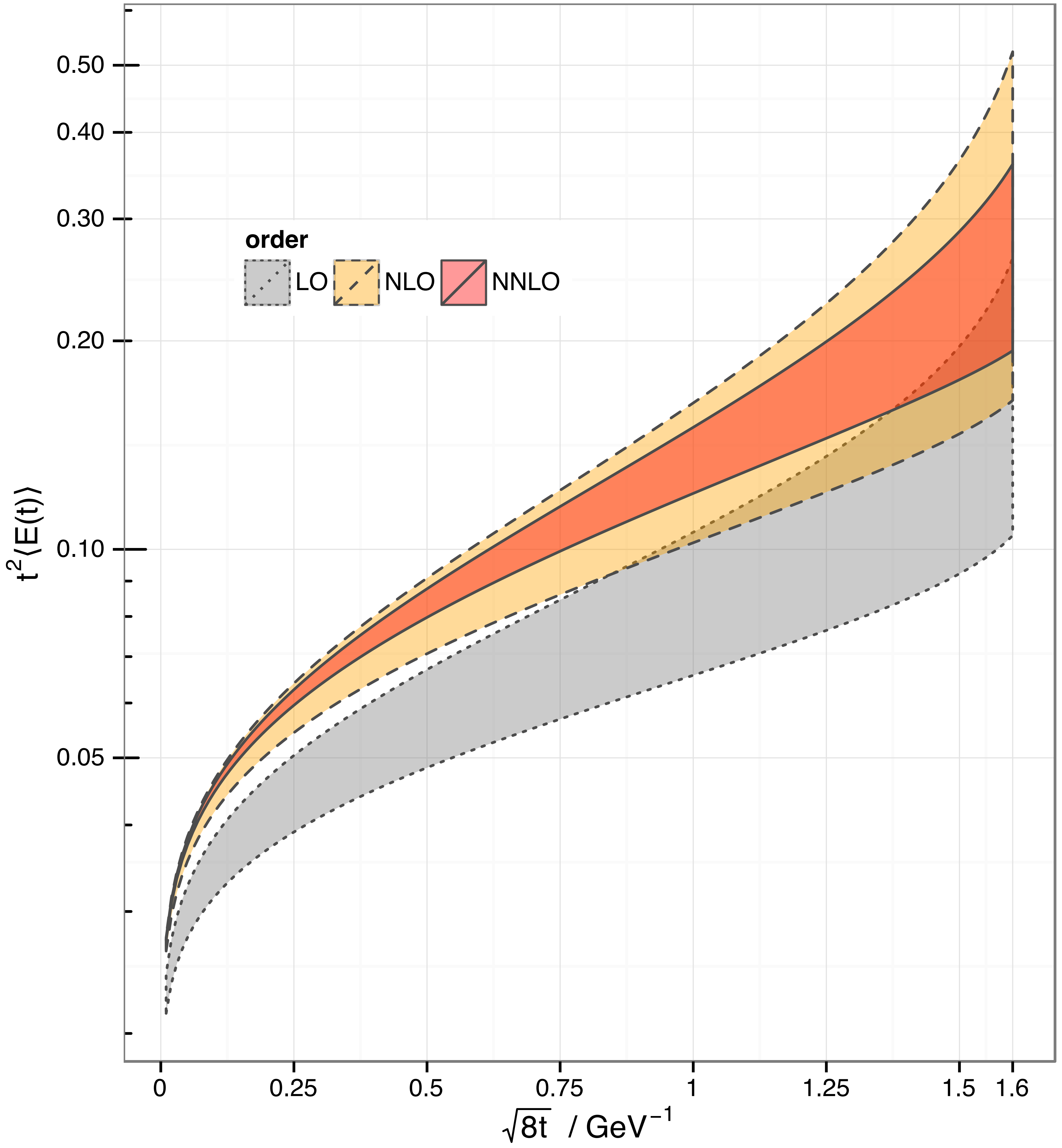
universal

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



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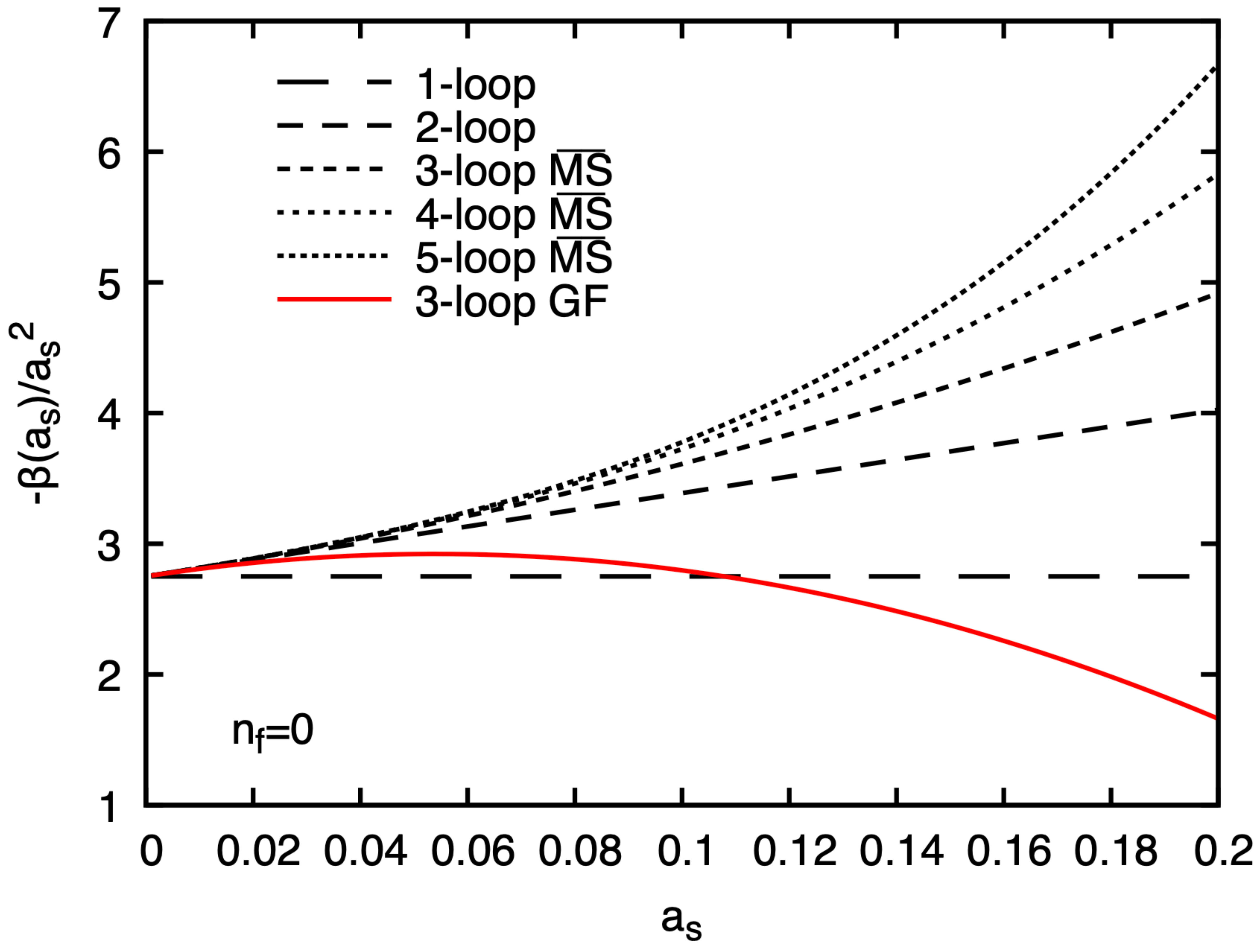
universal

GF specific depends on k_2

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



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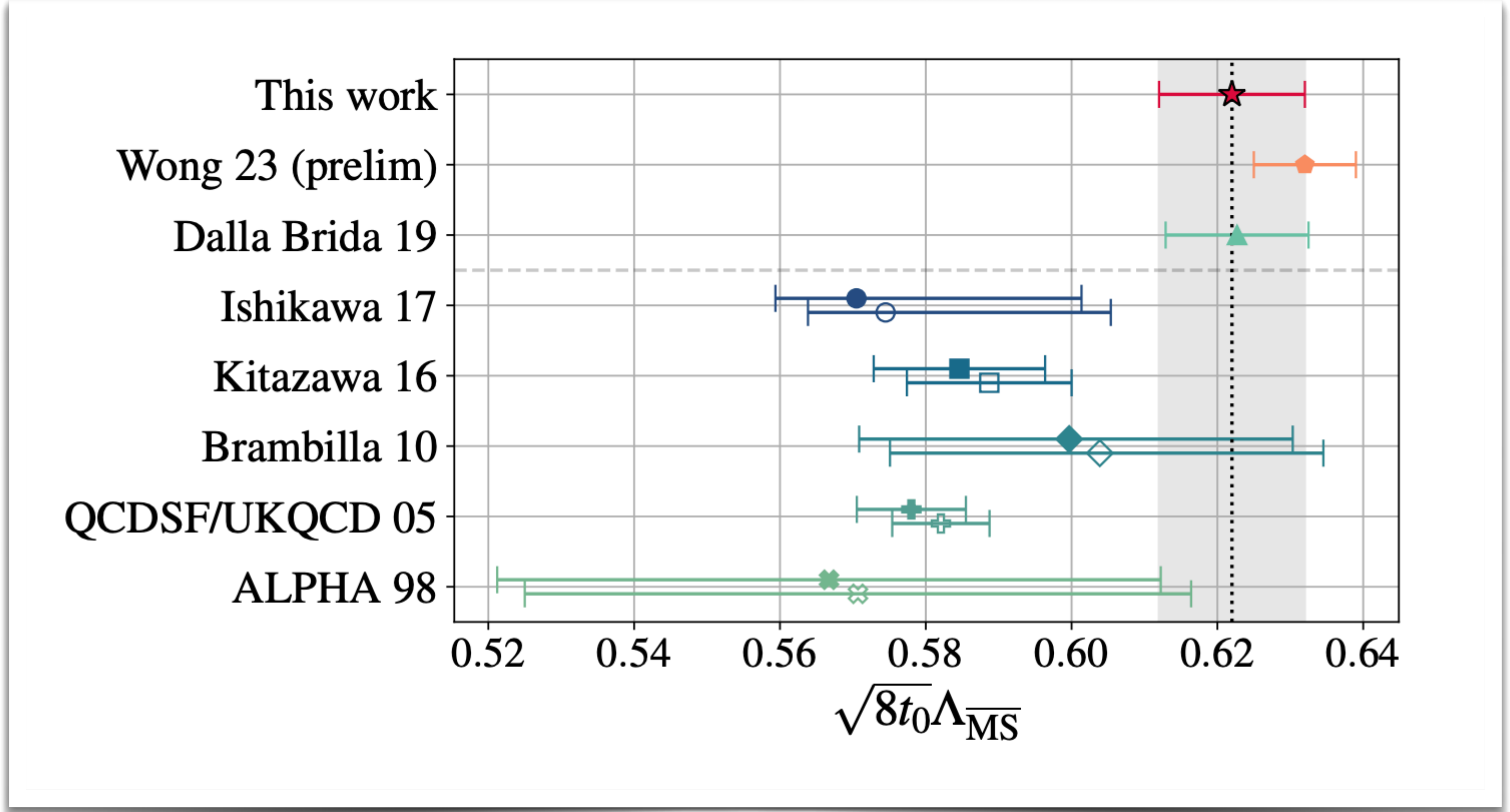
universal

GF specific
depends on k_2

Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel (2023)

see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory → C_n

lattice → $\langle \mathcal{O}_n \rangle$

match renormalization schemes?

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

match renormalization schemes?

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

gradient flow renormalization

Crucial observation: $\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite

Lüscher, Weisz 2011

$\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

Small flow-time expansion

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

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Small flow-time expansion

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

small flow-time expansion:

Lüscher, Weisz '11
Suzuki '13
Lüscher '13

$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

$$\tilde{C}_n(t) \overset{t\to 0}{\to} \sum_m C_m\, \zeta^{-1}_{mn}(t)$$

$\Rightarrow$ need $\zeta_{nm}(t)$ for small t $\Rightarrow$ perturbation theory

Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t) \, \langle \mathcal{O}_m \rangle$$

Determining $\zeta(t)$

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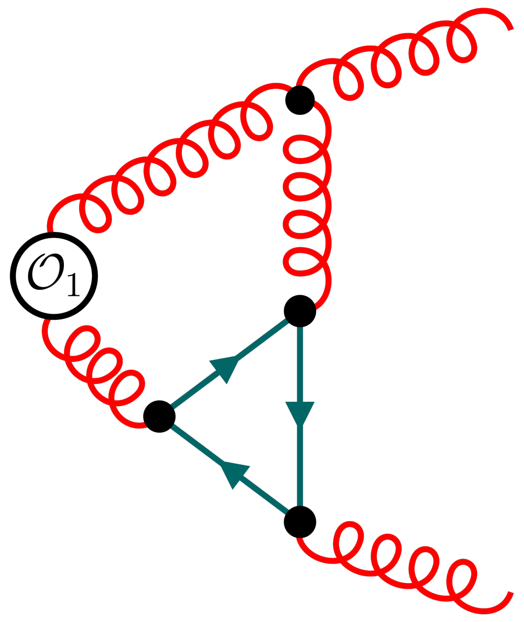

Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

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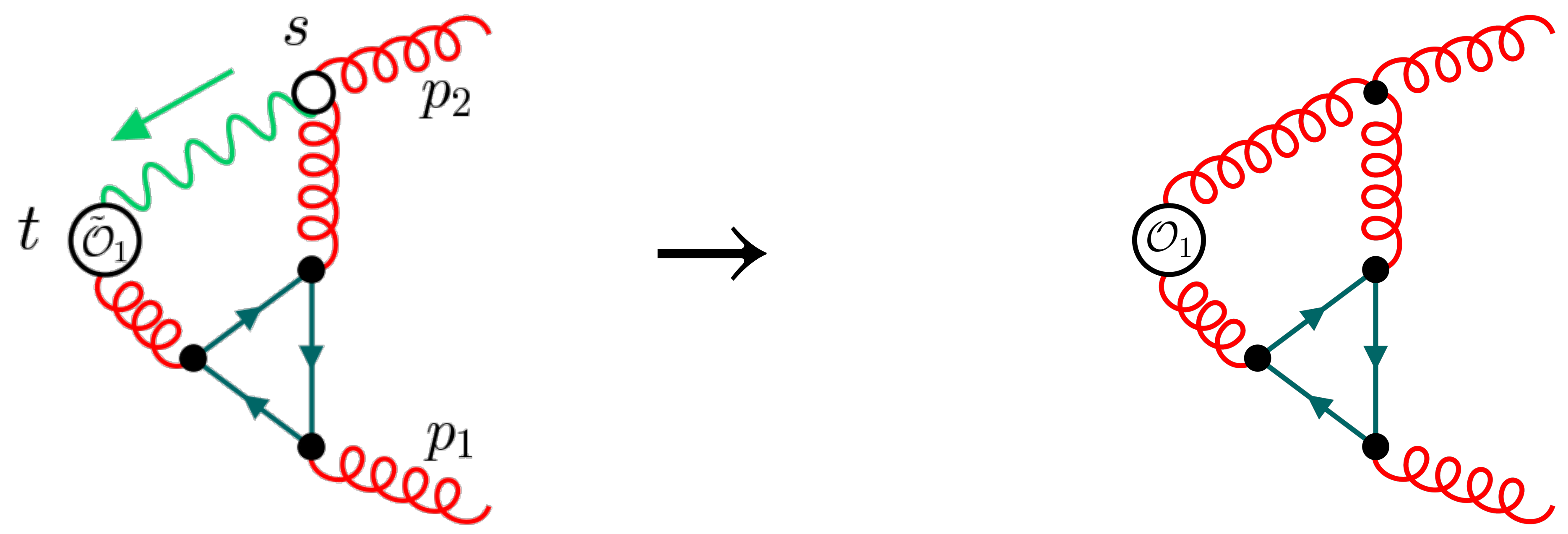

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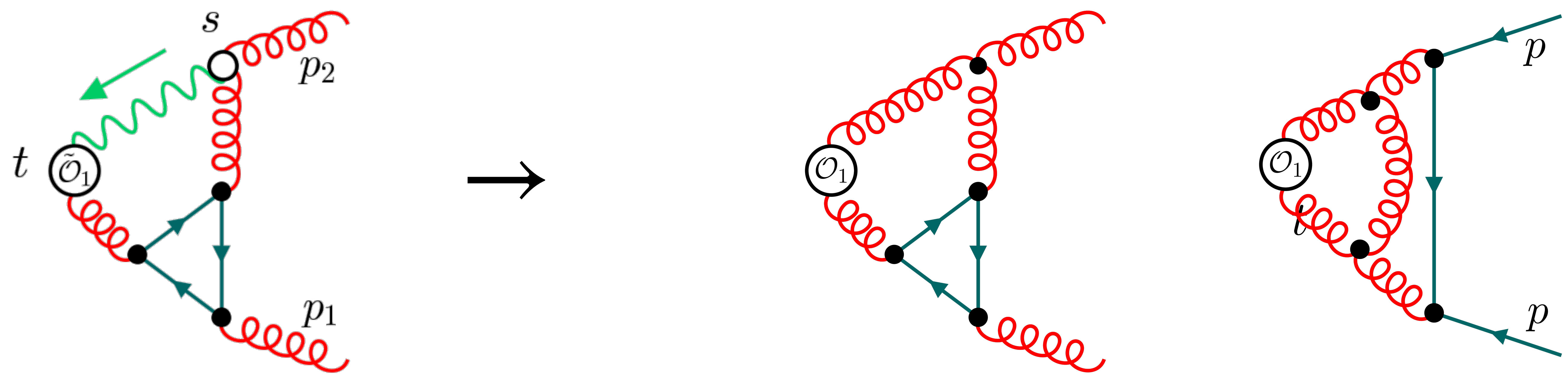

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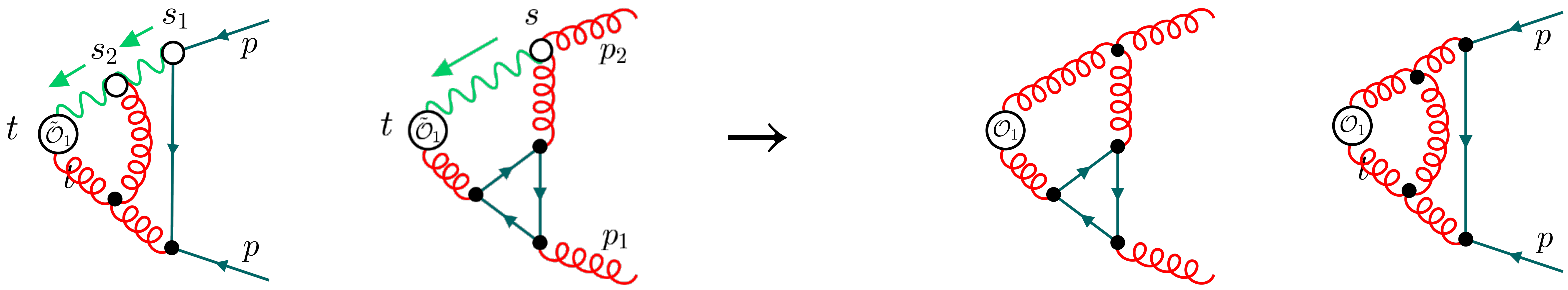

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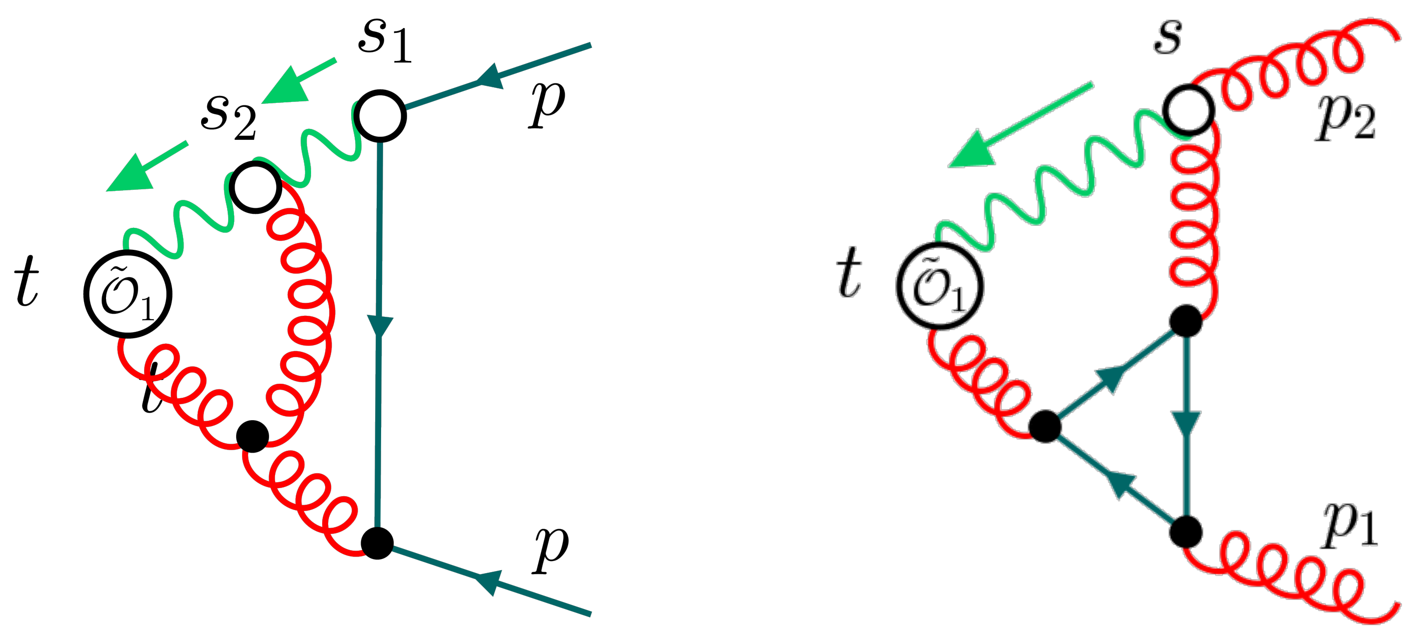
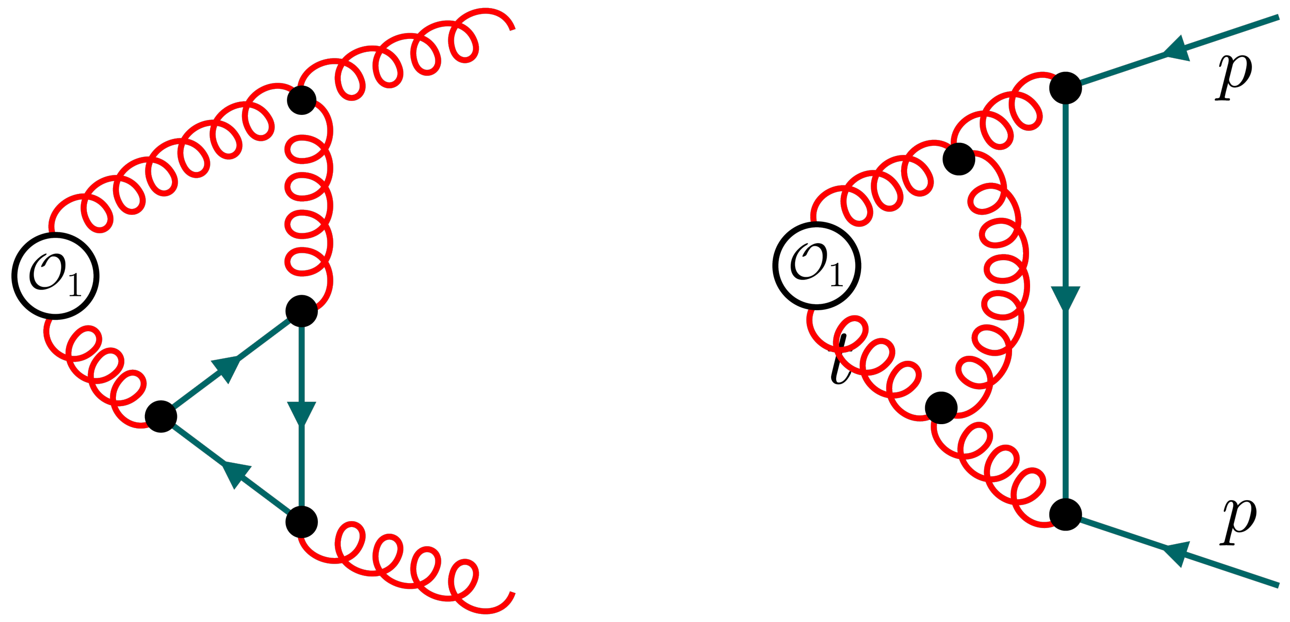


$p = m = 0$

$\rightarrow$



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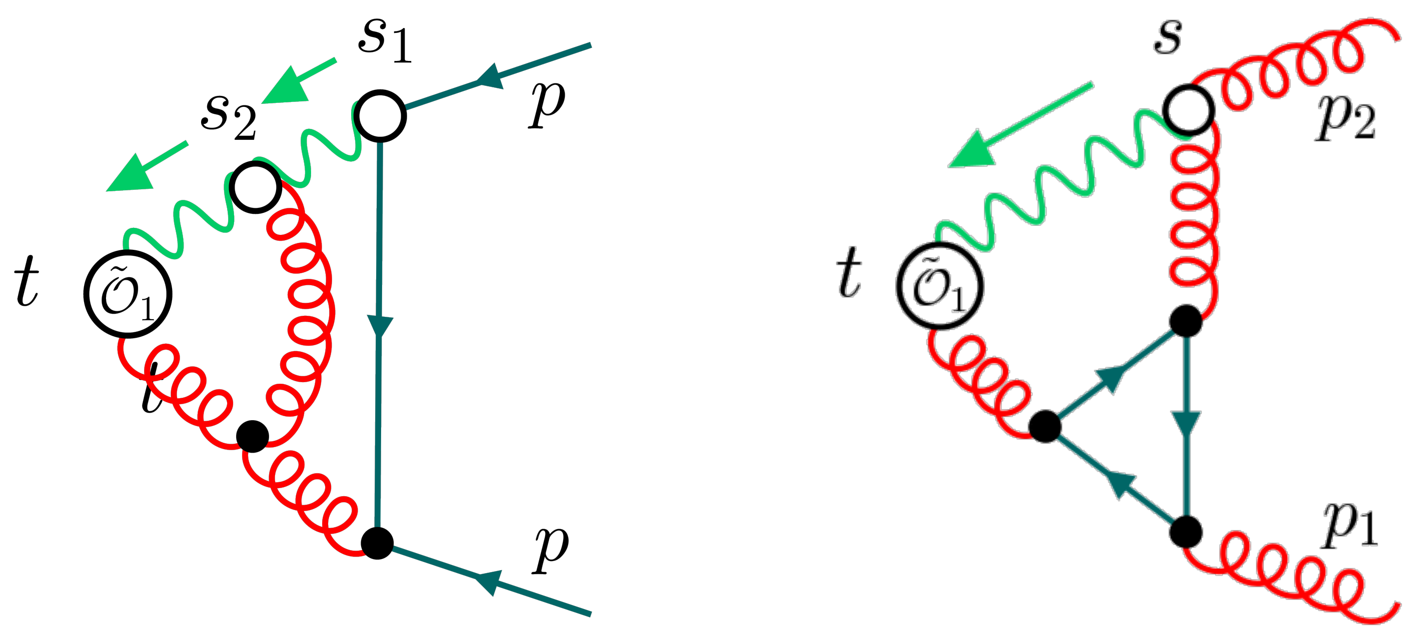

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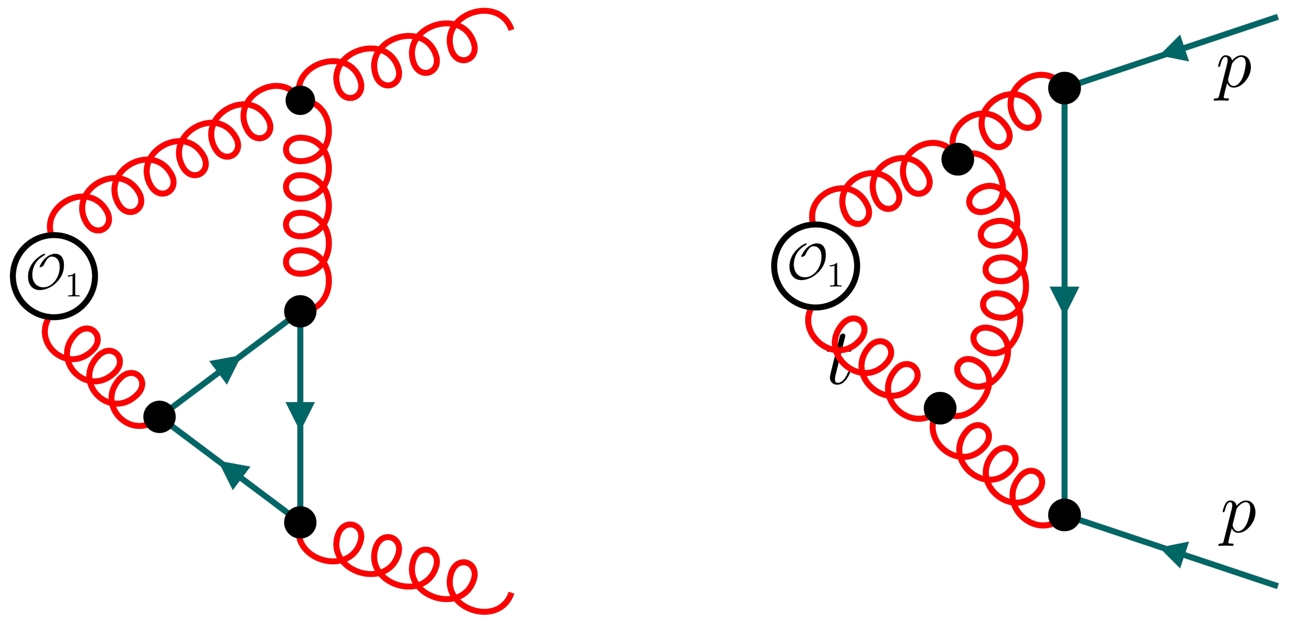
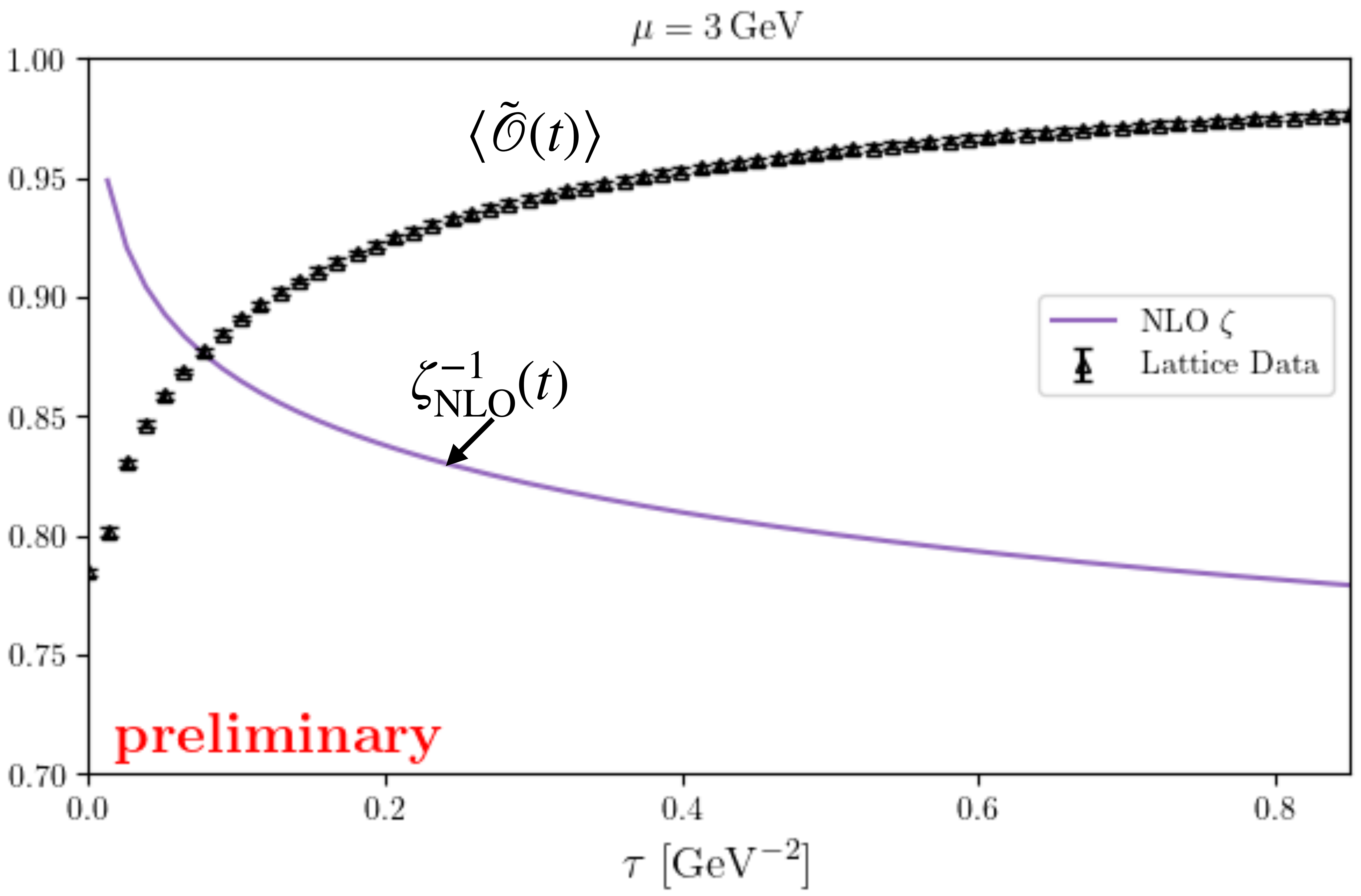


$p = m = 0$

only tree-level diagrams survive on r.h.s.

Gorishnii, Larin, Tkachov '83

Proof of principle



$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

$$= \sum_n C_n(\zeta^{-1}(t) \langle \tilde{\mathcal{O}}(t) \rangle)_n$$

→ see Fabian Lange's talk

Proof of principle

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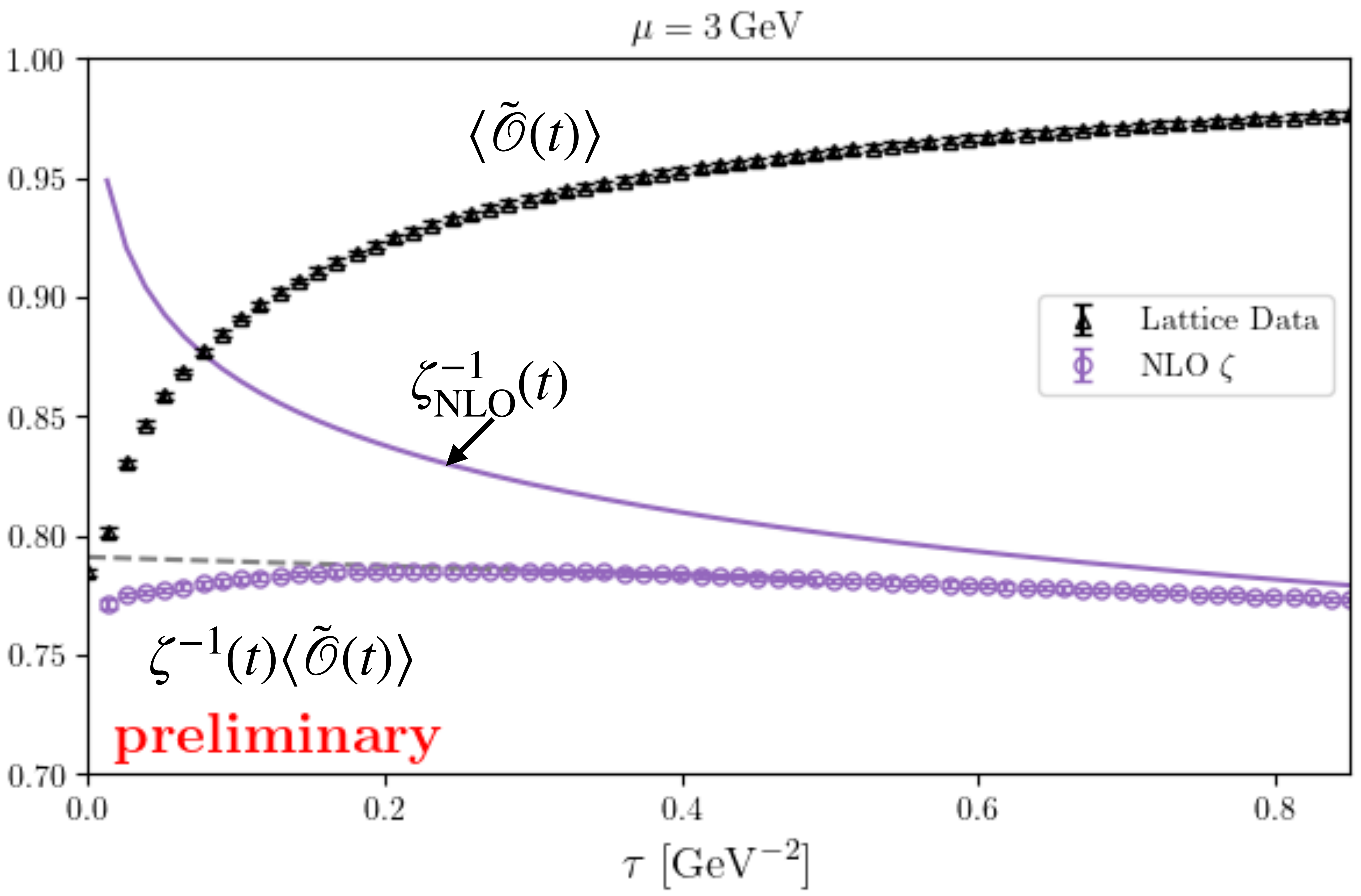

→ see Fabian Lange's talk

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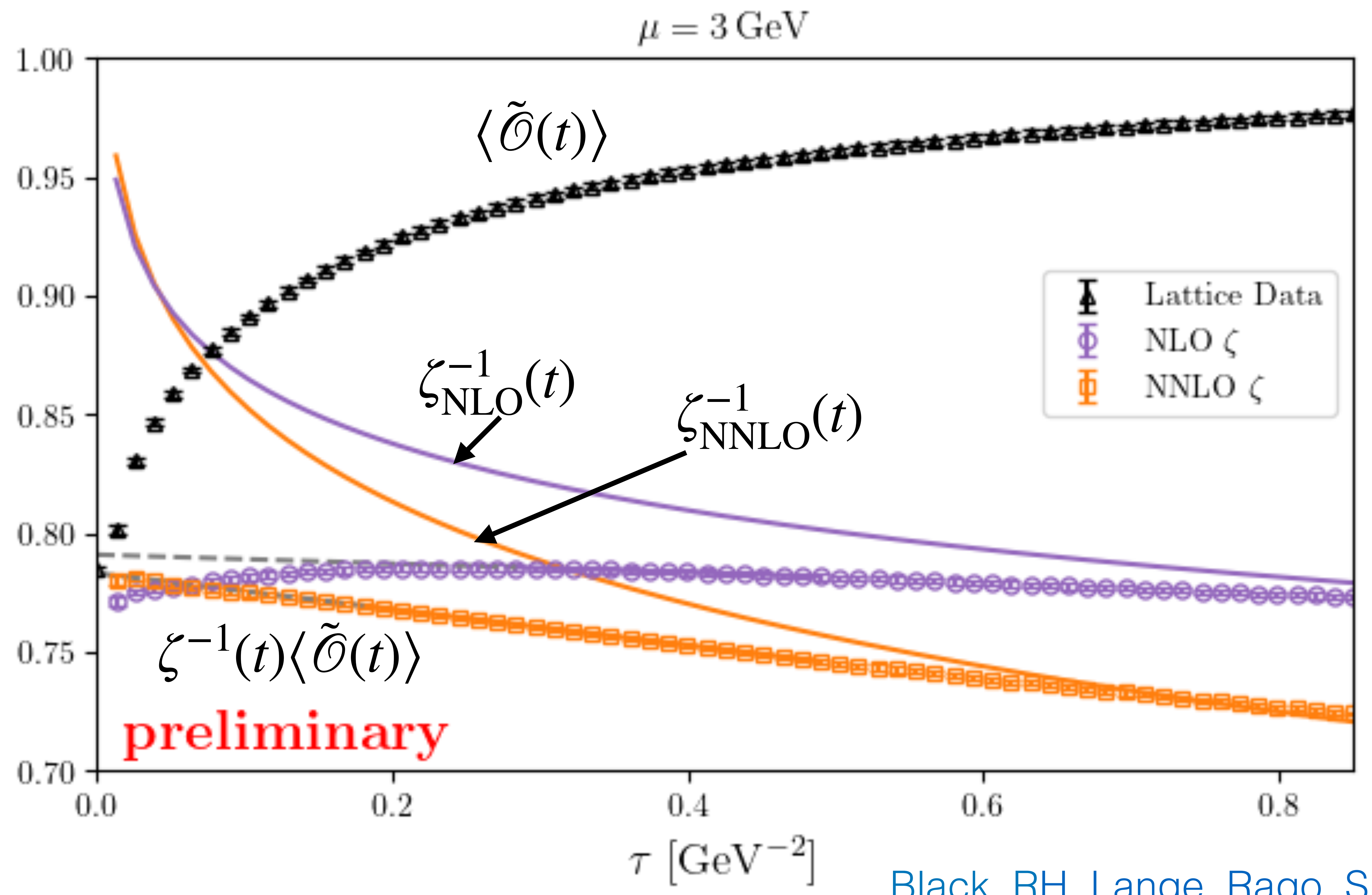

→ see Fabian Lange's talk

Black, RH, Lange, Rago, Shindler, Witzel (2023)

P H
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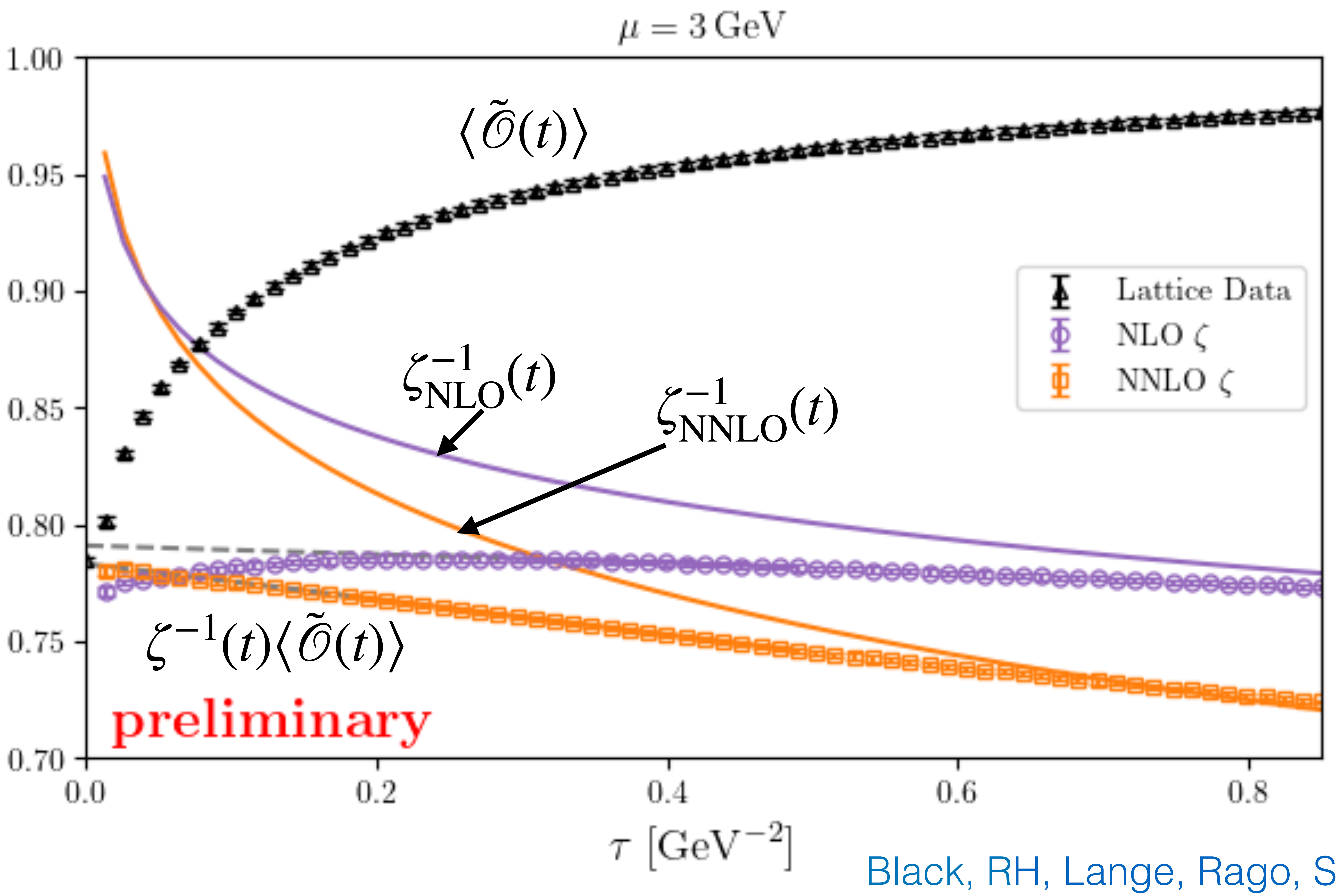
Proof of principle



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→ see Fabian Lange's talk

Black, RH, Lange, Rago, Shindler, Witzel (2023)

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CRC TRR 257

Energy momentum tensor

Suzuki (2013)
Suzuki, Makino (2014)

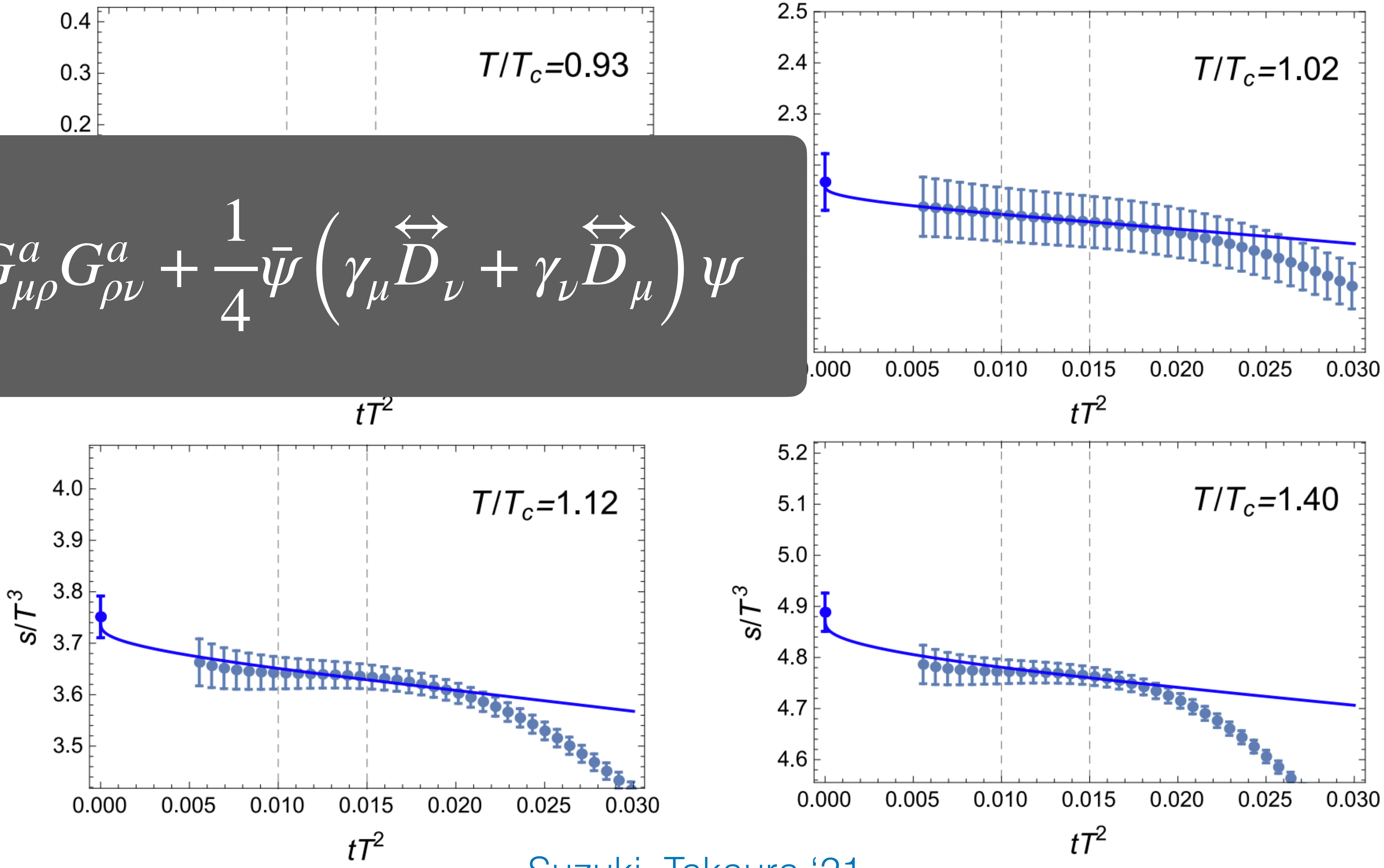
Entropy density:

ε

$$T_{\mu\nu} = -\frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma} + G^a_{\mu\rho}G^a_{\rho\nu} + \frac{1}{4}\bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$T_{\mu\nu} = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

NLO



Suzuki, Takaura '21

Energy momentum tensor

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Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

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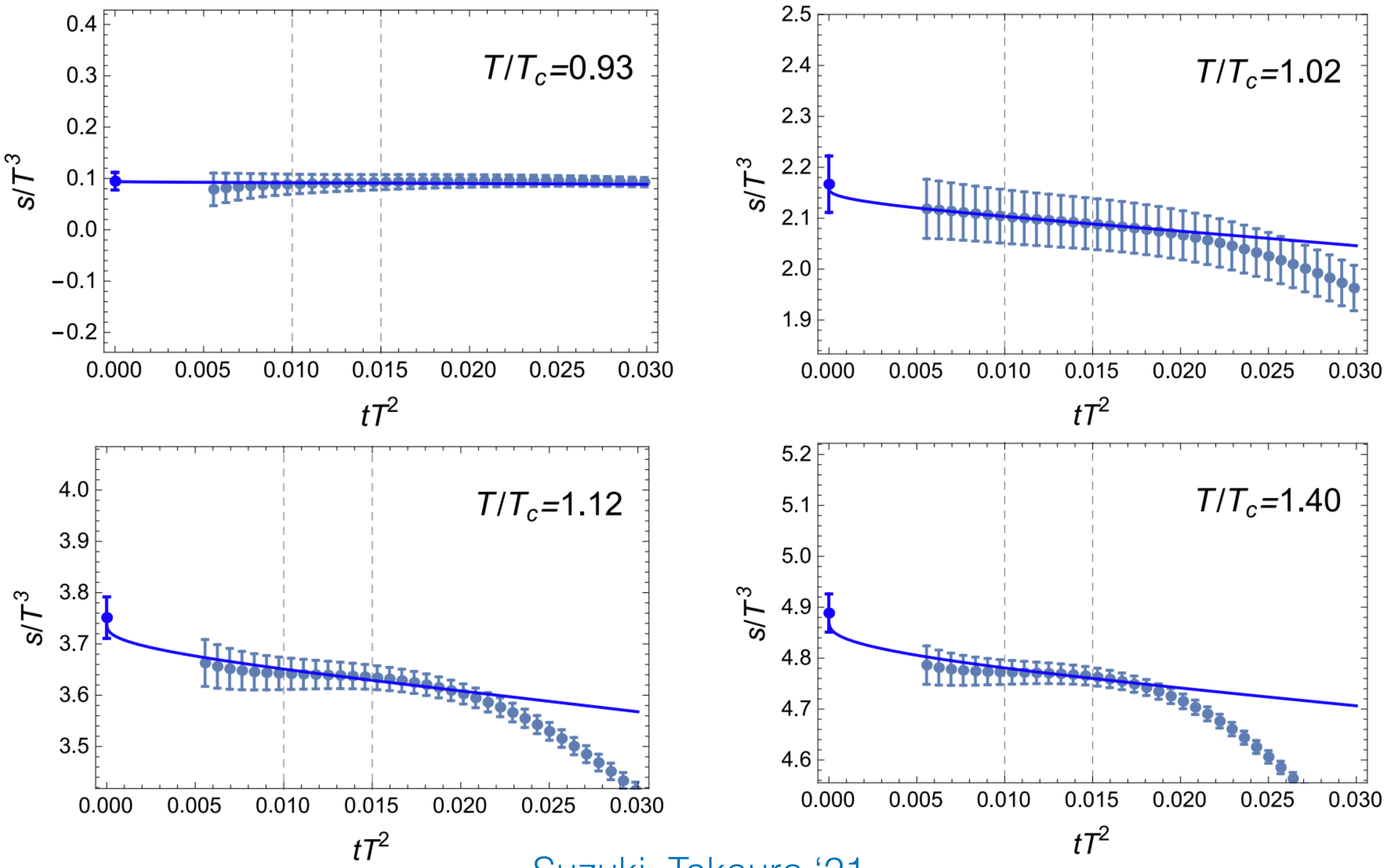

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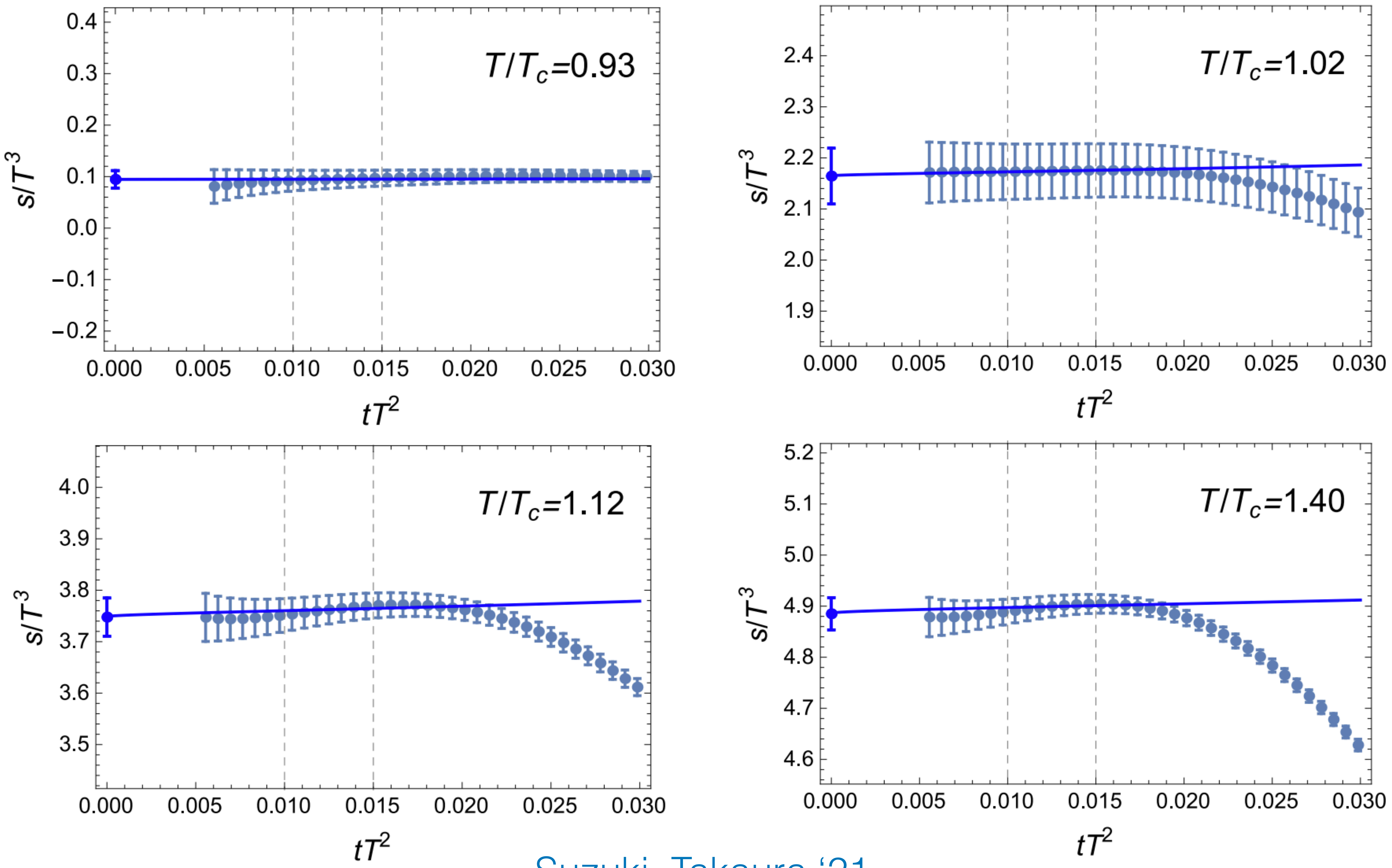

Suzuki, Takaura '21

Other applications

electric dipole operators

NLO: Mereghetti, Monahan, Rizik, Shindler, Stoffer (2022)
Crosas, Monahan, Rizik, Shindler, Stoffer (2023)
NNLO: Borgulat, RH, Rizik, Shindler (2022)

quark bilinears

NLO: Hieda, Suzuki (2016)
NNLO: Borgulat, RH, Kohnen, Lange (2023)

four-quark operators → see Fabian Lange's talk

NLO: A. Suzuki, Tanaguchi, H. Suzuki, Kanaya (2020)
NNLO: RH, Lange (2022)
Black, RH, Lange, Rago, Shindler, Witzel (2023, 2024)

LEFT operators → see Òscar Crosas' talk

moments of PDFs

NLO: A. Shindler (2024)

hadronic vacuum polarization

NNLO: RH, Lange, Neumann (2020)

$\overline{\text{MS}}$ renormalization of composite operators

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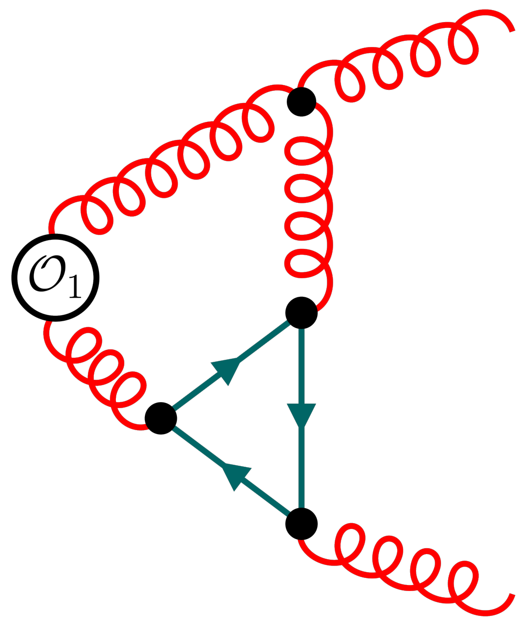

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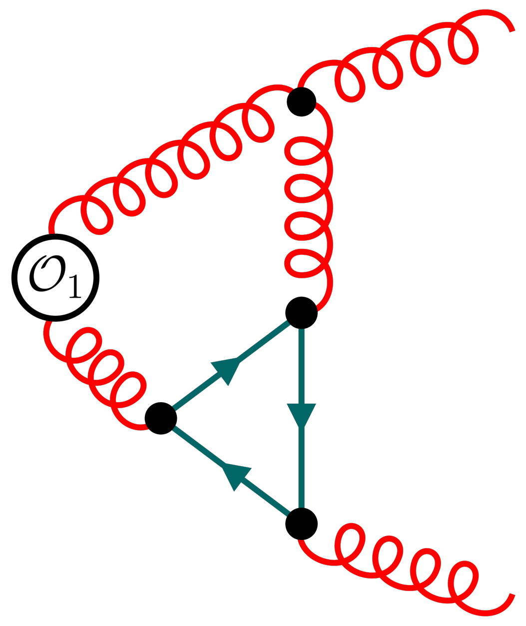

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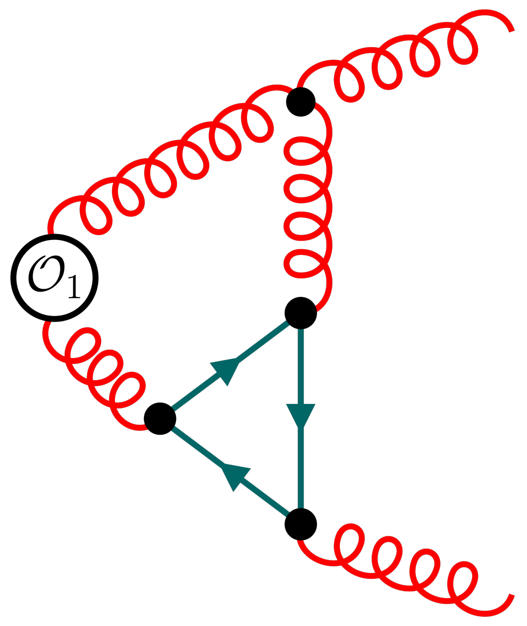

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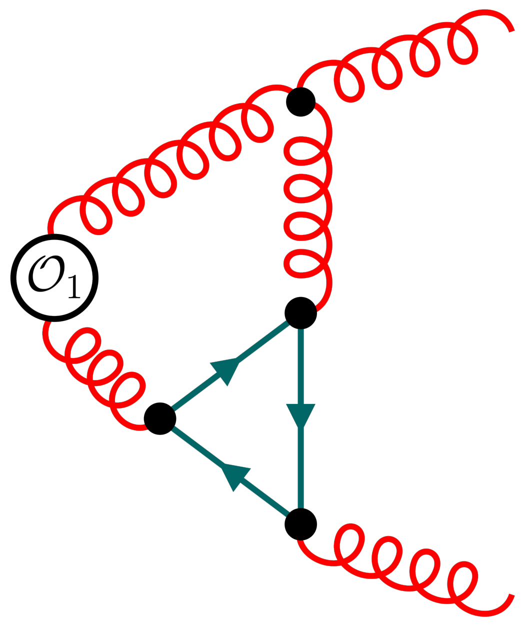

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But note:

$$\int d^D p\, p\, p^\alpha \sim \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} = 0$$

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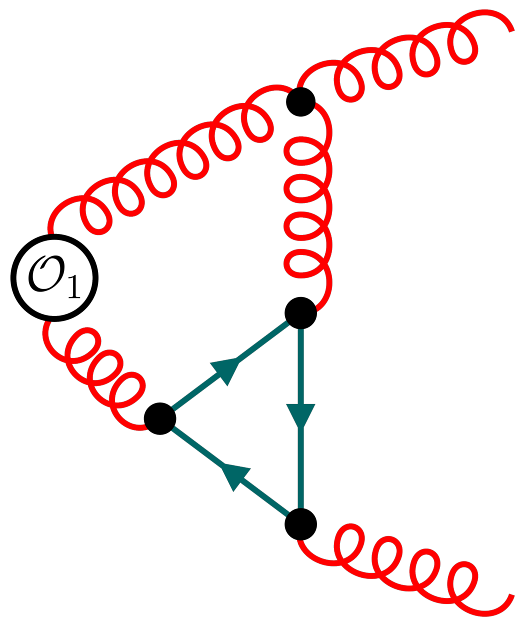

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Keep external momenta? How? Introduce (auxiliary?) masses? Gauge invariance?

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(SFTX)

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UV finite

⇒ calculation of $\zeta(t)$ also determines Z in $\overline{\text{MS}}$ scheme!

$\overline{\text{MS}}$ renormalization of composite operators

application: check for $\Delta F = 2$ four-quark operators Buras, Gorbahn, Haisch, Nierste (2006)

RH, Lange (2022)

$\overline{\text{MS}}$ renormalization of composite operators

application: check for $\Delta F = 2$ four-quark operators Buras, Gorbahn, Haisch, Nierste (2006)
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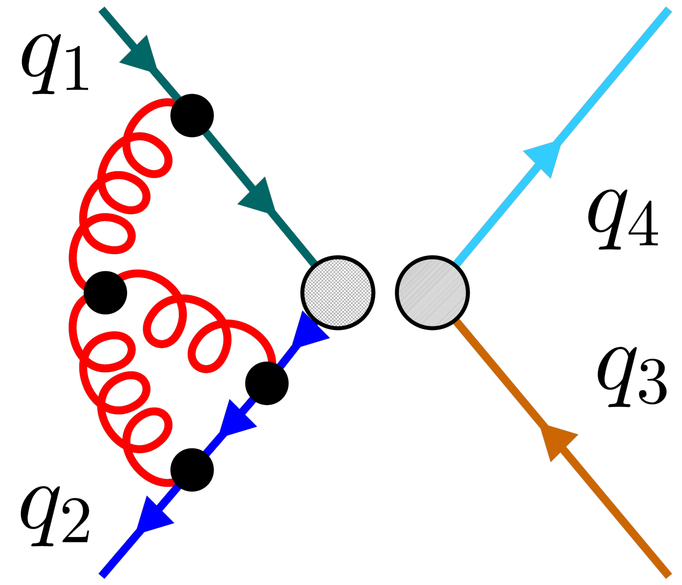

$\overline{\text{MS}}$ renormalization of composite operators

scalar QCD: $$\mathcal{L}_{\text{SQCD}} = \mathcal{L}_{\text{QCD}} + (D_\mu\phi)^\dagger(D_\mu\phi) - \frac{\lambda}{4}\phi^\dagger\phi$$

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flowed quark field: $$\frac{\partial}{\partial t}\chi(t,x) = \mathscr{D}^2\chi(t,x)$$ $$\chi(t=0,x) = \psi(x)$$

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Gorishnii, Kataev, Larin (1987)

Borgulat, Felten, RH, Kohnen (in prep)

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The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \mathscr{O}_n$$

needs renormalization:

$$= \sum_n (C\mathrm{Z})_n (\mathrm{Z}^{-1}\mathscr{O})_n = \sum_n C_n^{\mathrm{R}} \mathscr{O}_n^{\mathrm{R}}$$

anomalous dimension:

$$\mu \frac{d}{d\mu} C_n = \gamma_{nm} C_m \qquad \gamma_{nm} = \mu \frac{d}{d\mu} \ln \mathrm{Z}$$

gradient-flow scheme:

$$\mathscr{L}_{\text{eff}} = \sum_n \tilde{C}_n(t)\, \tilde{\mathscr{O}}_n(t)$$

The GF scheme

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gradient-flow scheme:

$$\mathscr{L}_{\text{eff}} = \sum_n \tilde{C}_n(t)\, \tilde{\mathcal{O}}_n(t) = \sum_n (C\zeta^{-1}(t))_n\, (\zeta(t)\mathcal{O})_n(t)$$

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$$t \frac{d}{dt} \tilde{C}(t) = \tilde{\gamma}\, \tilde{C}(t) \qquad \tilde{\gamma} = t \frac{d}{dt} \ln \zeta(t)$$

RH, Lange, Neumann (2020)

see also Hasenfratz, Monahan, Rizik, Shindler, Witzel (2021)

Conclusions and Outlook

- Gradient flow provides ideal basis for combining lattice and perturbation theory
- Many perturbative tools can be adapted
- Several proofs of principle already available
- Full potential still to be explored

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