

# The gradient flow and its applications

Robert Harlander

RWTH Aachen University

04 March 2026

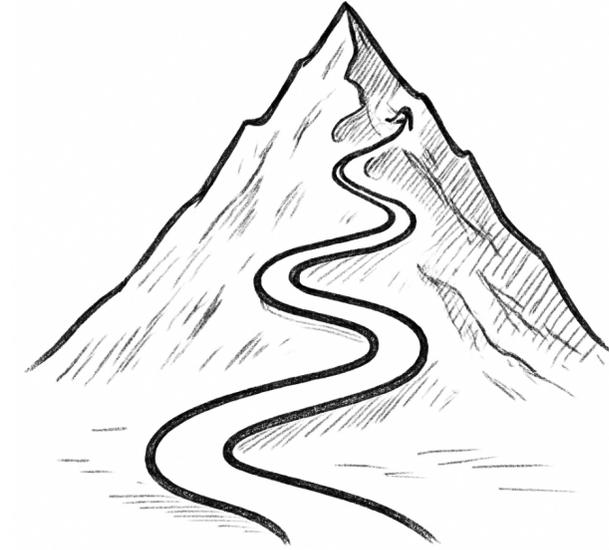
**Lattice Practices Workshop**

Jülich, 2-6 March 2026

# Motivation

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Consider walking in the alps along a path  $\vec{x}(t)$

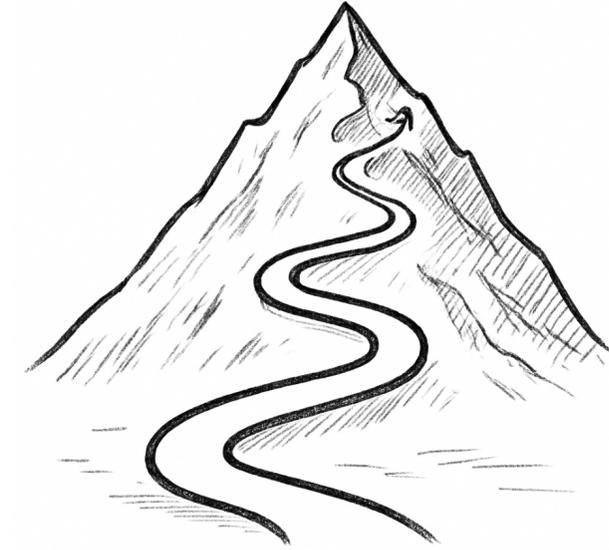


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Application to QCD:

$$\frac{\partial}{\partial t} B_\mu(\mathbf{t}) = - \frac{\delta S[B]}{\delta B_\mu(\mathbf{t})}$$

$\mathbf{t}$ : flow time

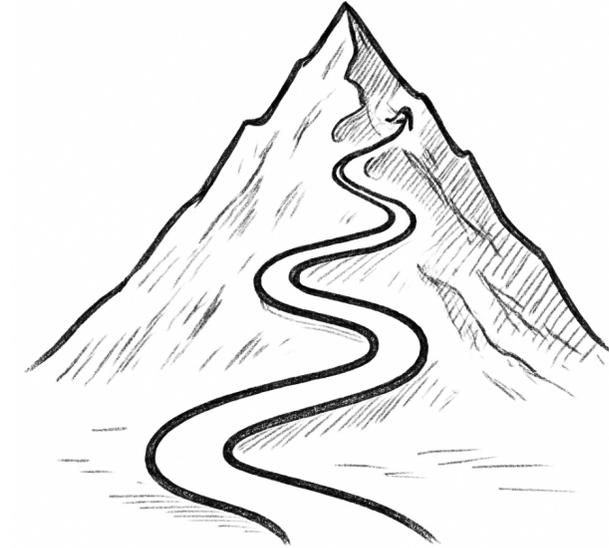


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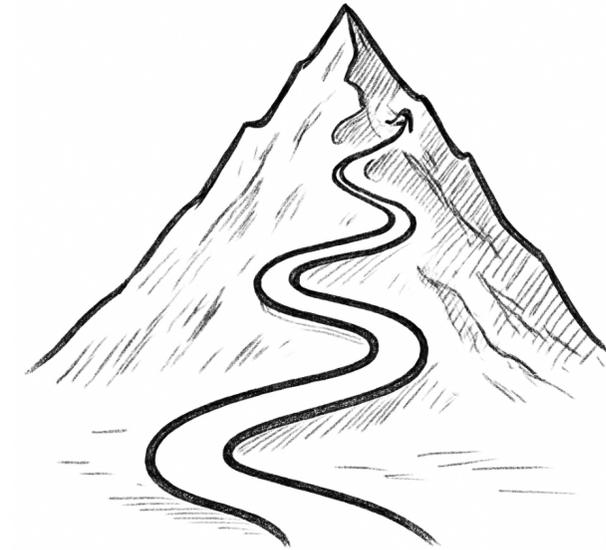
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$$B_\mu(t=0) = A_\mu$$

## Properties and uses of the Wilson flow in lattice QCD

#2

[Martin Lüscher](#) (CERN and Geneva U.) (Jun 23, 2010)

Published in: *JHEP* 08 (2010) 071, *JHEP* 03 (2014) 092 (erratum) • e-Print: [1006.4518](#) [hep-lat]



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## Trivializing maps, the Wilson flow and the HMC algorithm

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## Infinite N phase transitions in continuum Wilson loop operators

#5

[R. Narayanan](#) (Florida Intl. U.), [H. Neuberger](#) (Rutgers U., Piscataway) (Jan, 2006)

Published in: *JHEP* 03 (2006) 064 • e-Print: [hep-th/0601210](#) [hep-th]



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$$\partial_t B \sim (\partial + g_0 B)(\partial B + g_0 B^2) \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$$

# Perturbative solution

---

flow equation:

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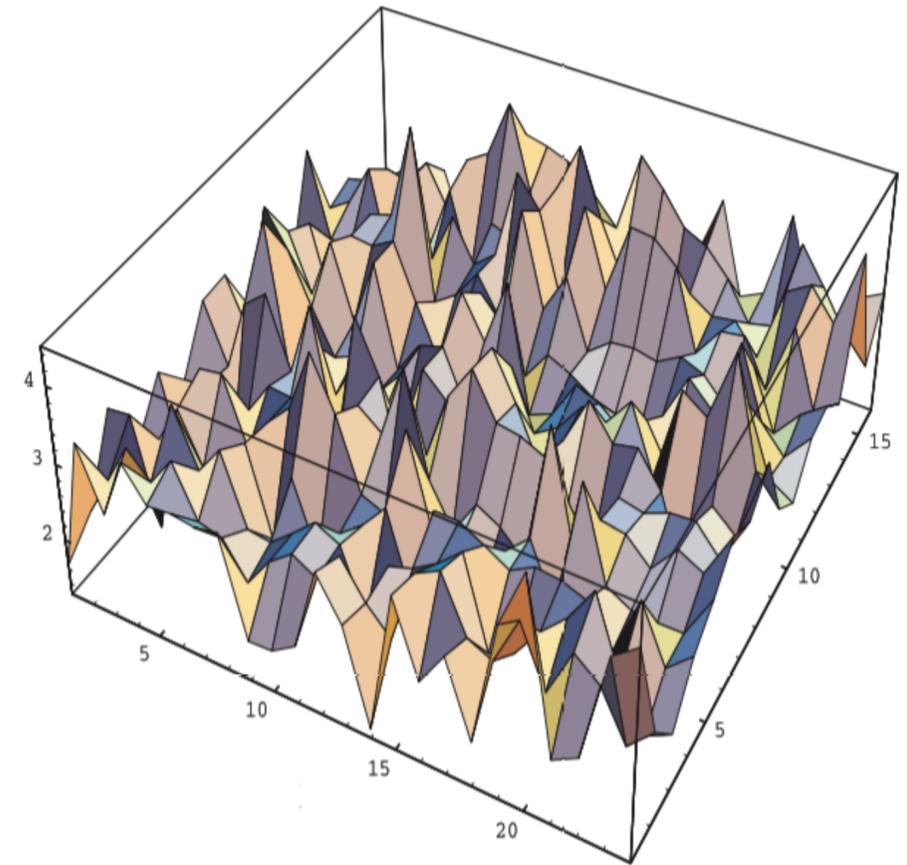
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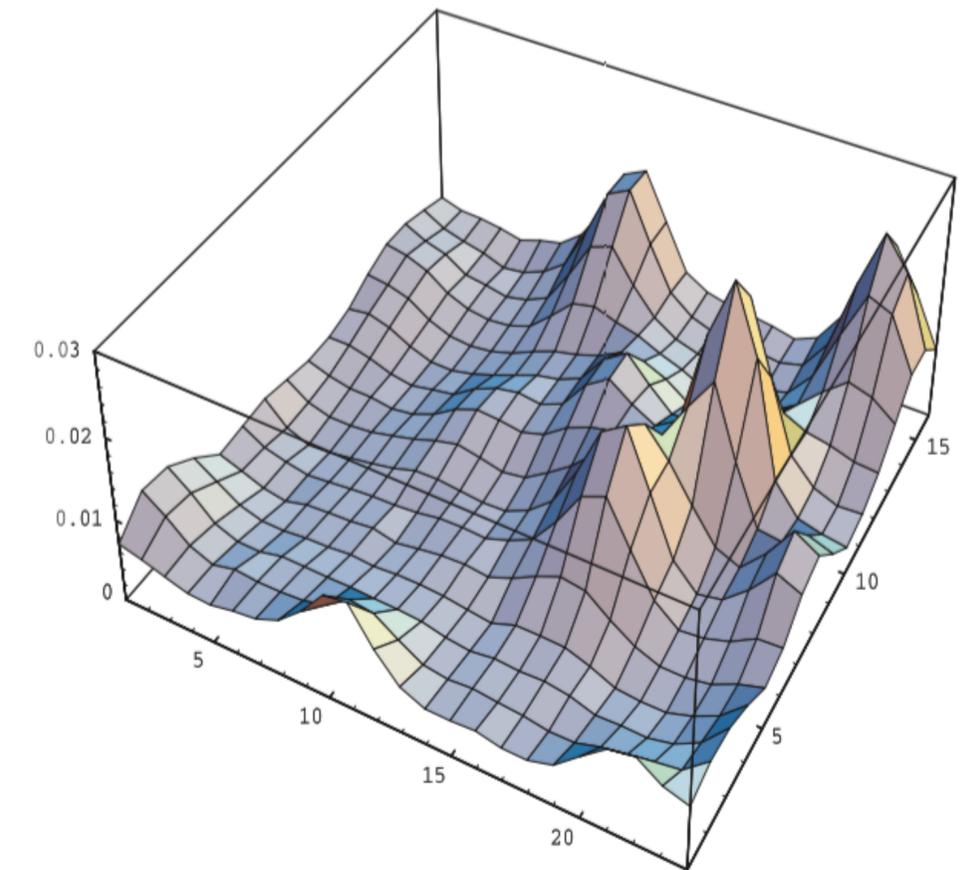
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momentum space:  $\tilde{B}_1(t, p) = e^{-tp^2} \tilde{A}(p)$   
 $\tilde{B}_2(t, p) = \int_0^t ds \int d^4 q K(t, s, p, q) A(p) A(p - q)$   
 $K(t, s, p, q) \sim \exp[-tp^2 - 2sq(q - p)]$

etc.

Exponential damping in momentum integrals!

# The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$

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$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

$$\mathcal{L}_B \sim \int_0^\infty dt L_\mu \left( \partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

$$\mathcal{L}_\chi \sim \int_0^\infty dt \bar{\lambda} \left( \partial_t - \mathcal{D}^2 \right) \chi + \text{h.c.}$$

Lüscher, Weisz 2011

Lüscher 2013

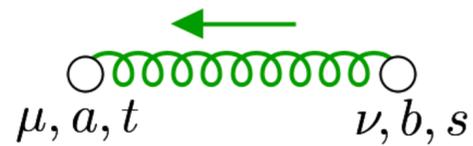
# Perturbative approach

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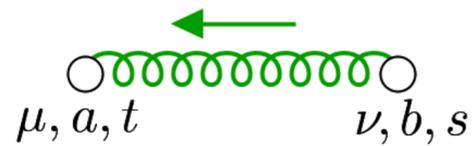
$$\delta_{ab} \delta_{\mu\nu} \theta(t-s) e^{-(t-s)p^2}$$

“gluon flow line”

$$\sim \langle 0 | T L_\mu^a(t, x) B_\nu^b(s, 0) | 0 \rangle$$

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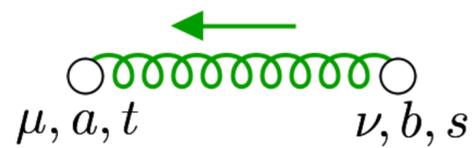


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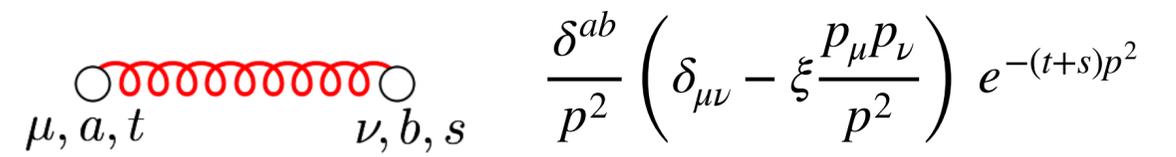
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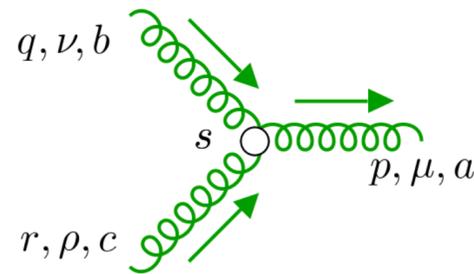
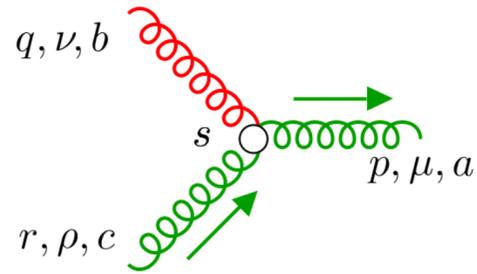
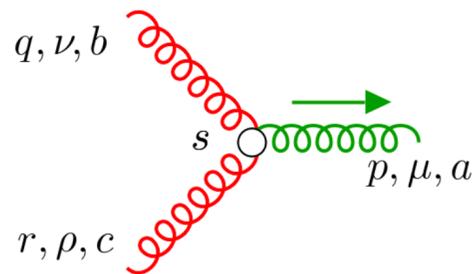
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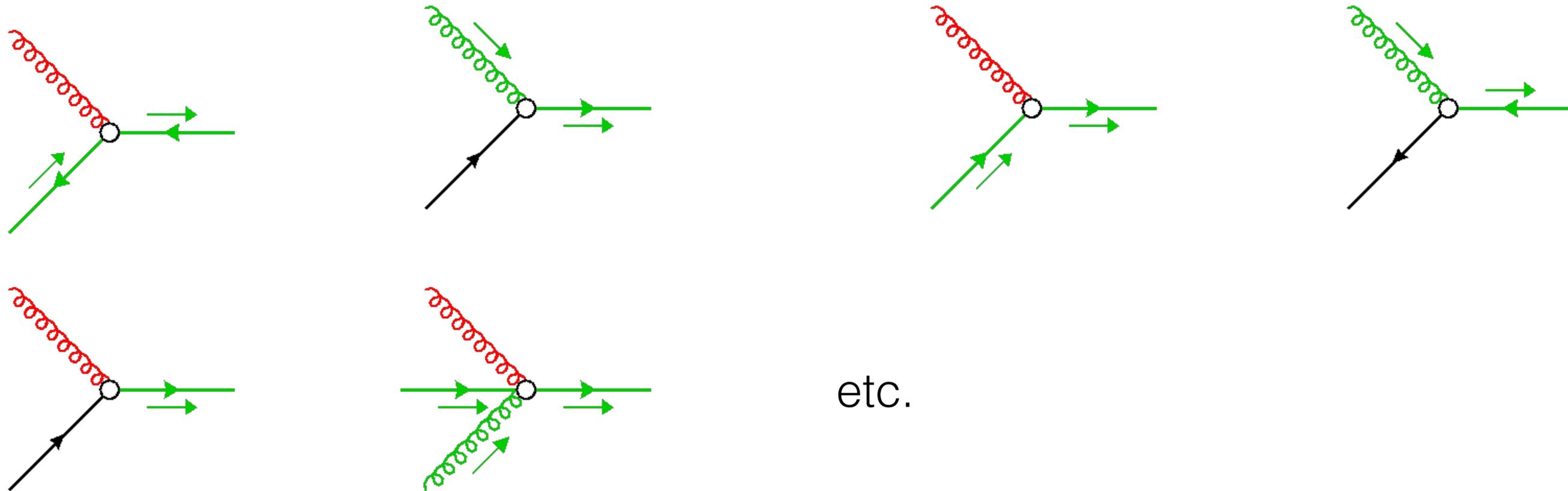


$$-ig f^{abc} \int_0^\infty ds \left( \delta_{\nu\rho} (r-q)_\mu + 2\delta_{\mu\nu} q_\rho - 2\delta_{\mu\rho} r_\nu + (\kappa - 1)(\delta_{\mu\rho} q_\nu - \delta_{\mu\nu} r_\rho) \right)$$

+ 4-gluon vertex

# Perturbative approach

$$\mathcal{L}_\chi \sim \int_0^\infty dt \bar{\lambda} (\partial_t - \mathcal{D}^2) \chi + \text{h.c.}$$



# Renormalization

bulk ( $t > 0$ ) is UV regulated  $\Rightarrow$   
renormalization of QCD parameters  
unaffected

renormalization of flowed fields:

$$B_{\mu}^{\text{R}}(t) = Z_B^{1/2} B_{\mu}(t)$$

$$\chi^{\text{R}}(t) = Z_{\chi}^{1/2} \chi(t)$$

$$Z_B = Z_g$$

Lüscher 2010

Lüscher, Weisz 2011

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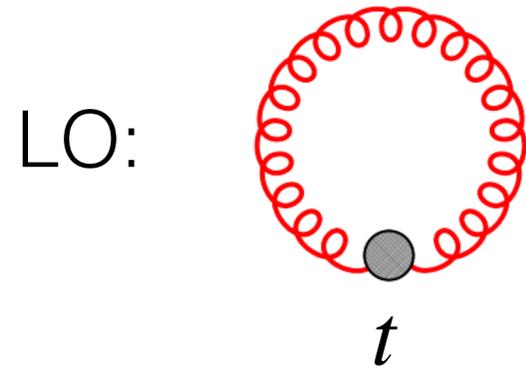
# Let's calculate

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$$E(t) \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$

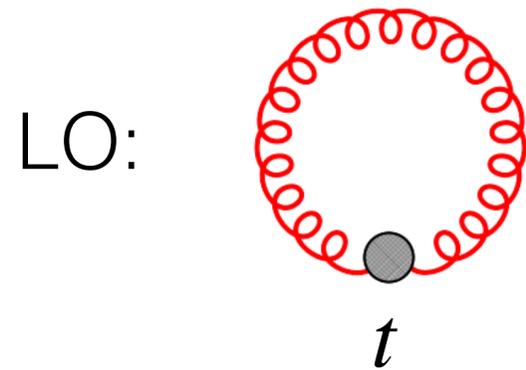
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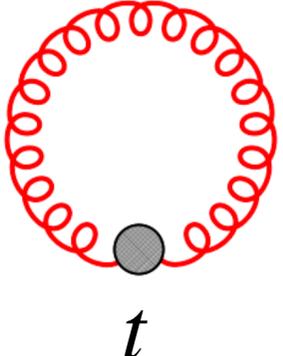
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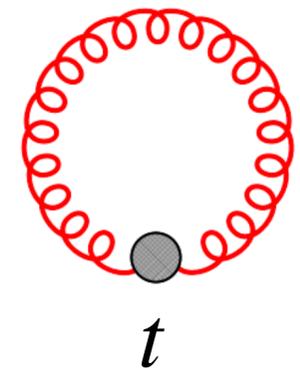
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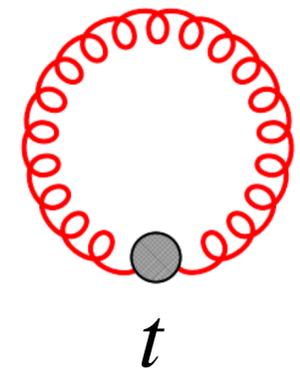
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$$E(t) = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

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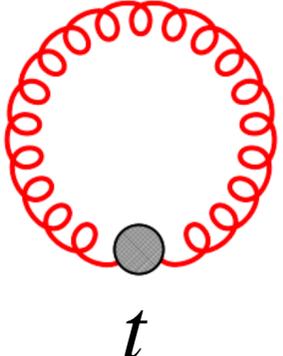
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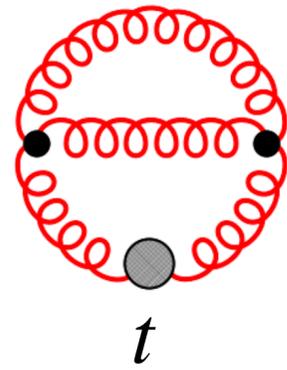
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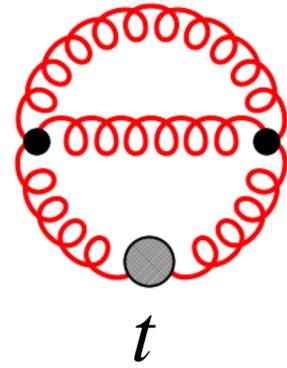
$$\alpha_s = \alpha_s(\mu)$$

# Higher orders

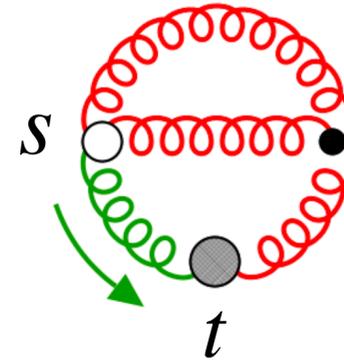


$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4 k^2 (p-k)^2}$$

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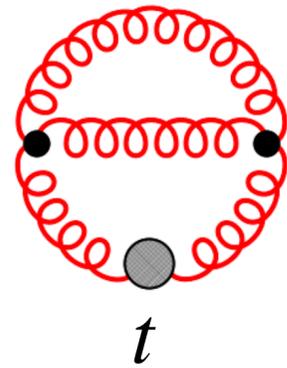


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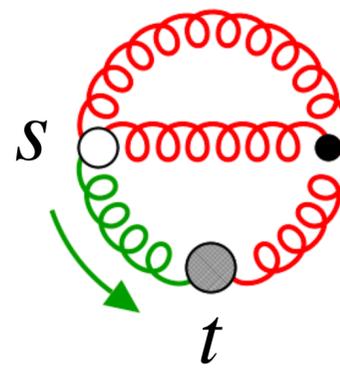


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

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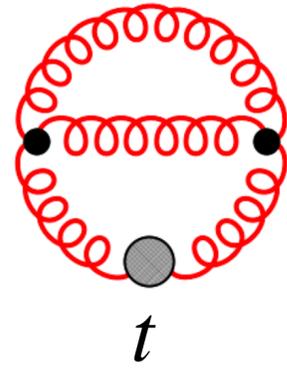
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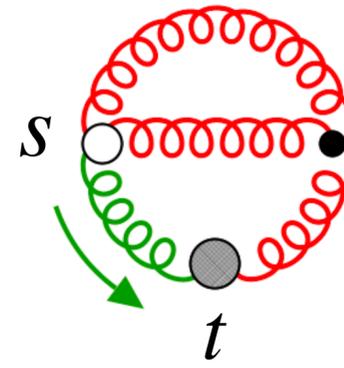
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- generalized loop integrals

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$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi} \left[ 1 + k_1(t, \mu) \alpha_s(\mu) \right]$$

Lüscher 2010

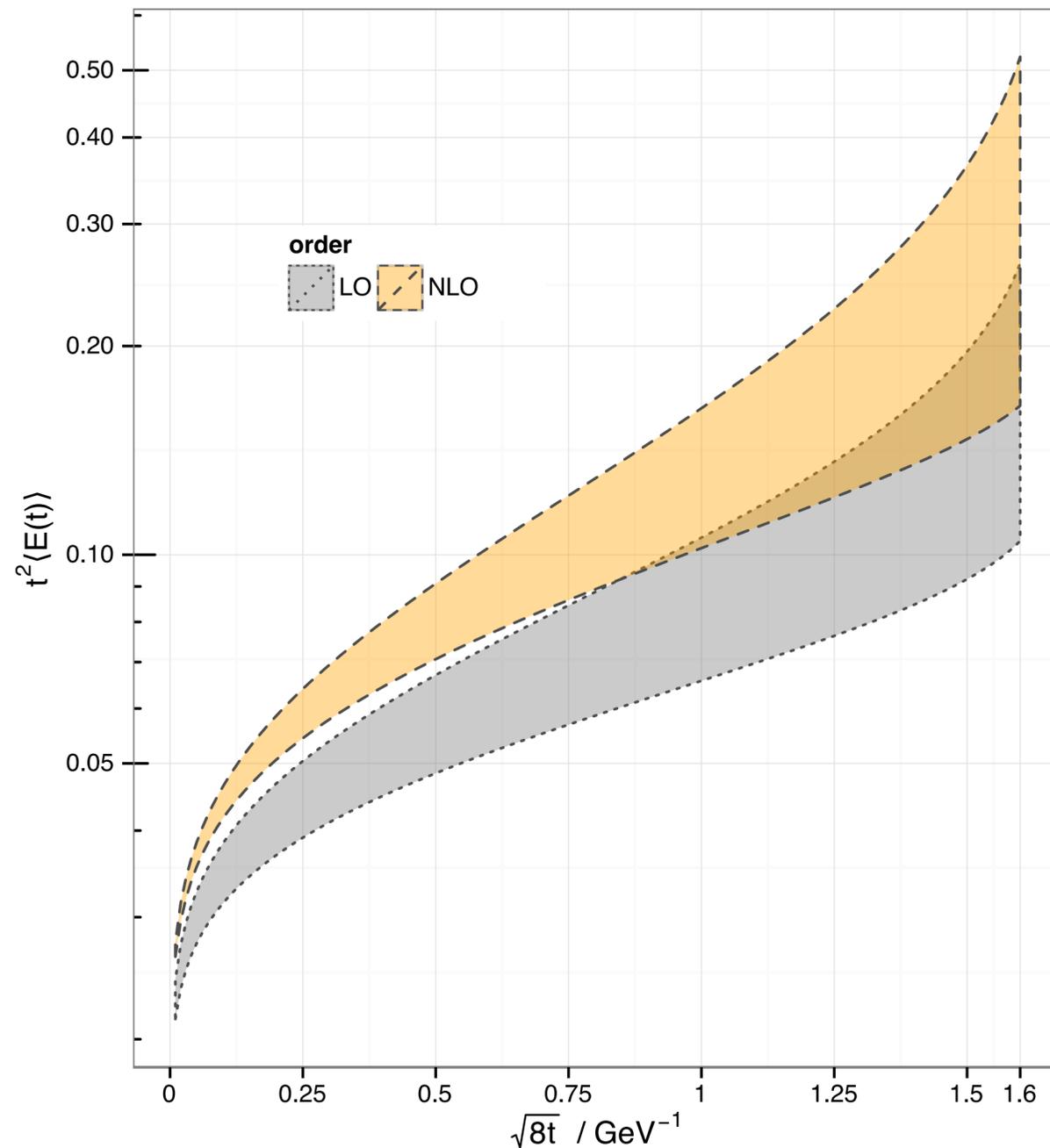
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$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative  
accuracy on  $\alpha_s$ :  $\pm 3-5\%$

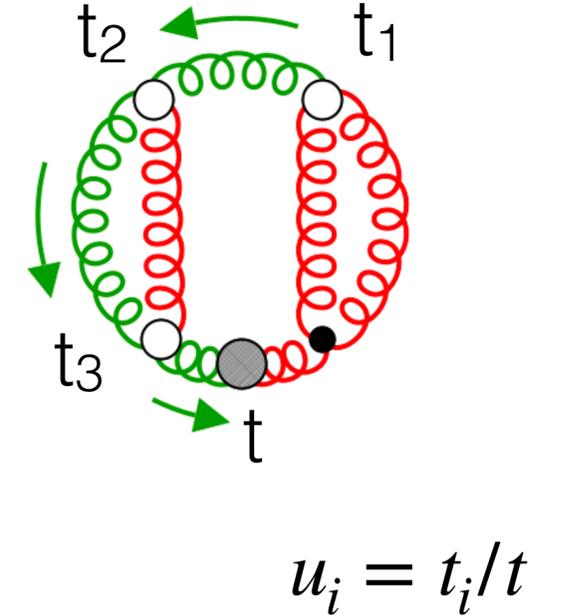
PDG:  $\pm 1\%$



# Three-loop calculation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\}) =$$

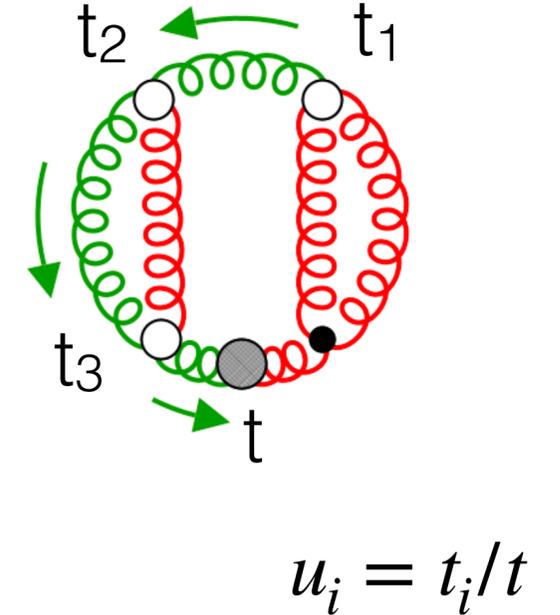
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IbP identities:  $\frac{\partial}{\partial p_i} \cdot p_j I(c, a, b) = D \delta_{ij} I(c, a, b) + \sum I(c', a, b')$

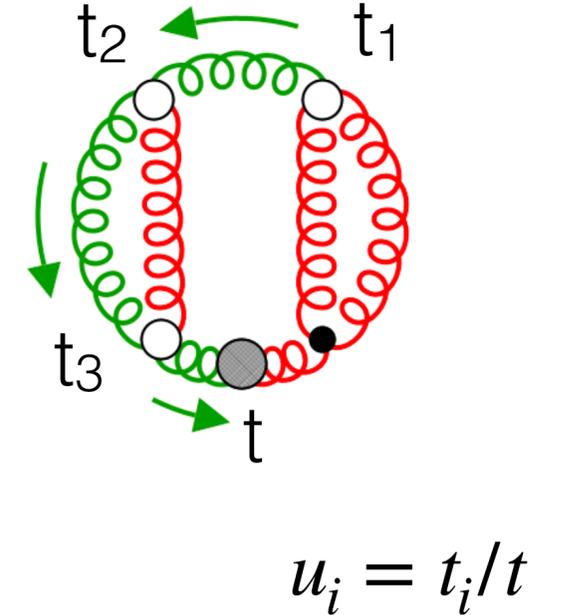
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Artz, RH, Lange, Neumann, Prausa 2019

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Artz, RH, Lange, Neumann, Prausa 2019

Huge systems of linear equations, solved by “master integrals”.

# The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation:

qgraf Nogueira 1993

diagram analyzation:

q2e/exp RH, Seidensticker, Steinhauser 1997

→ tapir/exp Gerlach, Herren, Lang 2022

algebraic manipulations:

FORM Vermaseren 2000, ...

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Kira ⊗ FireFly Usovitsch, Uwer, Maierhöfer 2017

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sector decomposition:

Binoth, Heinrich 2002

$$\int d^D k \int d^D p \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p)^2} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \dots$$

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$$c_1 = c_2 = 0$$

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$$b_1 = b_4 = 1$$

$$b_2 = b_3 = b_5 = b_6 = 0$$

$$m_1 = \dots = m_6 = 0$$

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ftint RH, Nellopoulos, Olsson, Wesle '24  
(based on pySecDec)  
Heinrich, Magerya, Kerner, Jones, ...

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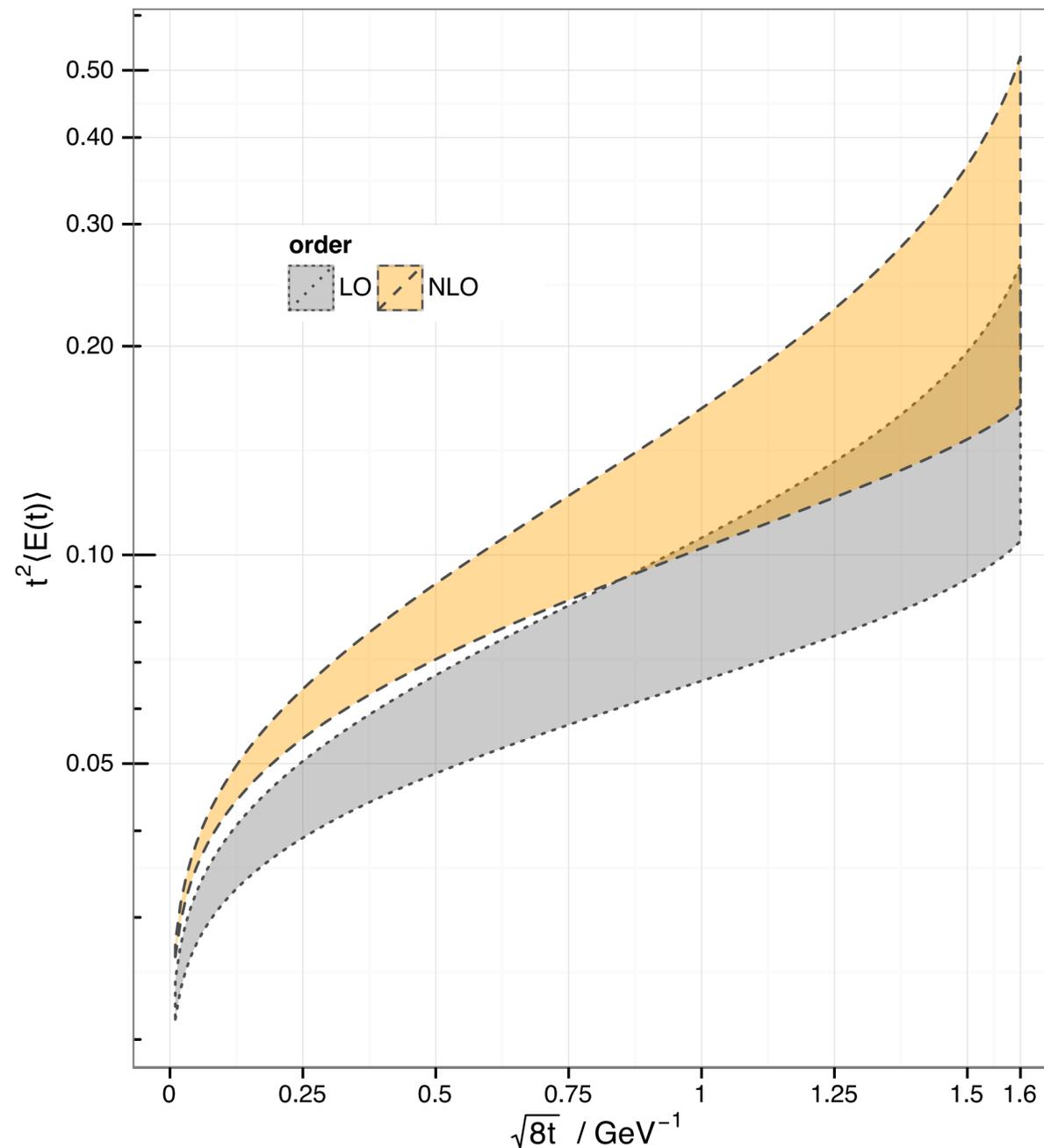
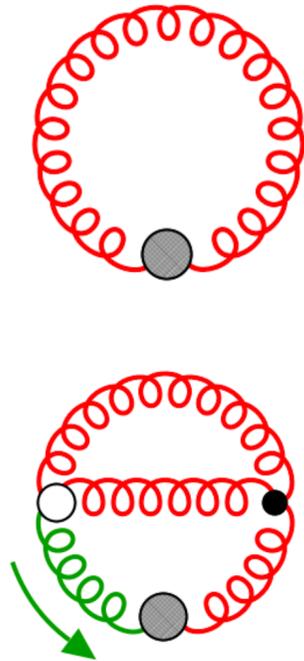
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```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}},{1,0,0,1,0,0}] -> (
+eps^-1*(+8.33333333333333343*10^-02+0.000000000000000000*10^+00*I)
+eps^-1*(+1.4433895444086145*10^-15+0.000000000000000000*10^+00*I)*plusminus
+eps^0*(+3.0238270284562663*10^-01+0.000000000000000000*10^+00*I)
+eps^0*(+1.6918362746499228*10^-08+0.000000000000000000*10^+00*I)*plusminus
+eps^1*(+6.5531010458012129*10^-01+0.000000000000000000*10^+00*I)
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),
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Lüscher 2010



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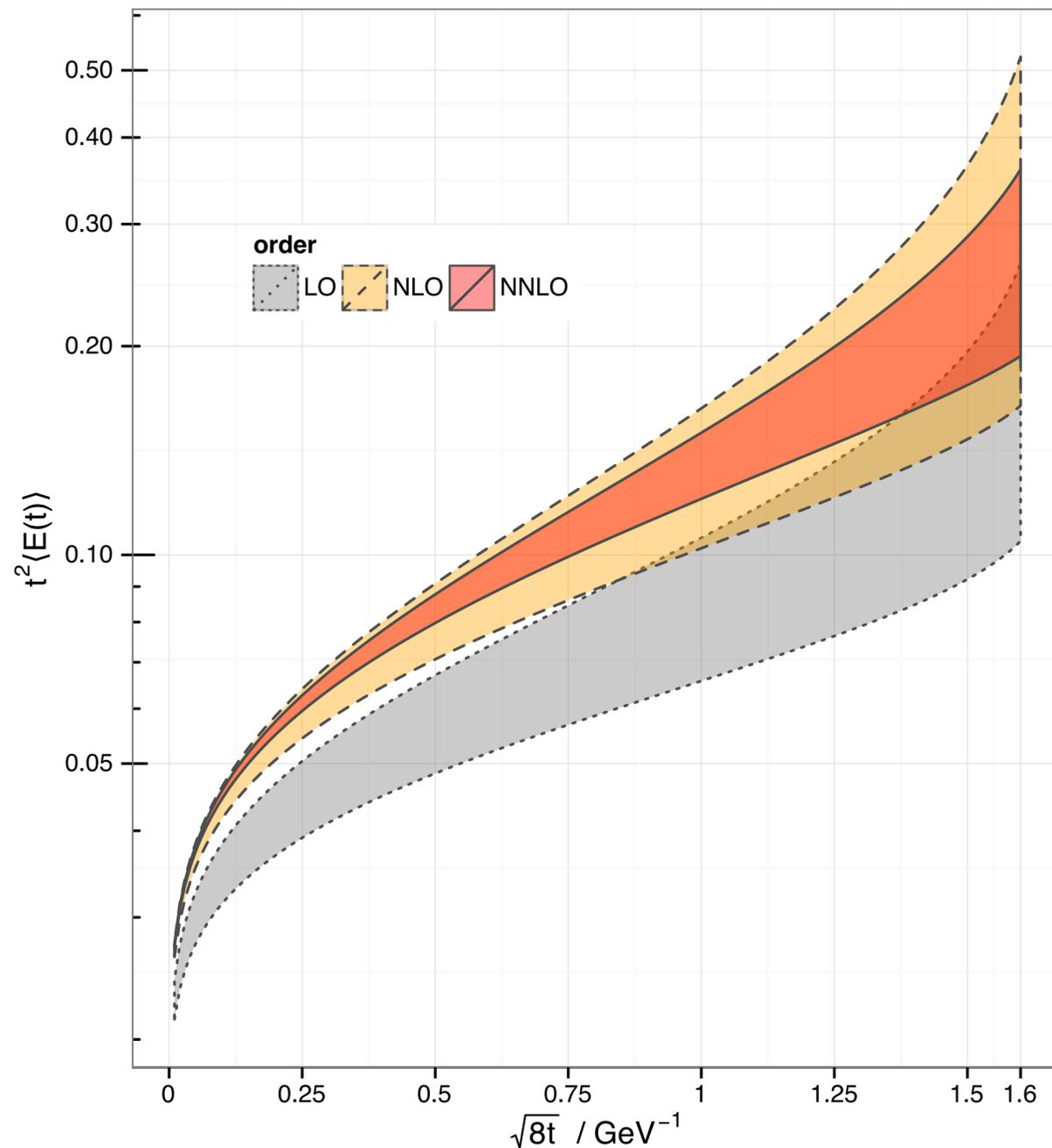
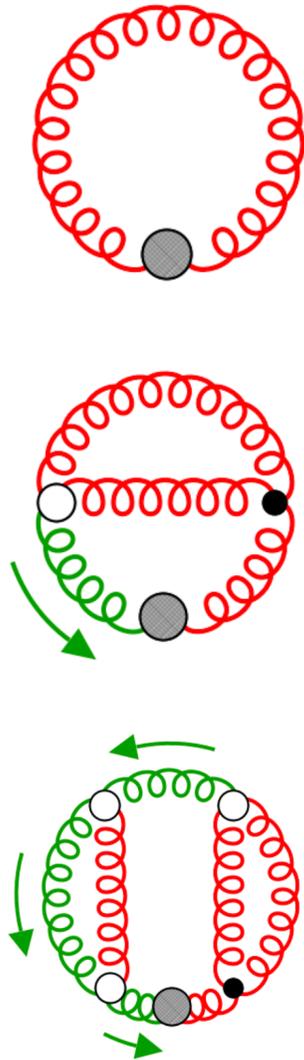
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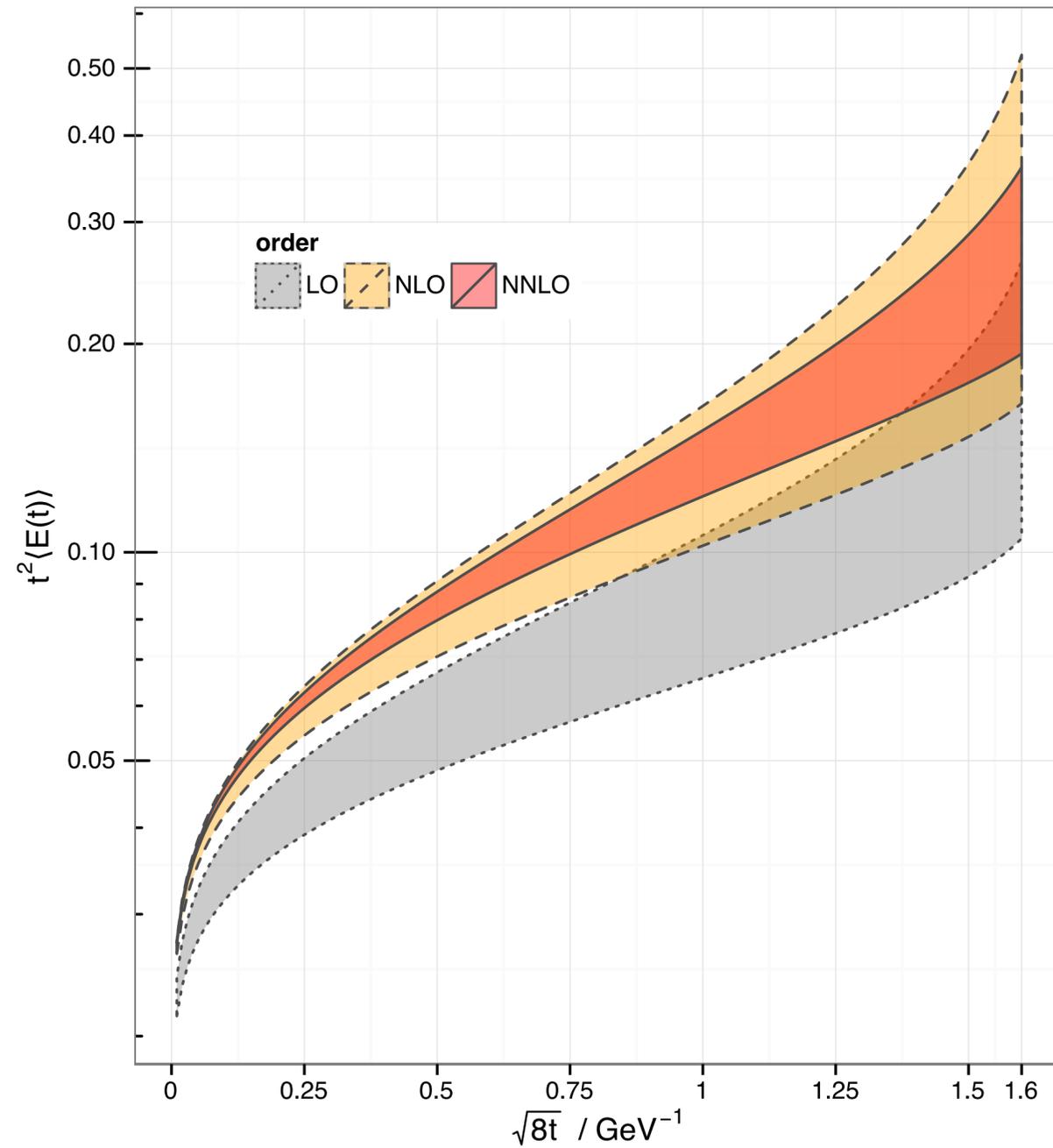
RH, Neumann 2016



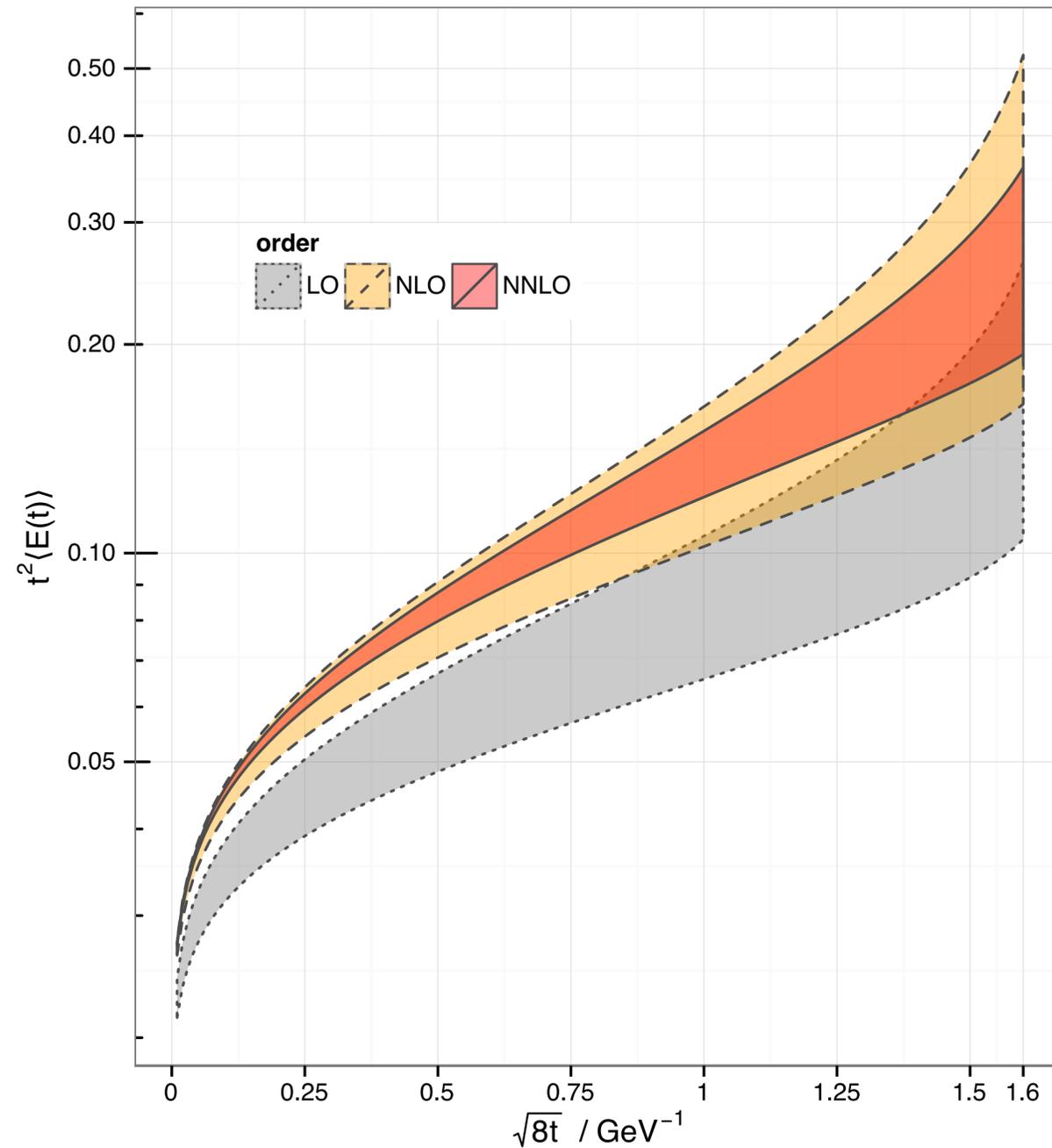
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accuracy on  $\alpha_s$ :  $O(1\%)$

PDG:  $\pm 1\%$

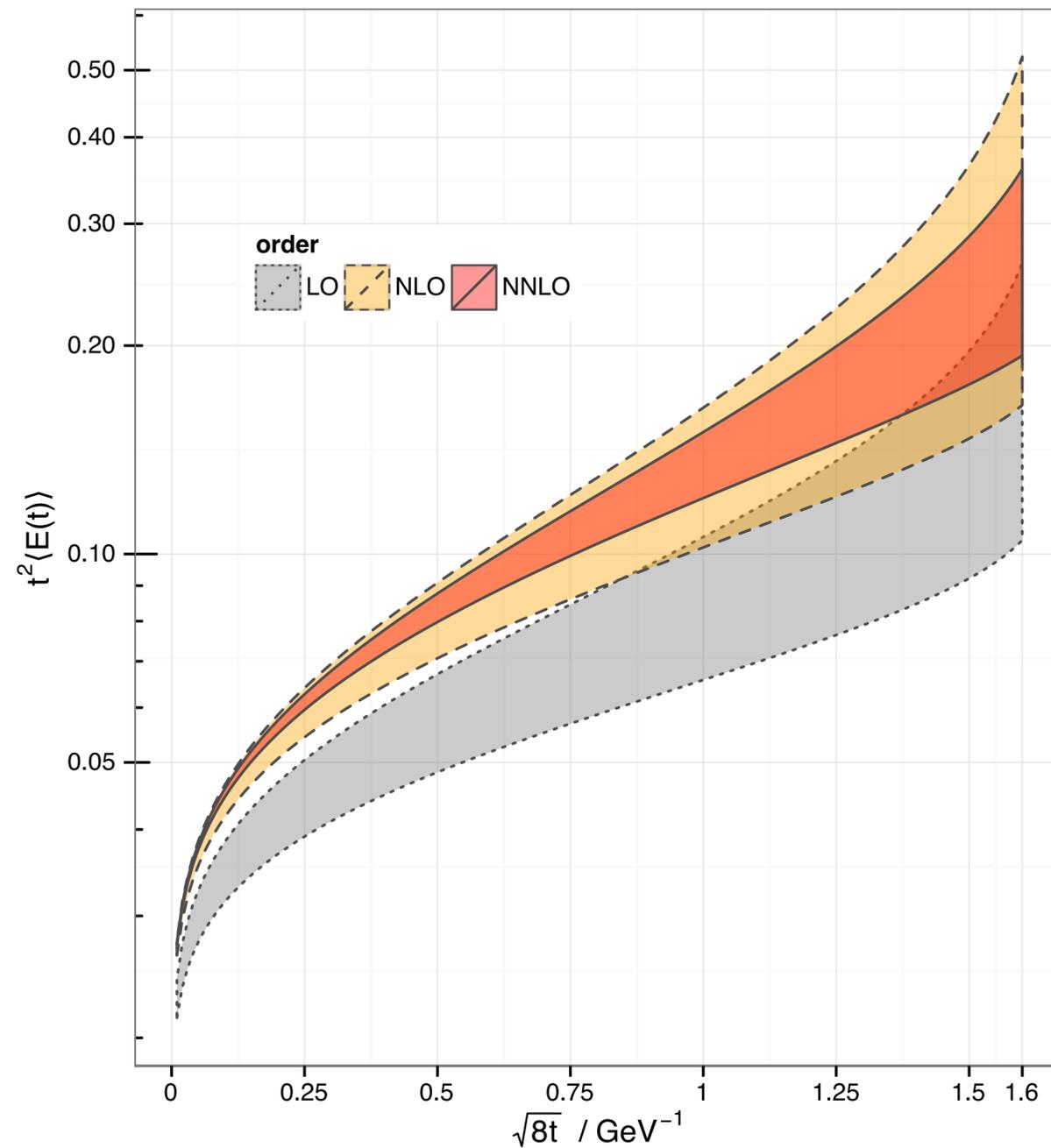
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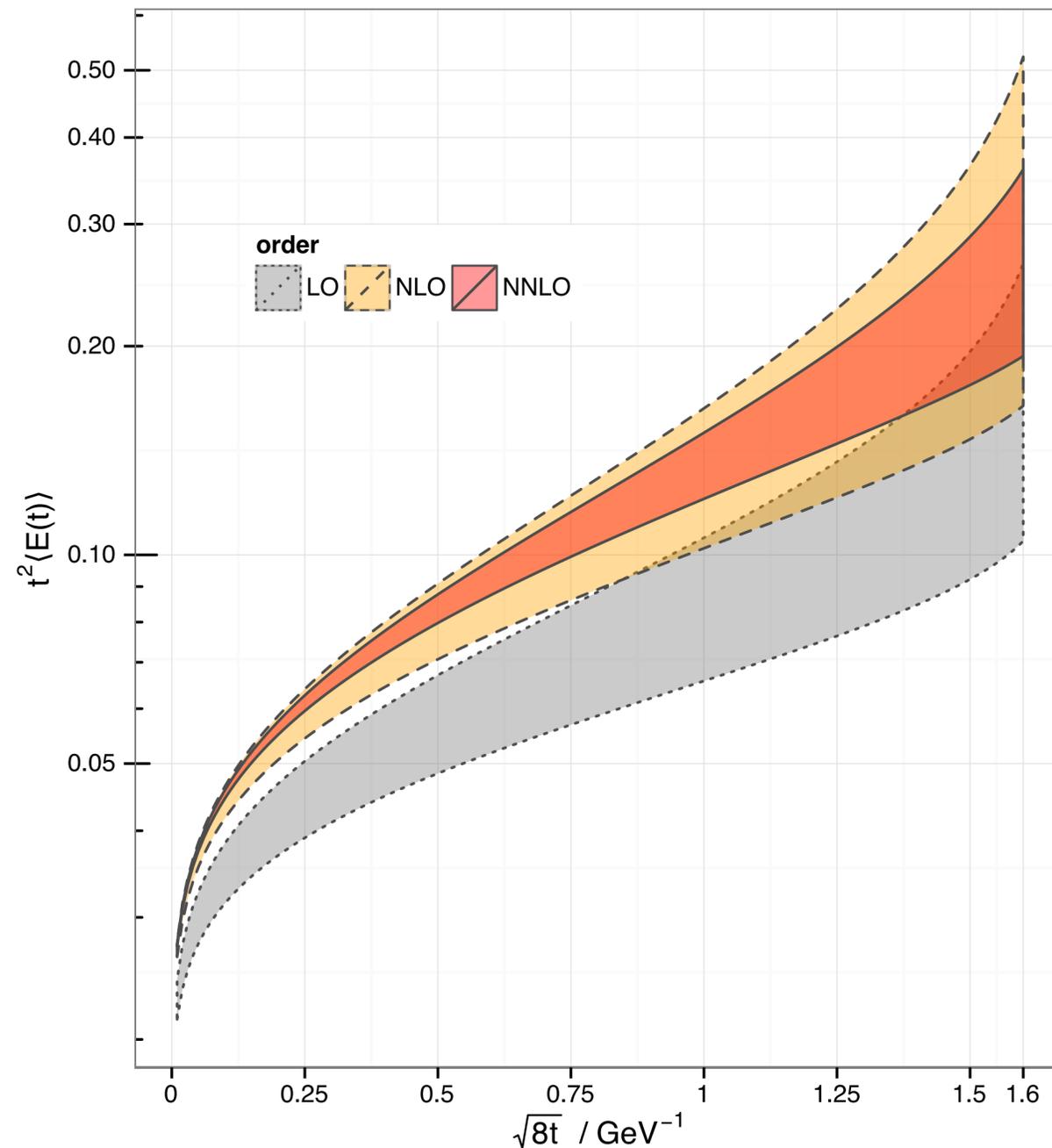


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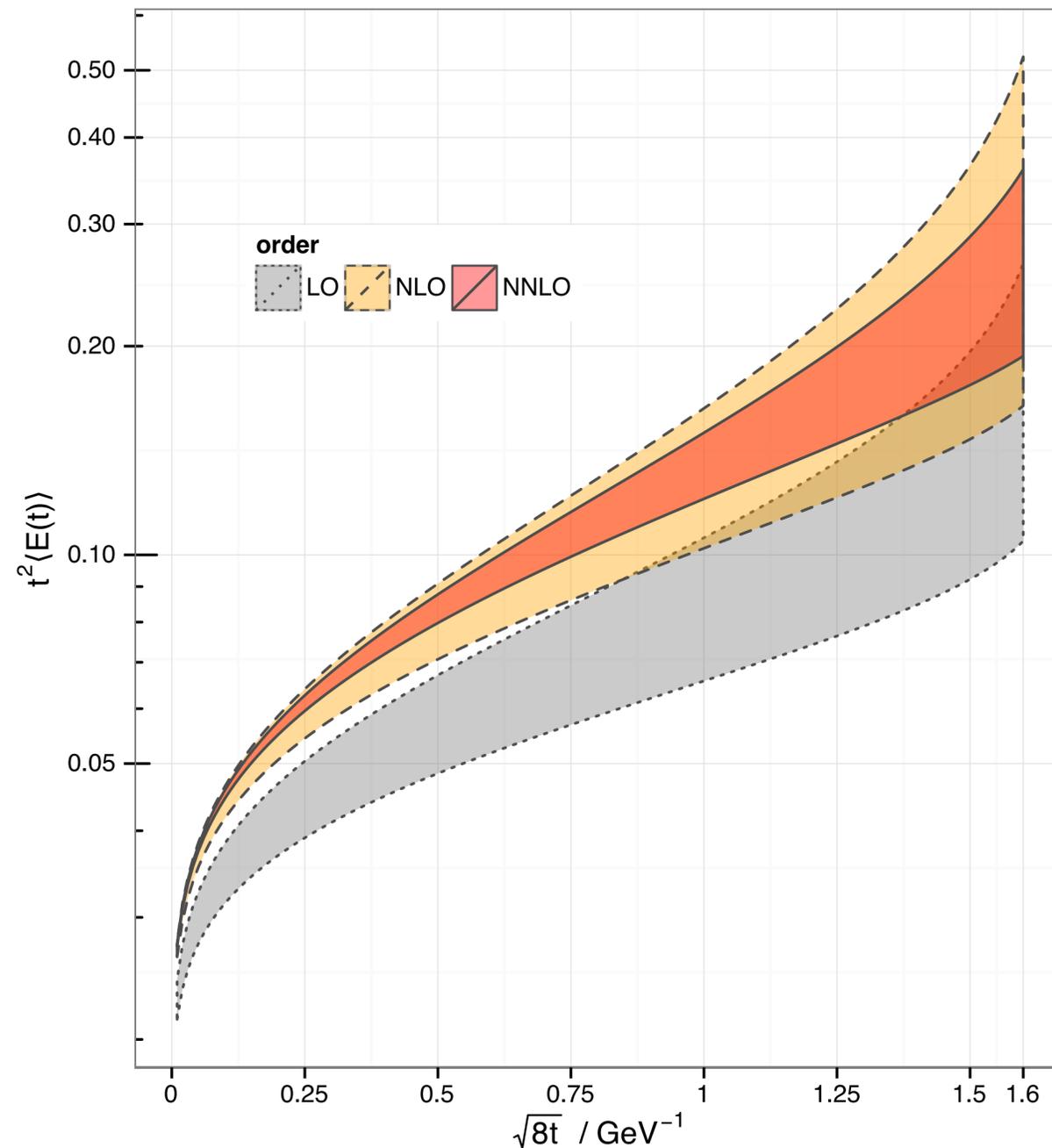
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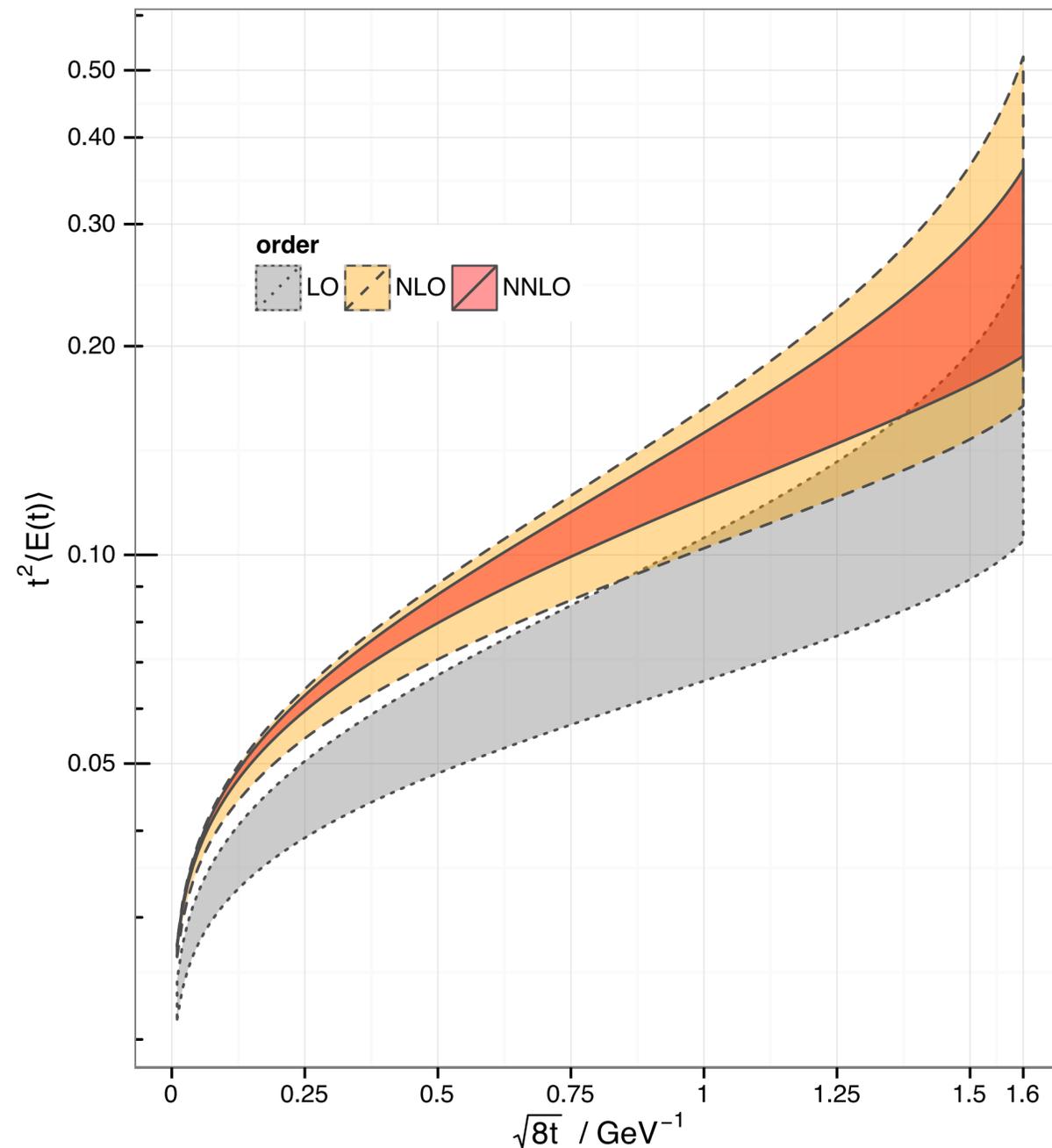


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↑      ↗  
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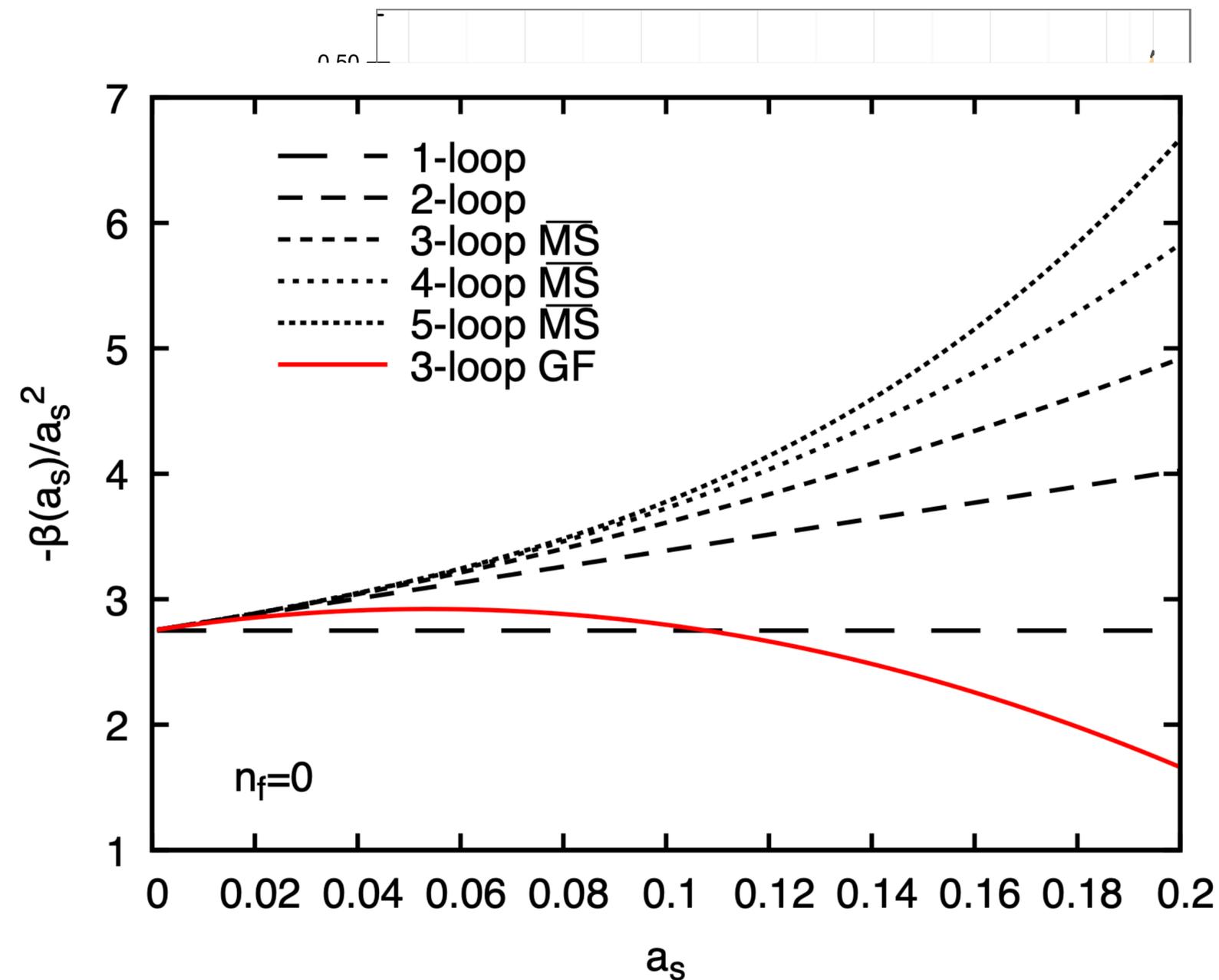
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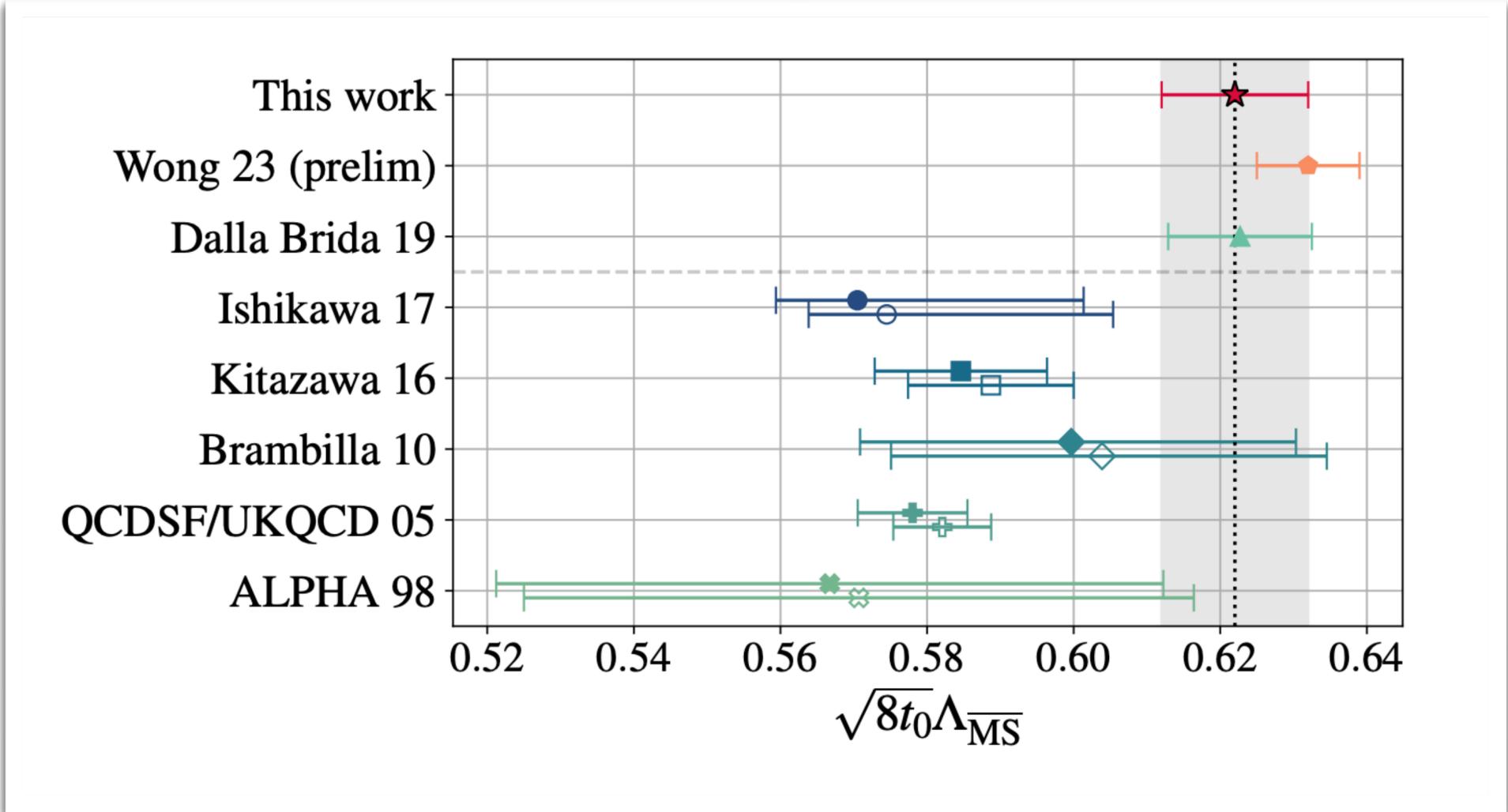
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# Determine $\Lambda_{\text{QCD}}$



Hasenfratz, Peterson, van Sickle, Witzel (2023)

see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

# Application to effective field theories

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation  
theory

lattice

match  
renormalization  
schemes?

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Instead:

$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

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$\rightarrow$  need  $\tilde{C}(t)$

# Short flow-time expansion

---

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short flow-time expansion:

Lüscher, Weisz '11

Suzuki '13

Lüscher '13

$$\tilde{\mathcal{O}}_n(t) \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \mathcal{O}_m$$

$$\tilde{C}_n(t) \xrightarrow{t \rightarrow 0} \sum_m C_m \zeta_{mn}^{-1}(t)$$

$\Rightarrow$  need  $\zeta_{nm}(t)$  for small  $t$   $\Rightarrow$  perturbation theory

# Determining $\zeta(t)$

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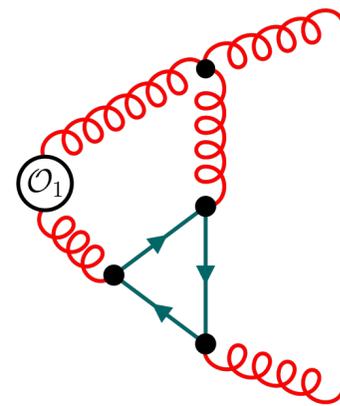
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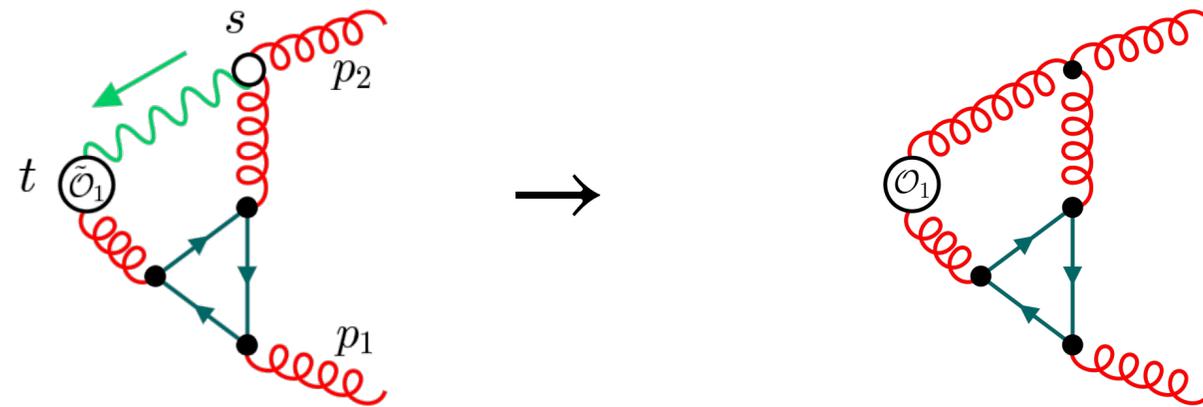
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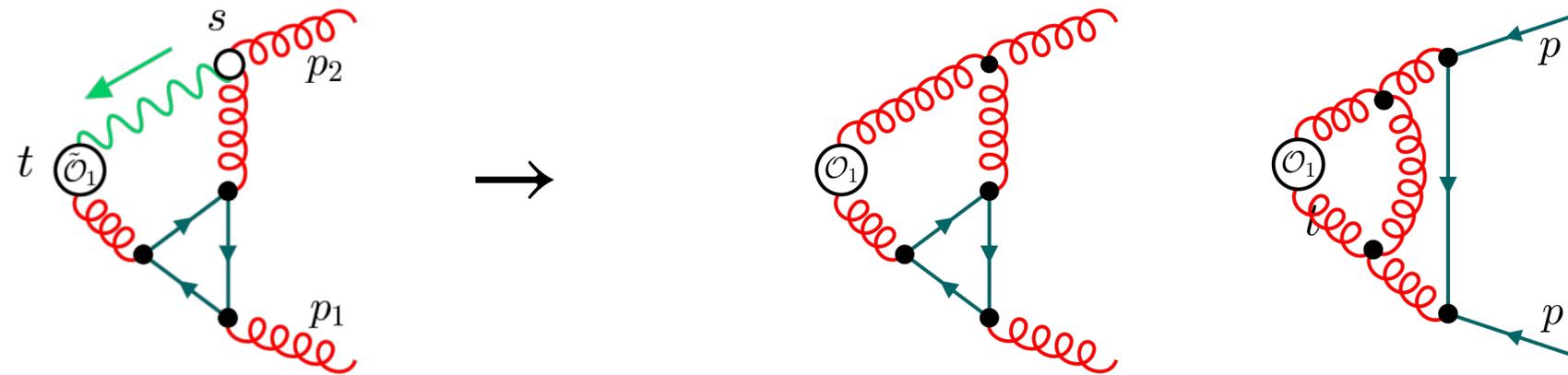
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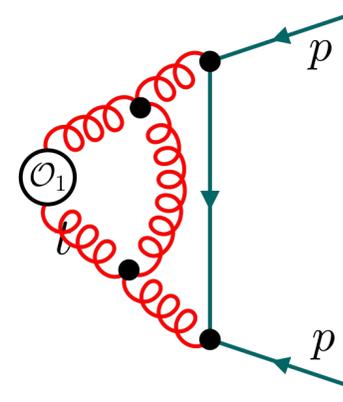
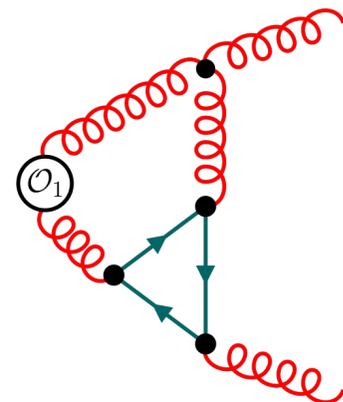
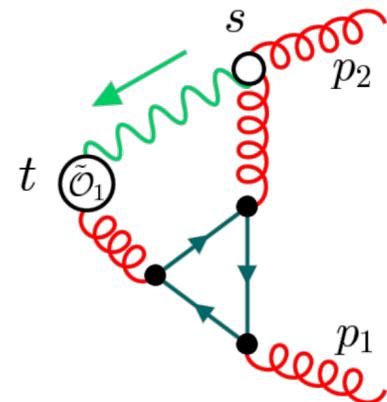
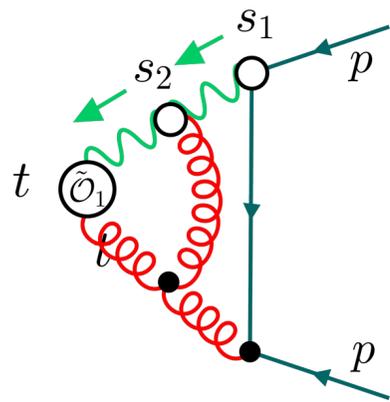
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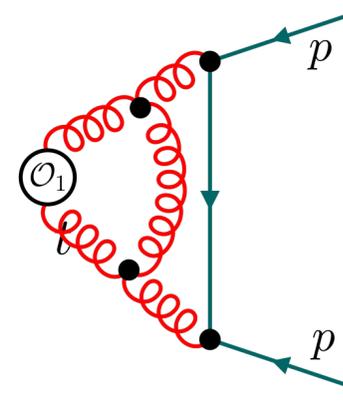
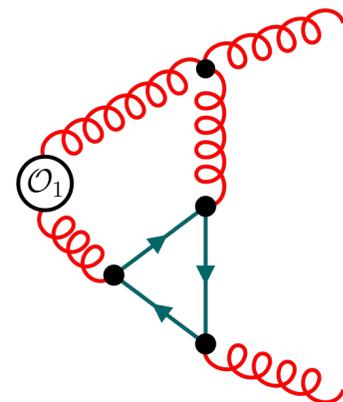
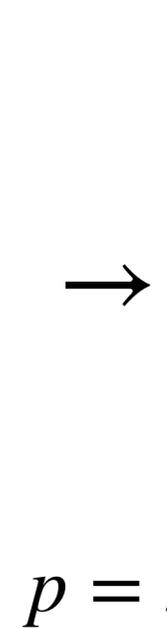
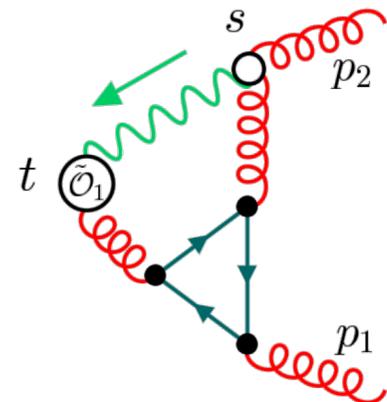
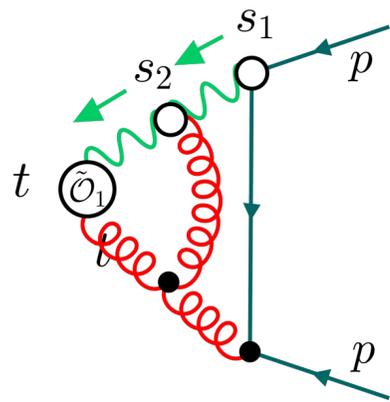
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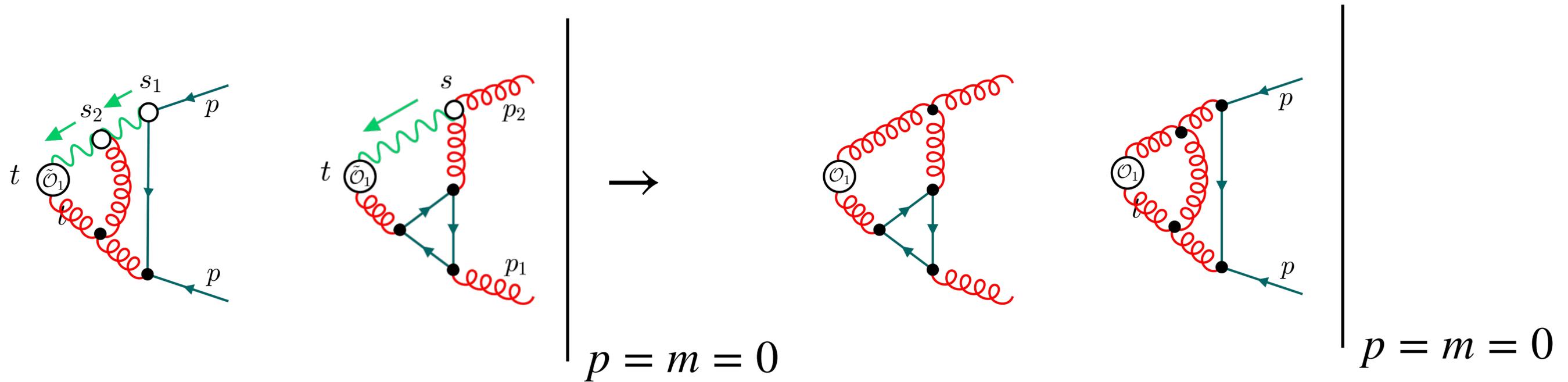
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Matching: calculate a set of suitable Green's functions and solve for  $\zeta_{nm}(t)$

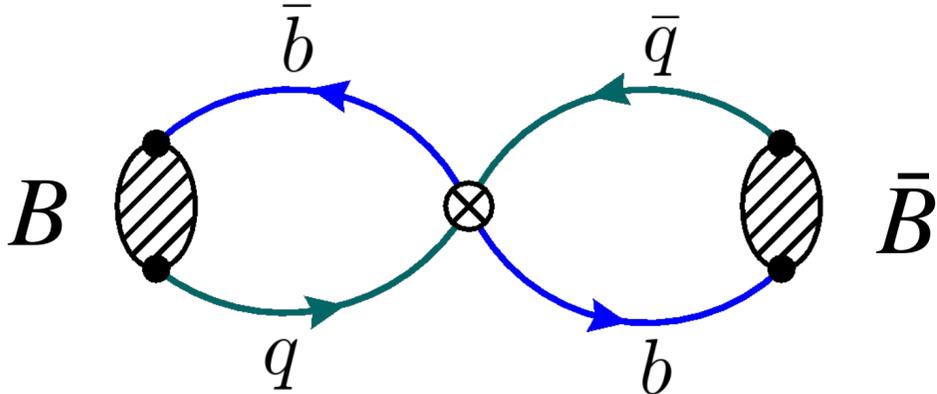
$$\langle \tilde{\mathcal{O}}_n(t) \rangle \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \langle \mathcal{O}_m \rangle$$



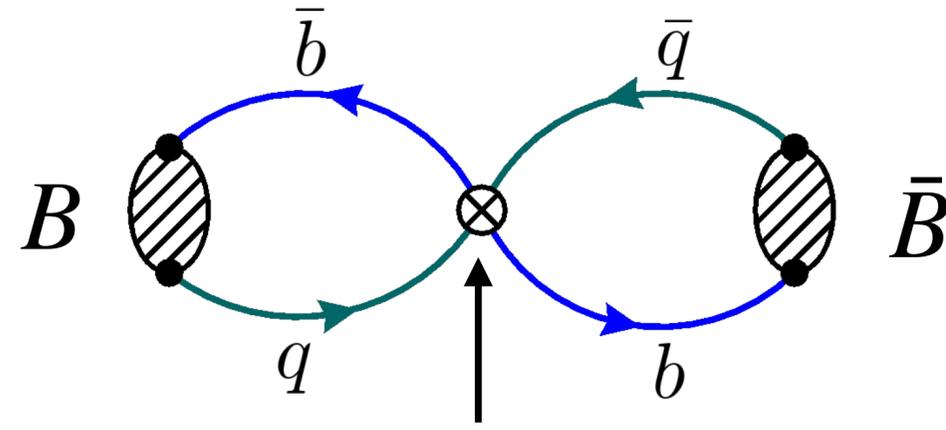
only tree-level diagrams survive on r.h.s.

Gorishnii, Larin, Tkachov '83

# Example: Meson mixing

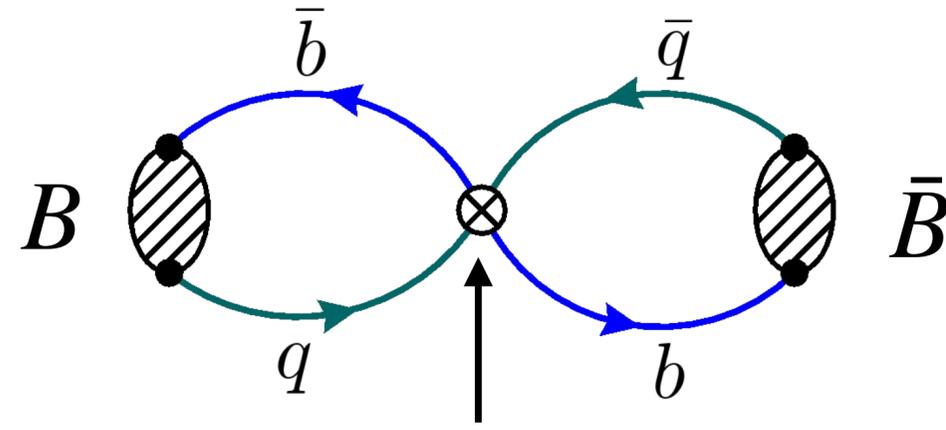


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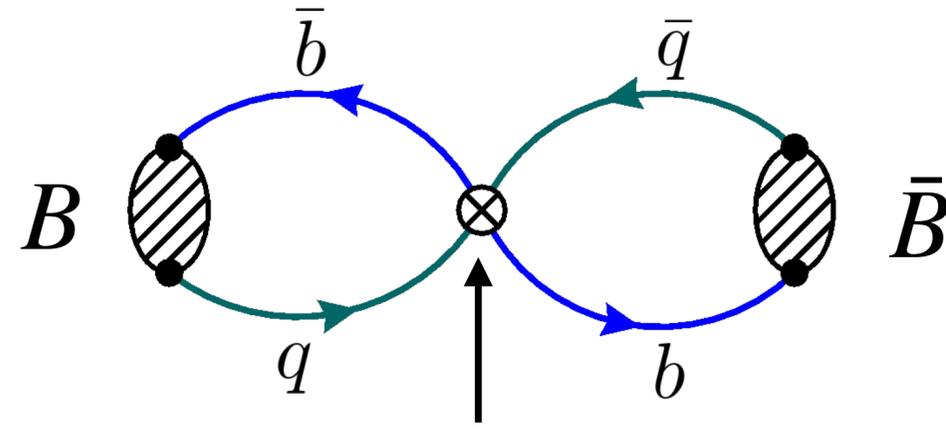
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# Example: Meson mixing



$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}_n \equiv \sum_n \tilde{C}(t)_n \tilde{\mathcal{O}}(t)_n$$

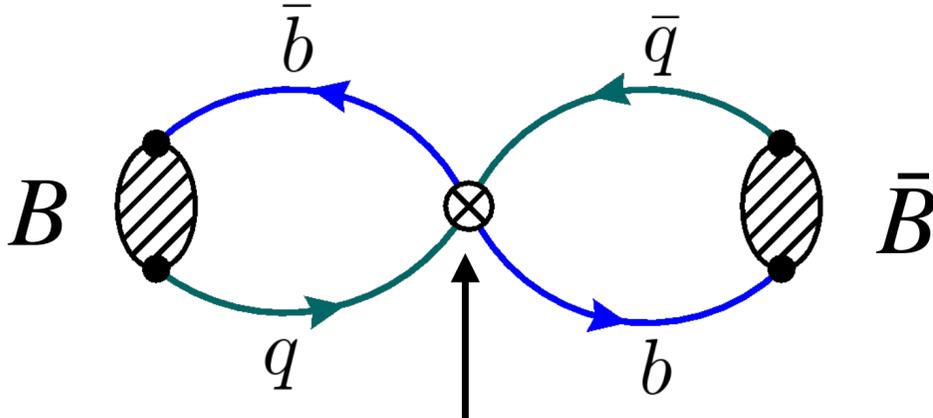
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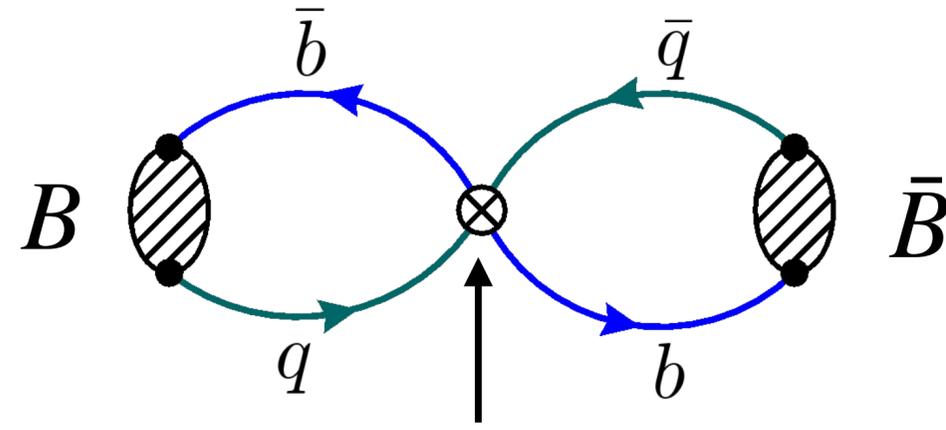
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# Example: Meson mixing

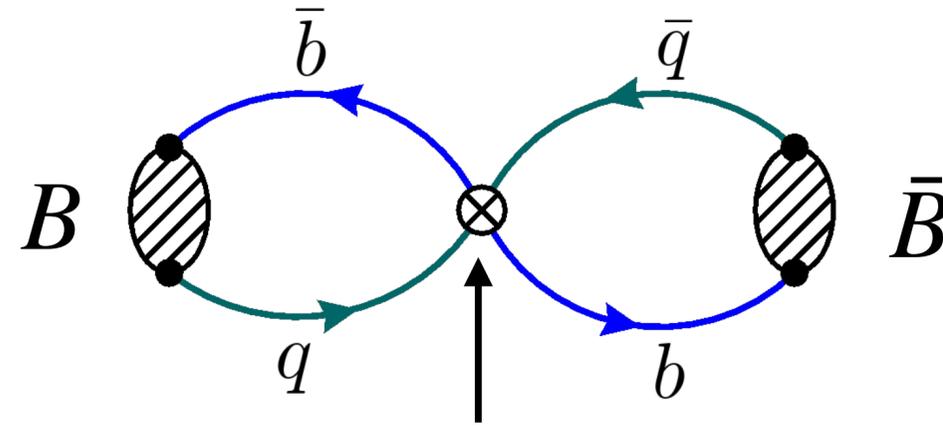


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# Example: Meson mixing



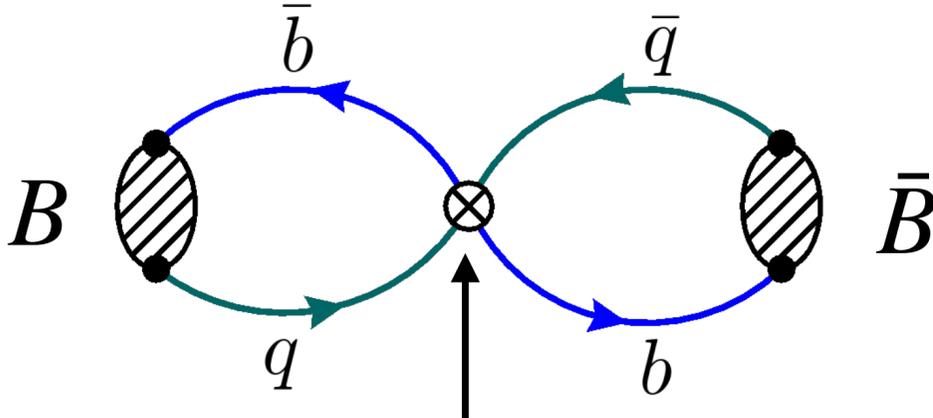
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perturbative  $\nearrow$

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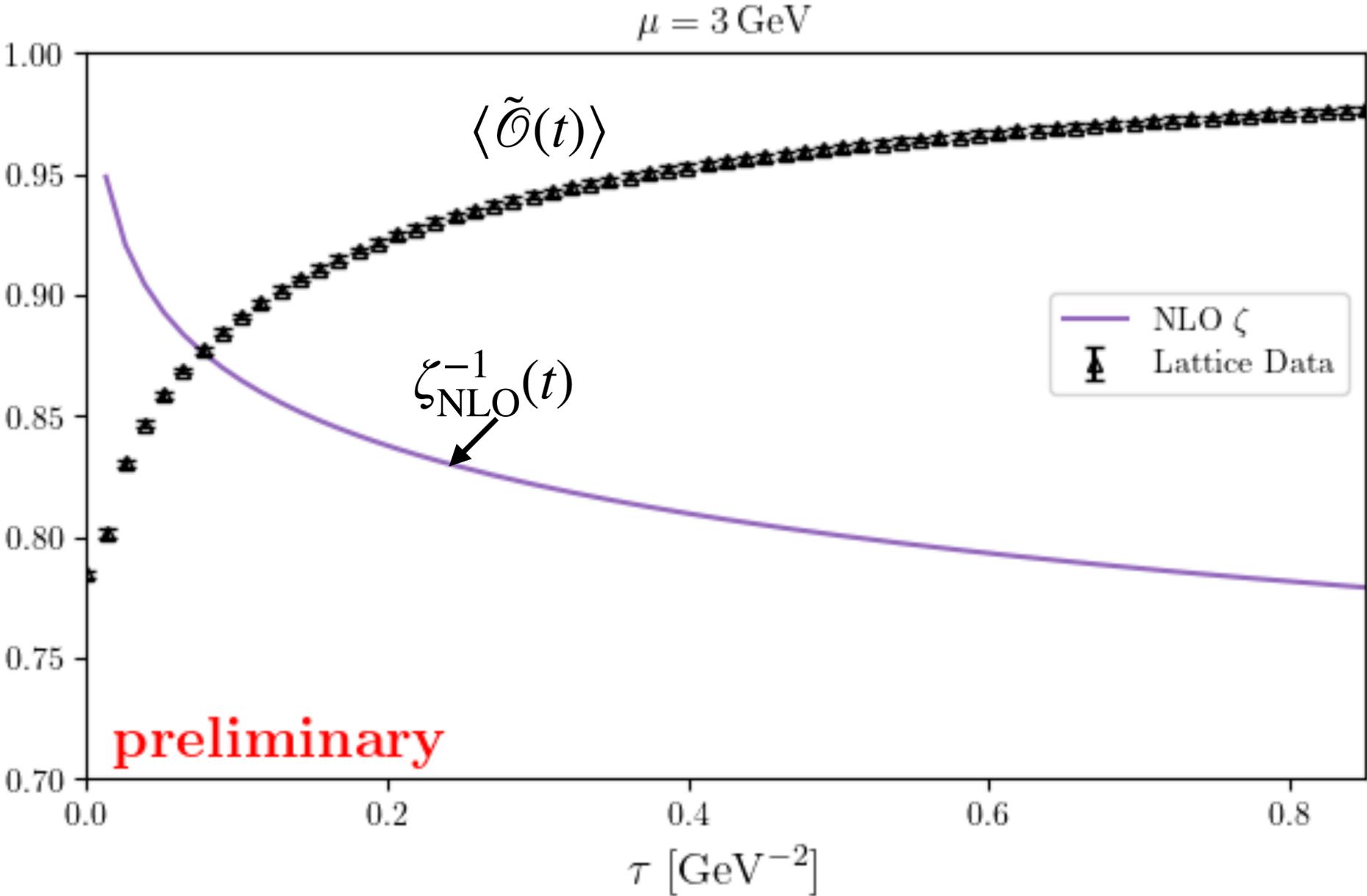
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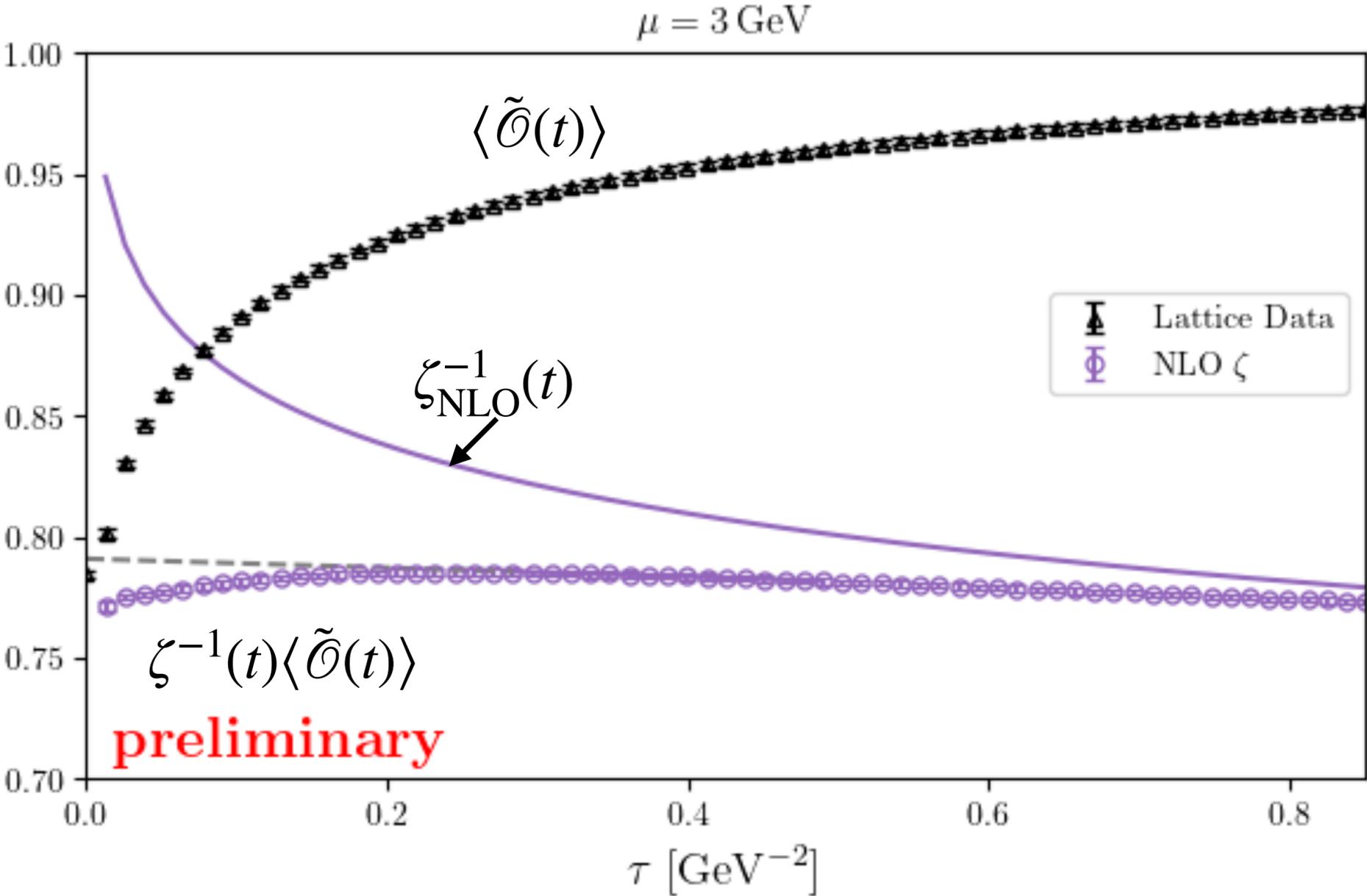
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perturbative
lattice

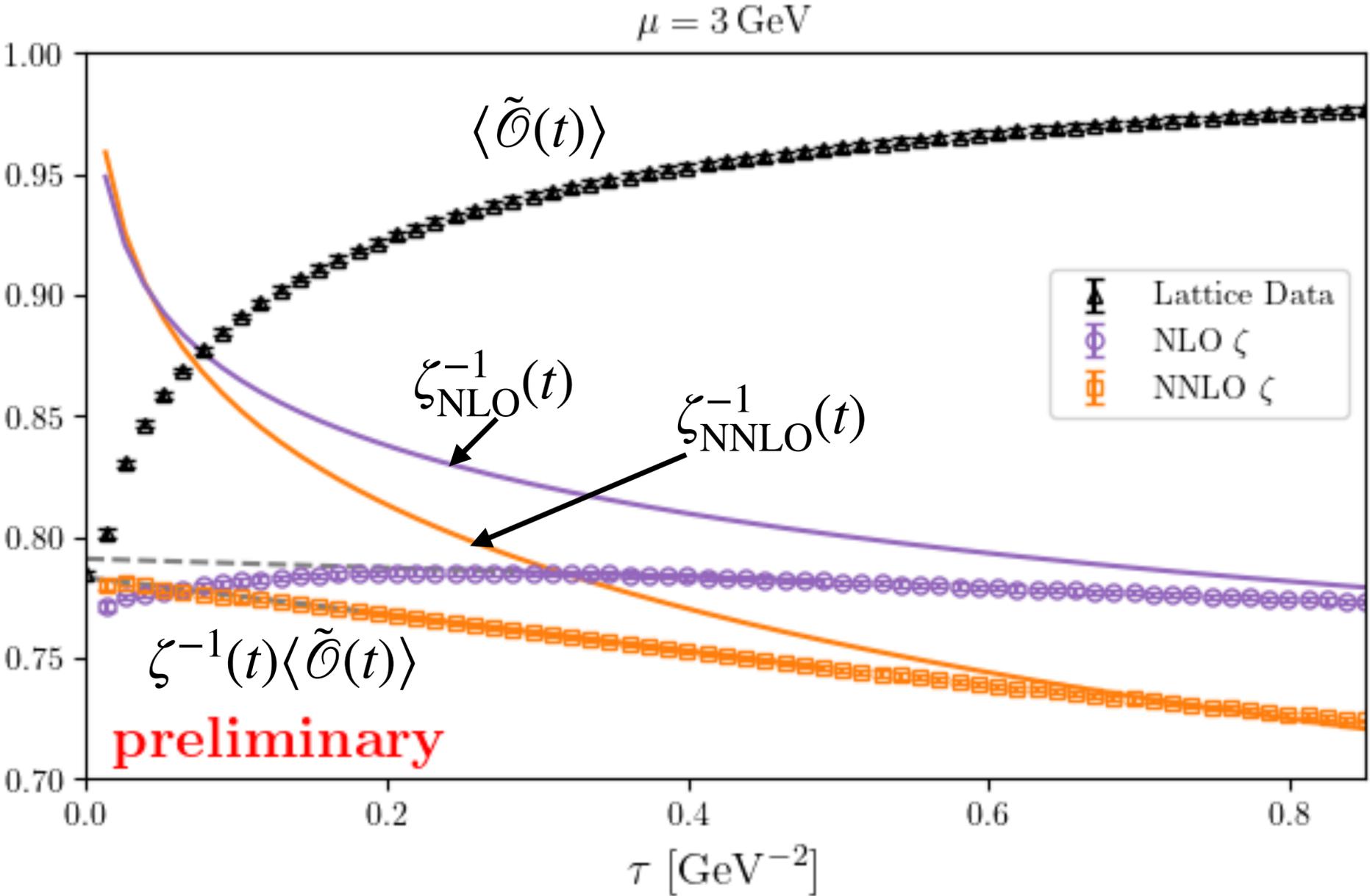
# Bag parameter



# Bag parameter



# Bag parameter



Black, RH, Lange, Rago, Shindler, Witzel (2023)



# Application to effective field theories

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead:

$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$  is UV finite  $\Rightarrow \lim_{a \rightarrow 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$  exists!

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- application: energy-momentum tensor Suzuki '13
- parton density functions Shindler '24

# Energy-momentum tensor

---

$$T_{\mu\nu}(x) = \frac{2}{|g(x)|^{1/2}} \frac{\delta S}{\delta g^{\mu\nu}(x)}$$

here:  $S = S_{\text{QCD}}$  and  $g^{\mu\nu} = \text{flat metric}$

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Noether current of space-time translations

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Noether current of space-time translations

→ ill-defined on the lattice!

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$$T_{\mu\nu}(x) = \sum_{n=1}^4 c_n(t) \tilde{\mathcal{O}}_{n,\mu\nu}(t, x)$$

idea and NLO result: H. Suzuki '14

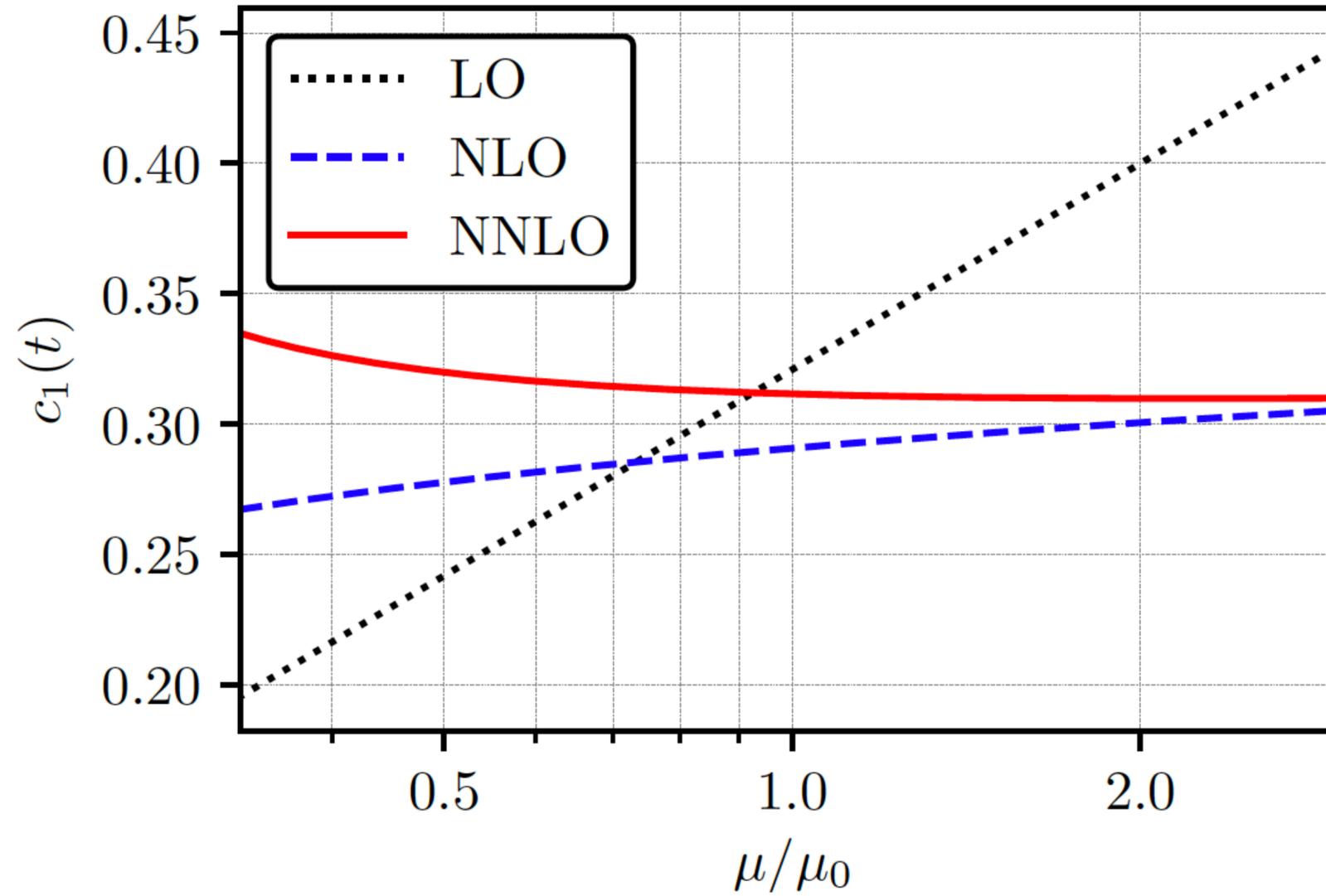
# NNLO result

$$\begin{aligned} c_1(t) = & \frac{1}{g^2} \left\{ 1 + \frac{g^2}{(4\pi)^2} \left[ -\frac{7}{3}C_A + \frac{3}{2}T_F - \beta_0 L(\mu, t) \right] \right. \\ & + \frac{g^4}{(4\pi)^4} \left[ -\beta_1 L(\mu, t) + C_A^2 \left( -\frac{14482}{405} - \frac{16546}{135} \ln 2 + \frac{1187}{10} \ln 3 \right) \right. \\ & + C_A T_F \left( \frac{59}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{10873}{810} + \frac{73}{54} \pi^2 - \frac{2773}{135} \ln 2 + \frac{302}{45} \ln 3 \right) \\ & \left. \left. + C_F T_F \left( -\frac{256}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{2587}{108} - \frac{7}{9} \pi^2 - \frac{106}{9} \ln 2 - \frac{161}{18} \ln 3 \right) \right] \right. \\ & \left. + \mathcal{O}(g^6) \right\}, \quad L(\mu, t) \equiv \ln(2\mu^2 t) + \gamma_E \end{aligned}$$

etc.

RH, Kluth, Lange '18

$\mu_0 = 3 \text{ GeV}$

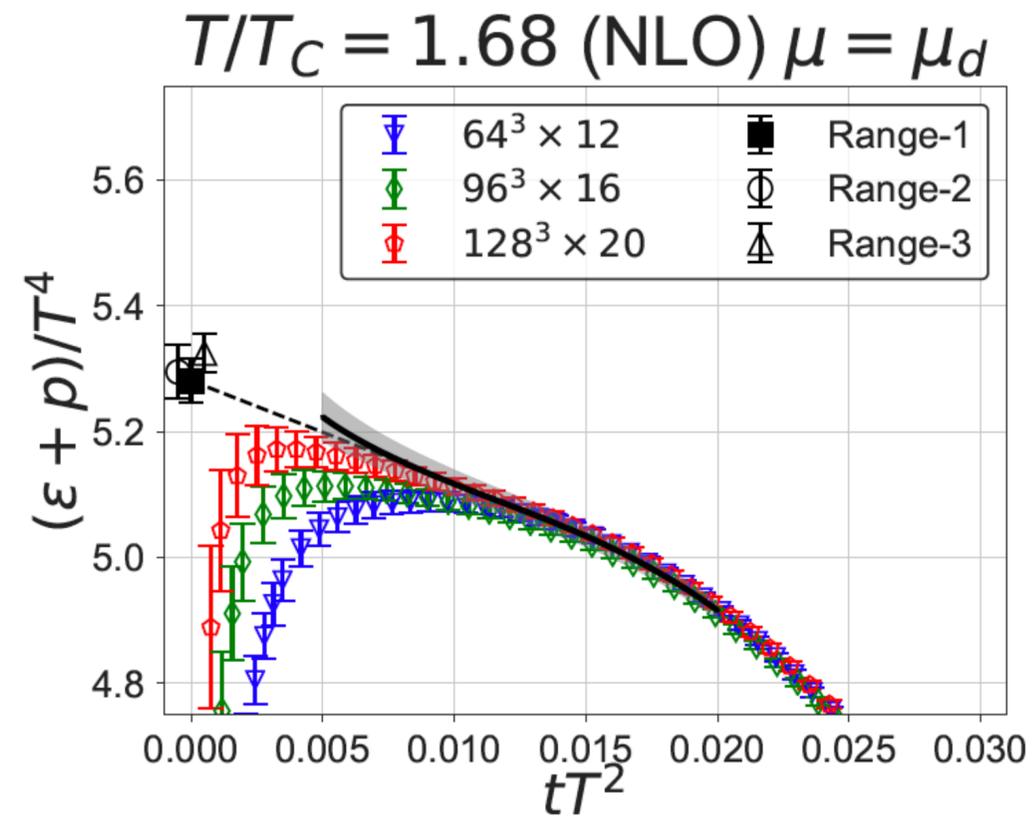
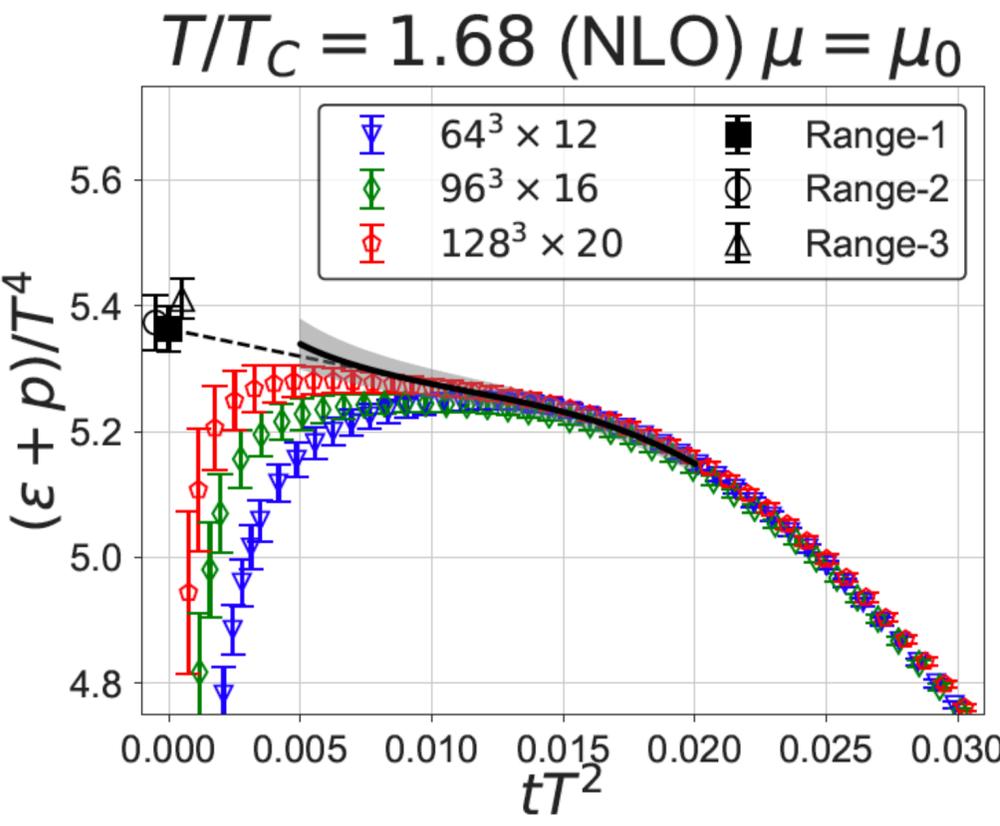


# Application

Entropy density:  $\varepsilon + p = -\frac{4}{3} \left\langle T_{00}(x) - \frac{1}{4} T_{\mu\mu}(x) \right\rangle$

$\mu_0 = \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$

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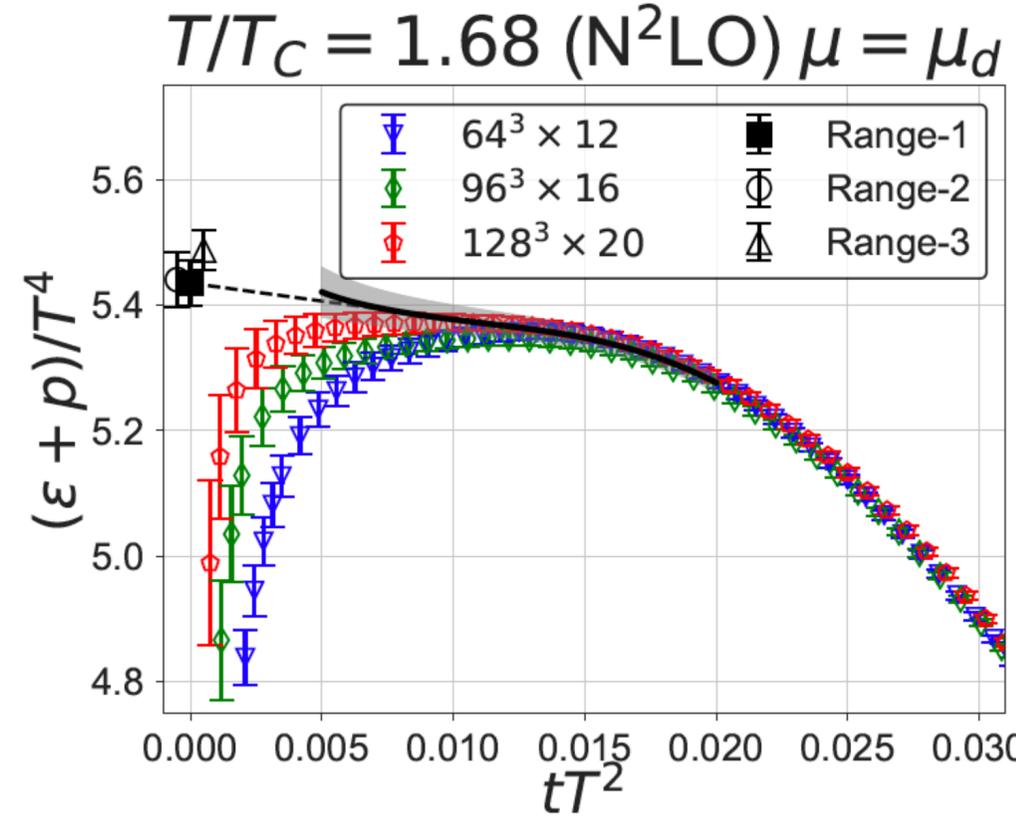
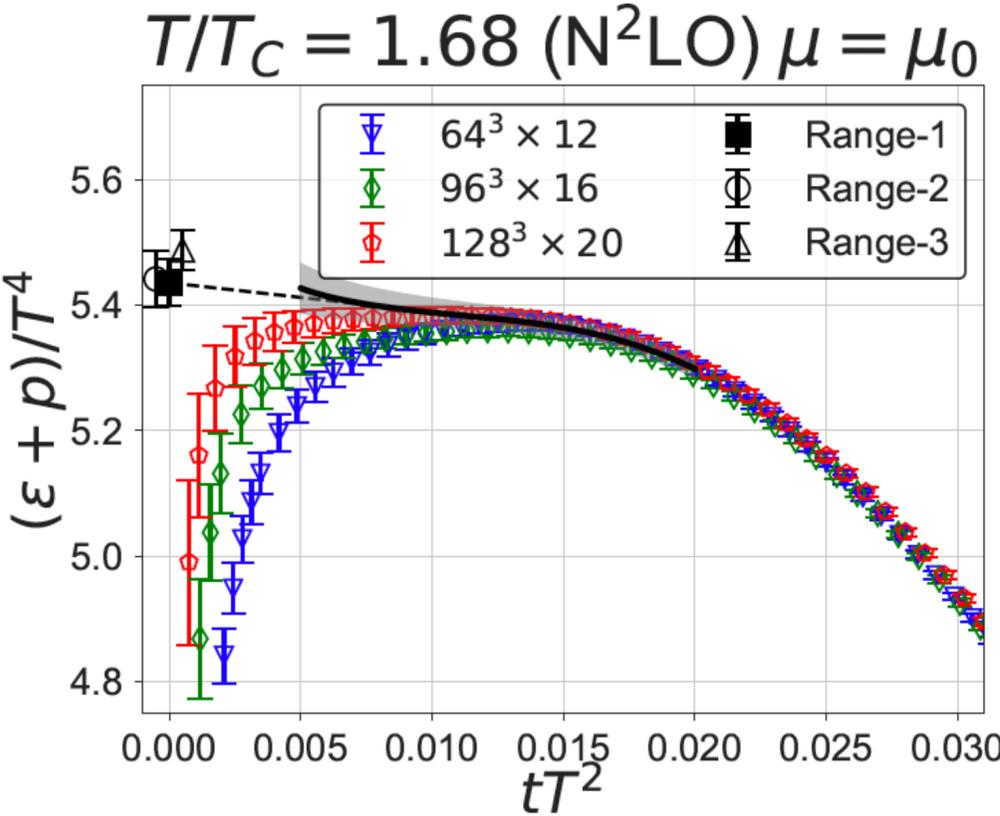
Iritani, Kitazawa, Suzuki, Takaura 2019

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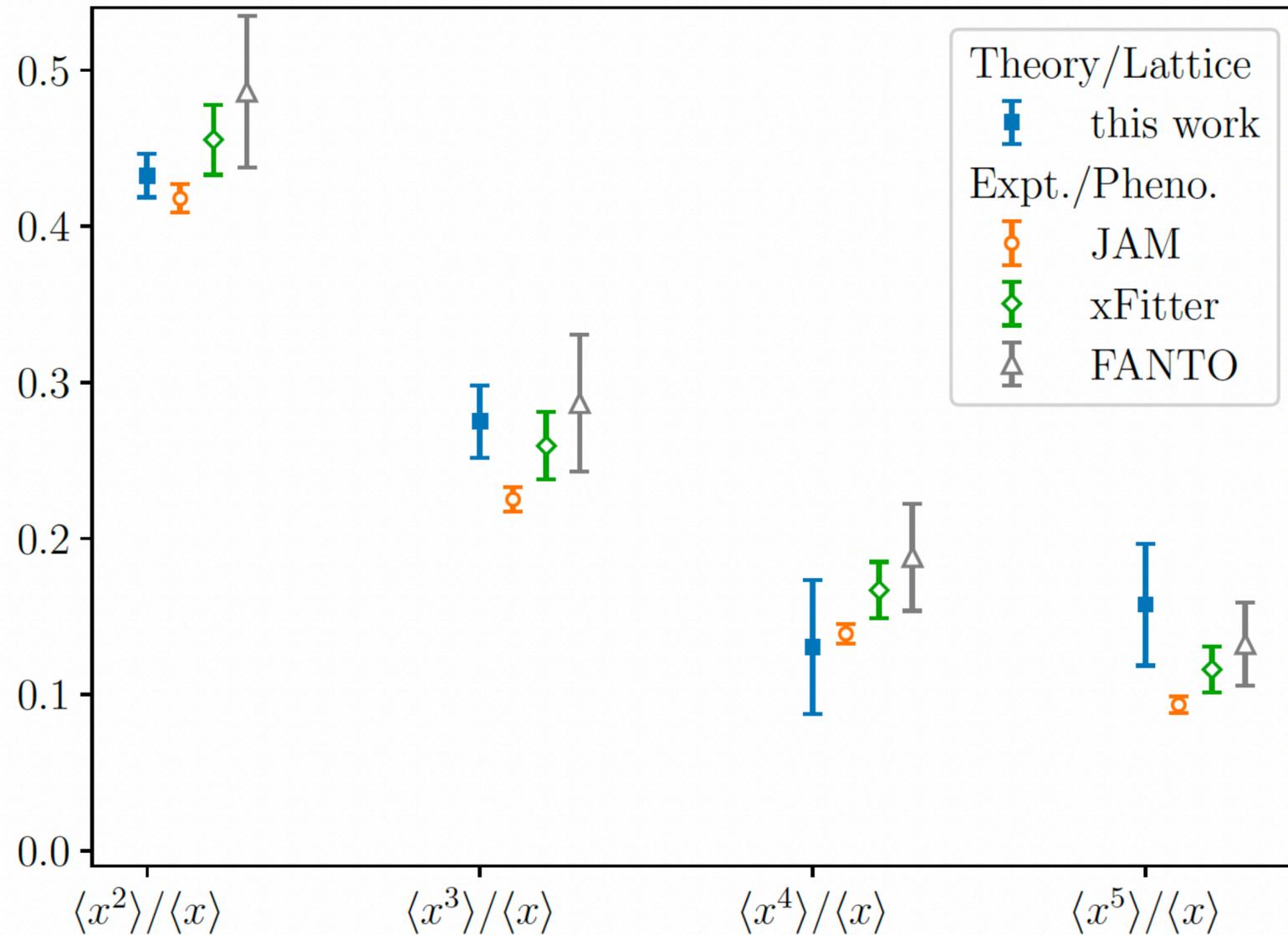
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# Gradient flow for parton distribution functions: first application to the pion

Anthony Francis,<sup>1</sup> Patrick Fritzsche,<sup>2</sup> Robert V. Harlander,<sup>3</sup> Rohith Karur,<sup>4,5</sup>  
 Jangho Kim,<sup>6</sup> Jonas T. Kohnen,<sup>3</sup> Giovanni Pederiva,<sup>7,8</sup> Dimitra A. Pefkou,<sup>4,5,\*</sup>  
 Antonio Rago,<sup>9</sup> Andrea Shindler,<sup>3,5,4,†</sup> André Walker-Loud,<sup>5,4</sup> and Savvas Zafeiropoulos<sup>10</sup>

<sup>1</sup>*Institute of Physics, National Yang Ming Chiao Tung University, 30010 Hsinchu, Taiwan*

$\mu = 2 \text{ GeV}$



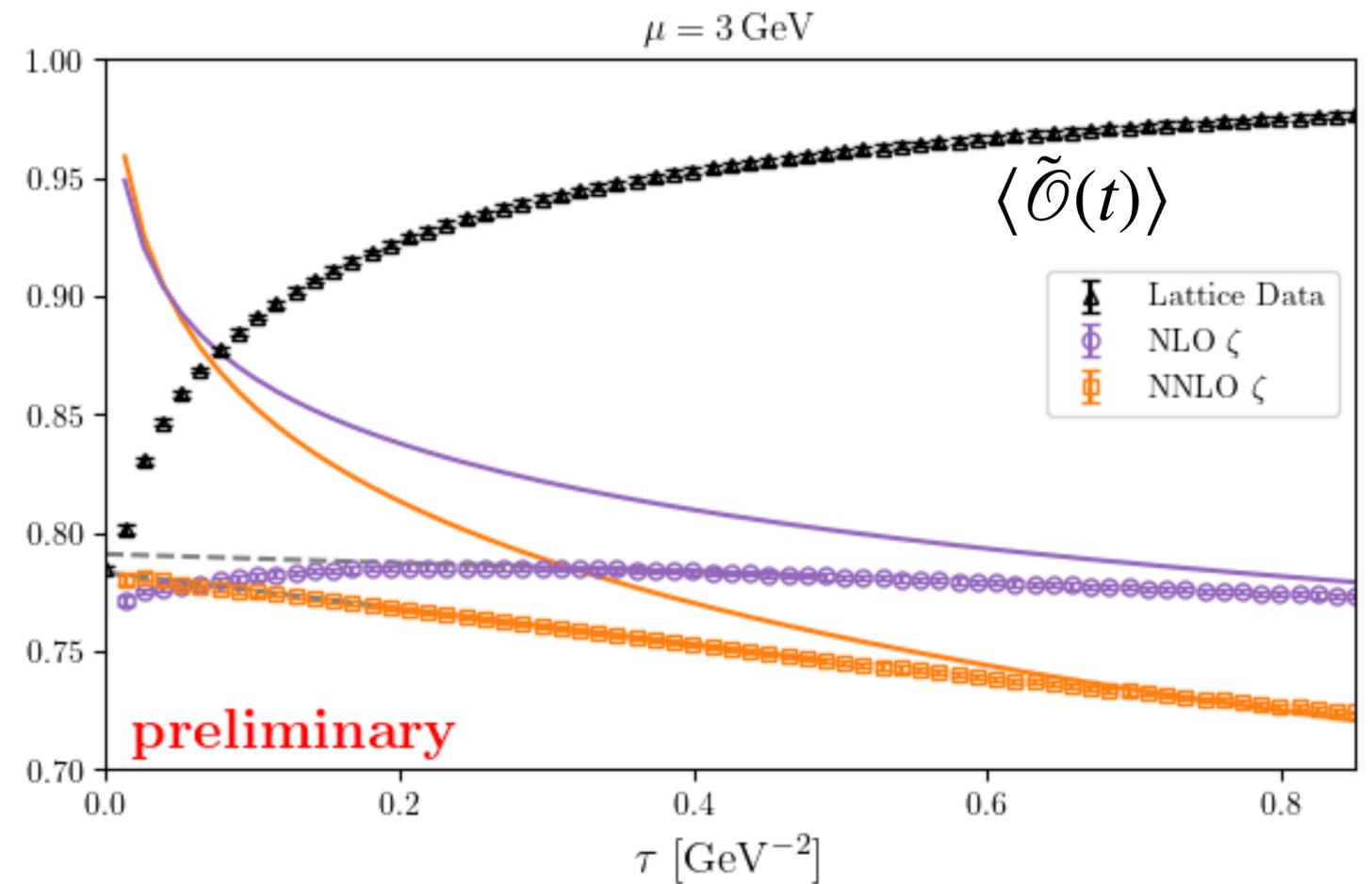
$$\langle x^{n-1} \rangle_q(\mu) = \int dx x^{n-1} q(x, \mu)$$

obtained from matrix elements of:

$$O_n^{rs}(x) \equiv i^{n-1} \bar{\psi}^r \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} \} \psi^s(x)$$

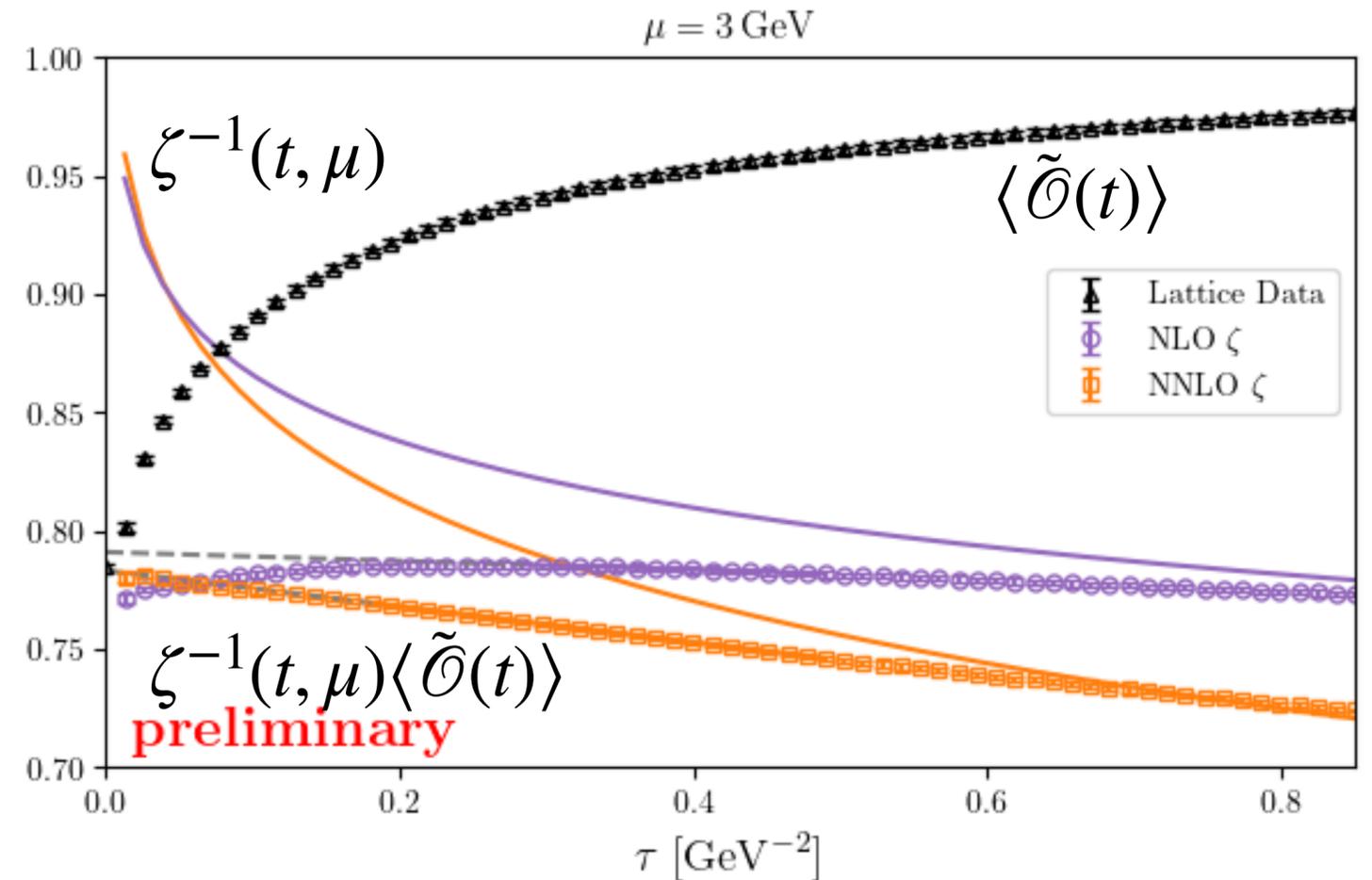
# The gradient flow scheme

$$\begin{aligned}\Delta\Gamma &\sim \sum_n C_n^{\text{R}}(\mu) \langle \mathcal{O}_n^{\text{R}} \rangle(\mu) \\ &= \sum_n C_n^{\text{R}}(\mu) \left[ \zeta^{-1}(t, \mu) \langle \tilde{\mathcal{O}}_n \rangle(t) \right]\end{aligned}$$



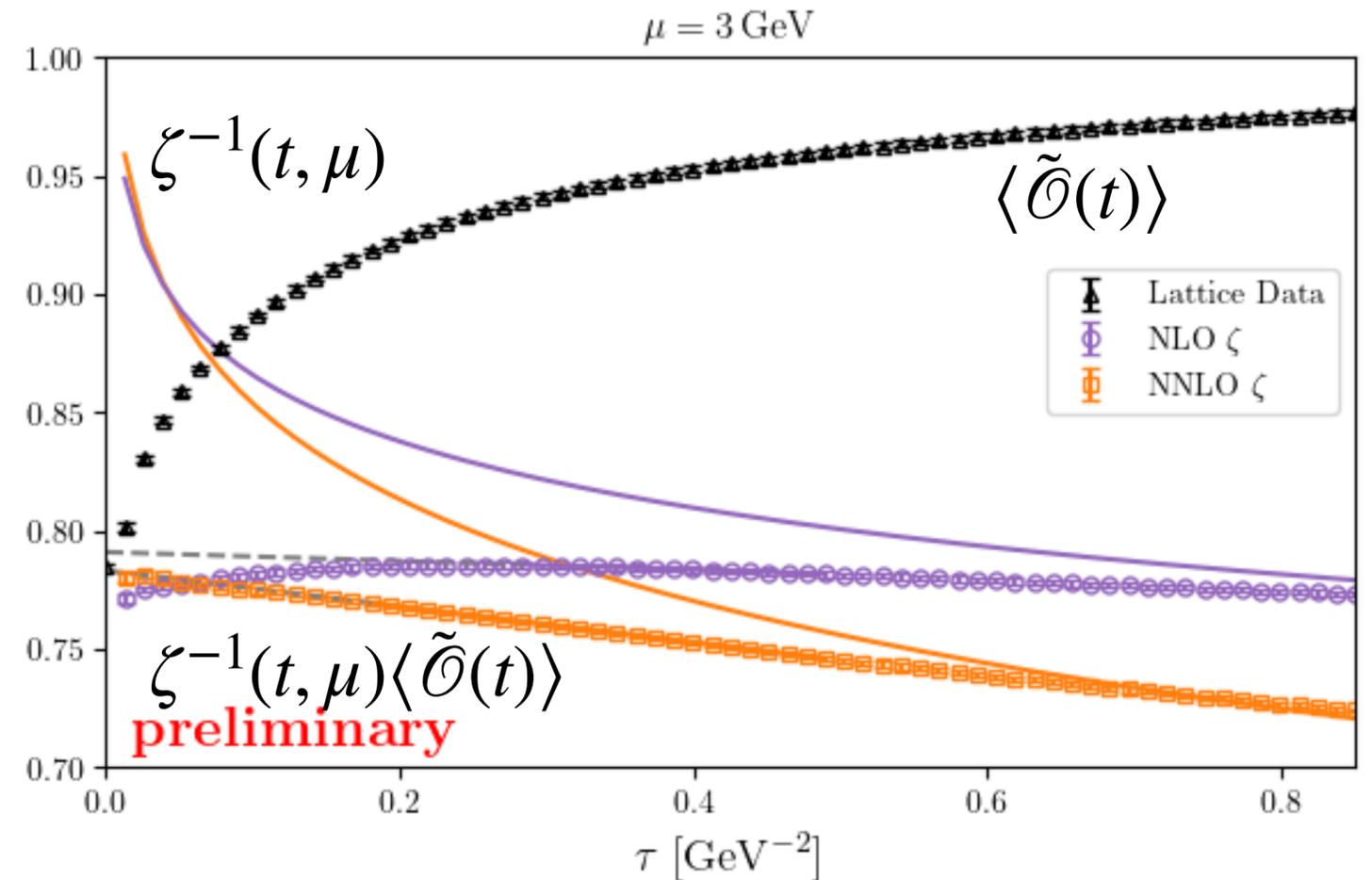
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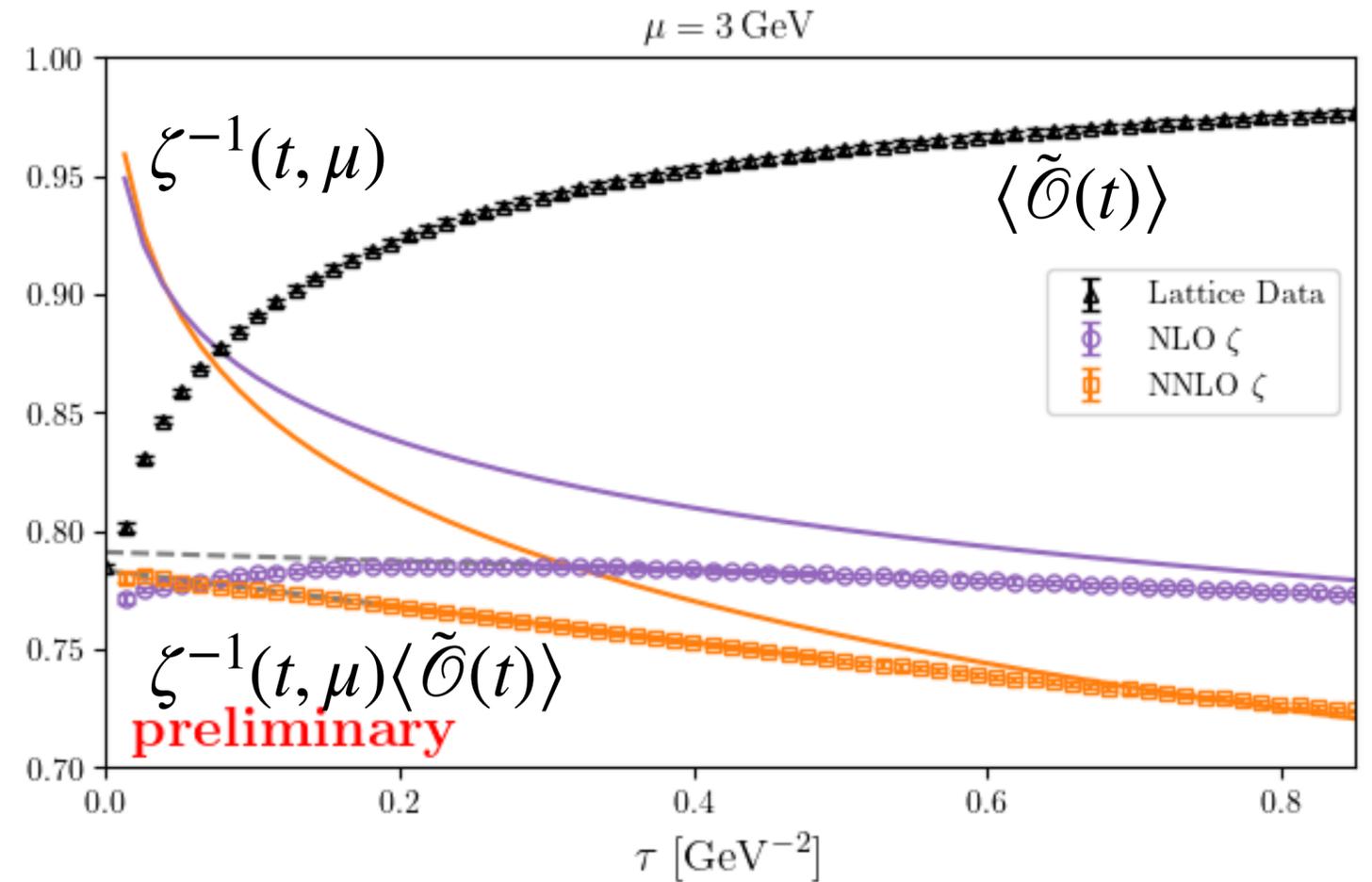
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GF

$\overline{\text{MS}}$

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$$\tilde{\gamma} = -t \frac{d}{dt} \ln \zeta(t)$$

$$\gamma_{nm} = -\mu \frac{d}{d\mu} \ln Z$$

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$$= \sum_n (C \zeta^{-1}(t))_n \langle \zeta(t) \mathcal{O} \rangle_n$$

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$$= \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

$$= \sum_n C_n^{\text{R}}(\mu) \langle \mathcal{O}_n^{\text{R}} \rangle(\mu)$$

$$t \frac{d}{dt} \tilde{C}(t) = \tilde{C}(t) \tilde{\gamma}$$

$$\mu \frac{d}{d\mu} C^{\text{R}}(\mu) = C^{\text{R}}(\mu) \gamma$$

$$\tilde{\gamma} = -t \frac{d}{dt} \ln \zeta(t)$$

$$\gamma_{nm} = -\mu \frac{d}{d\mu} \ln Z$$

RH, Lange, Neumann '20

Borgulat, Felten, RH, Kohnen '25

# Application to QED

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flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$

$$B_\mu(t = 0, x) = A_\mu(x)$$

# Application to QED

flowed gauge field:

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$U(1)$   
→

$$\frac{\partial}{\partial t} B_\mu = \left( \square g_{\mu\nu} - \partial_\mu \partial_\nu \right) B_\nu$$

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exact solution:

$$\tilde{B}_\mu(t, p) = e^{-tp^2} \tilde{A}(p)$$

# Application to QED

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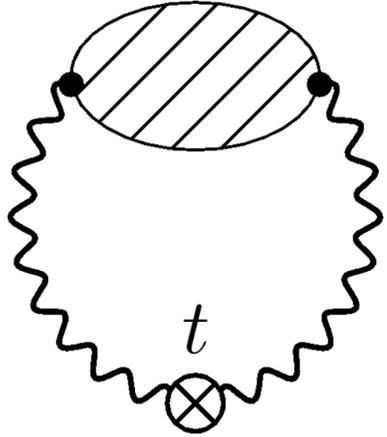
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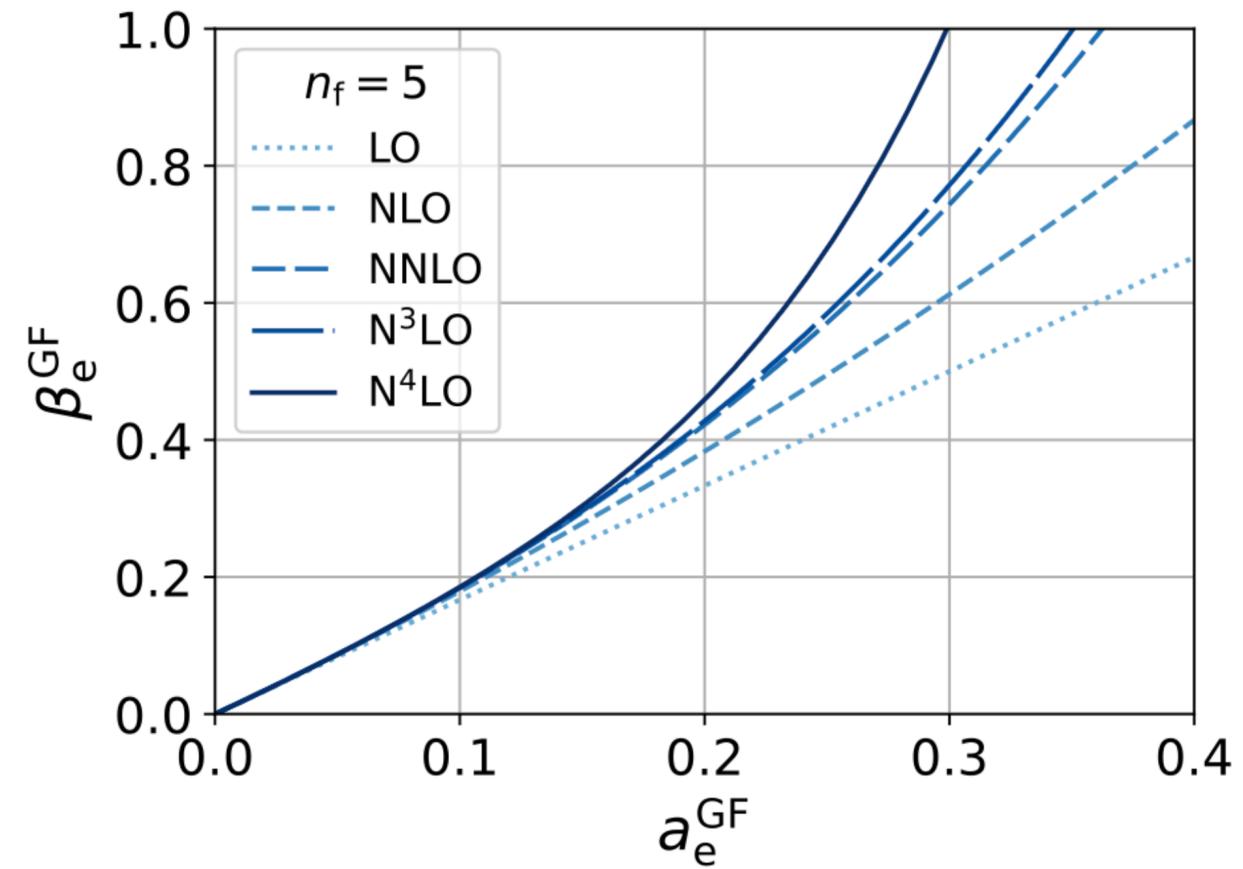
$$\alpha^{\text{GF}}(\mu) \sim \alpha(\mu) \int d^4p \frac{e^{-2tp^2}}{1 + \Pi_R(p)} \Big|_{t=(c/\mu)^2}$$

=

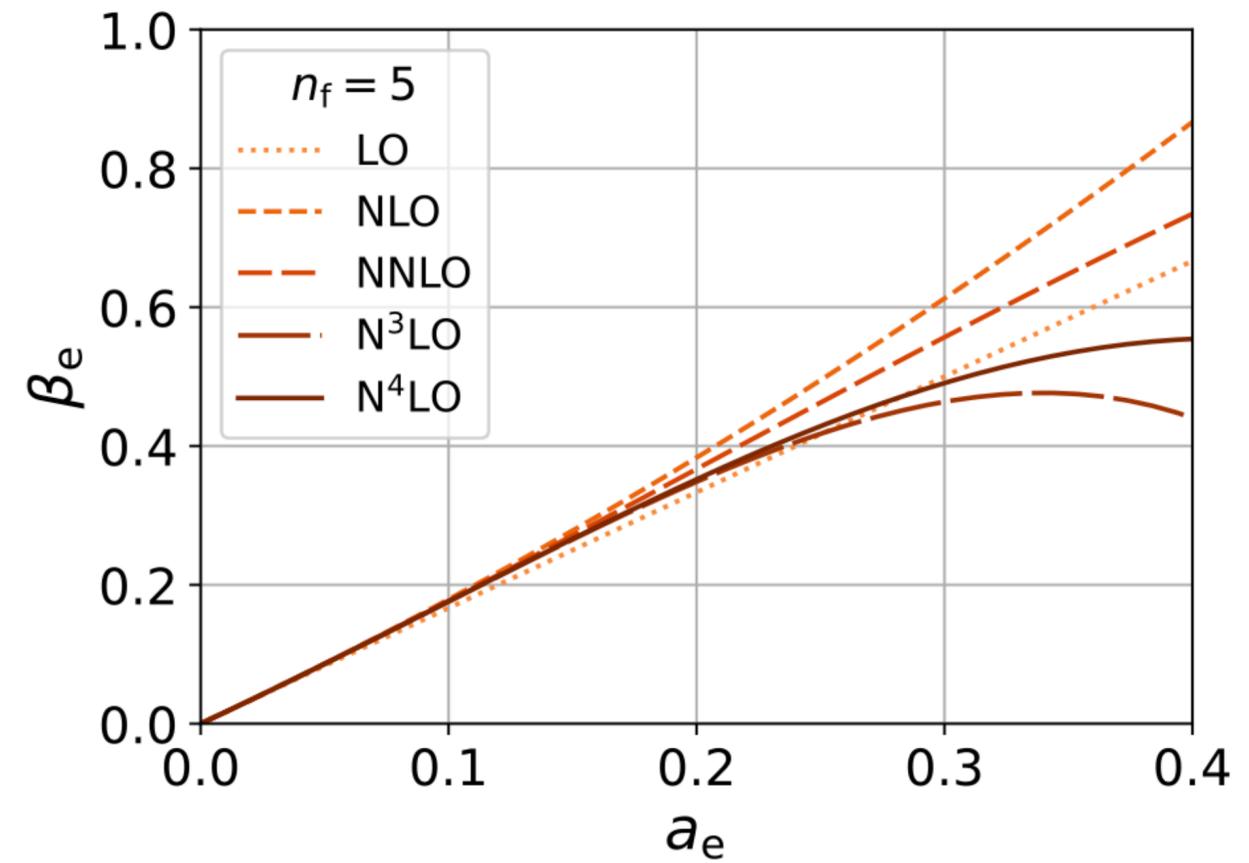


regular QED polarization function

## GF scheme



## $\overline{\text{MS}}$ scheme



Georg, RH, Mason 2025

# Application to QED<sub>3</sub>

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$\alpha$  mass dimension = 1       $\hat{\alpha} \sim \alpha/p$

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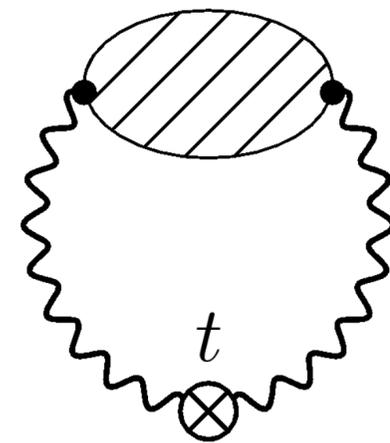
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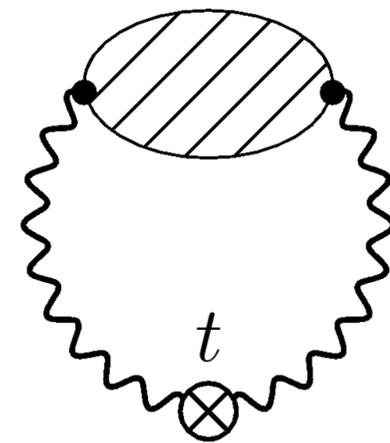
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regular QED polarization function

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$$\frac{1}{1 + \Pi_R(q)} = c_1 \frac{\alpha}{p} + c_2 \left( \frac{\alpha}{p} \right)^2 + \dots \quad \Rightarrow \quad \alpha^{\text{GF}} \text{ IR divergent beyond NNLO}$$

# Application to QED<sub>3</sub>

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$\alpha^{\text{GF}}$  IR divergent beyond NNLO

large- $n_f$  limit:

$$\Pi_R(p) = \frac{\alpha_e}{\pi p} h_f$$

$$h_f = \frac{\pi^2 n_f}{2} \left( 1 + \mathcal{O}(1/n_f) \right)$$

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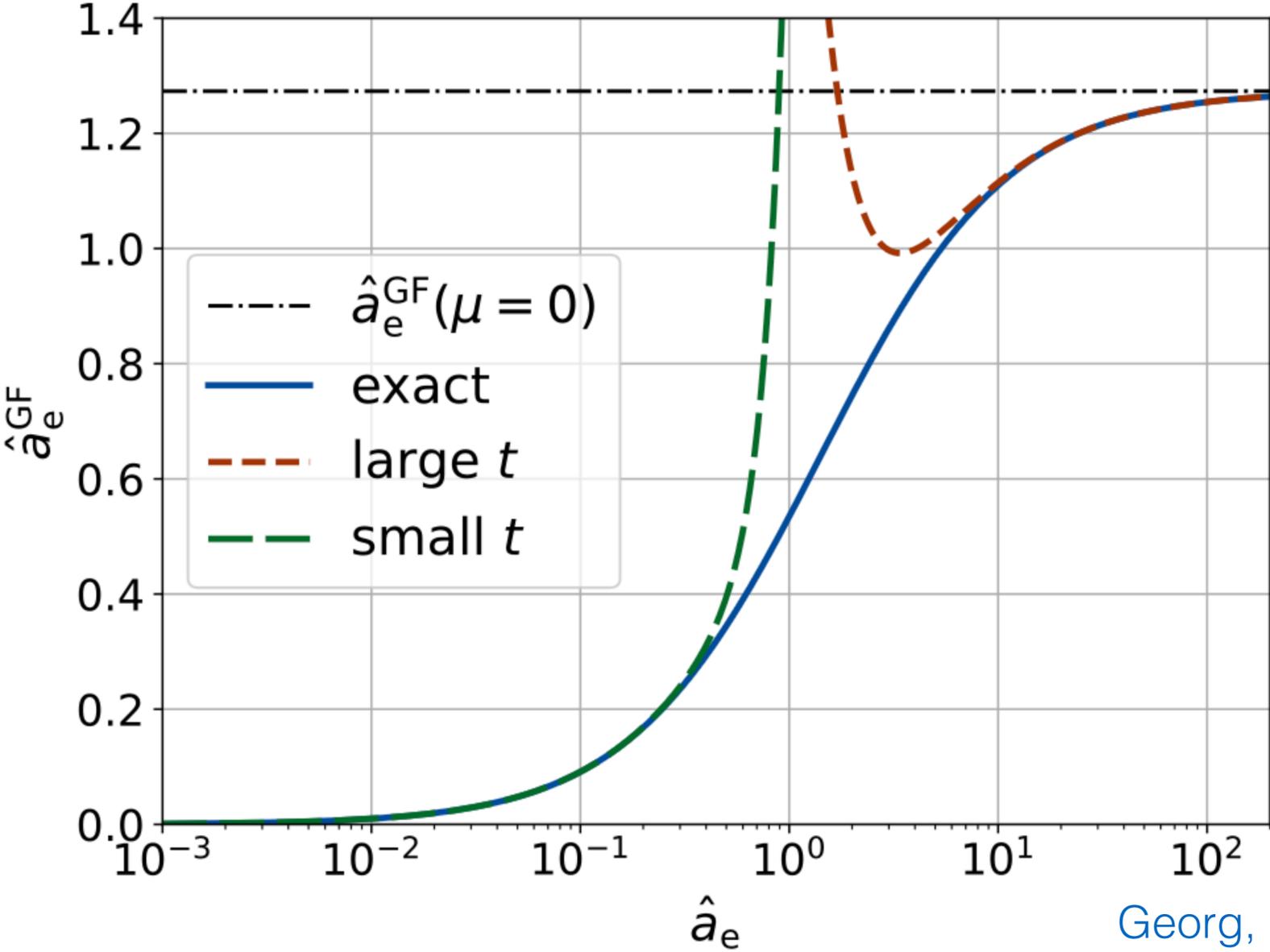
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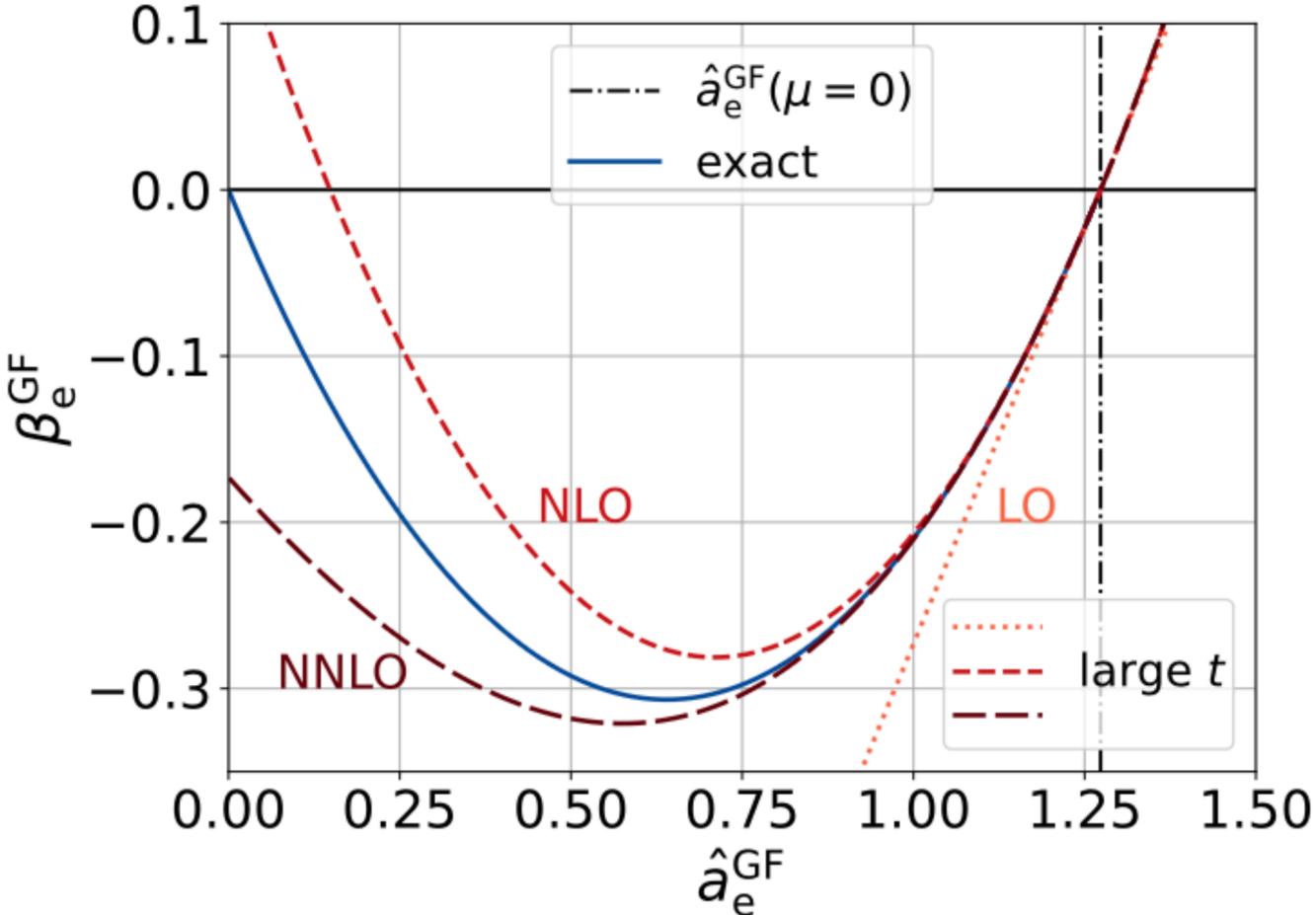
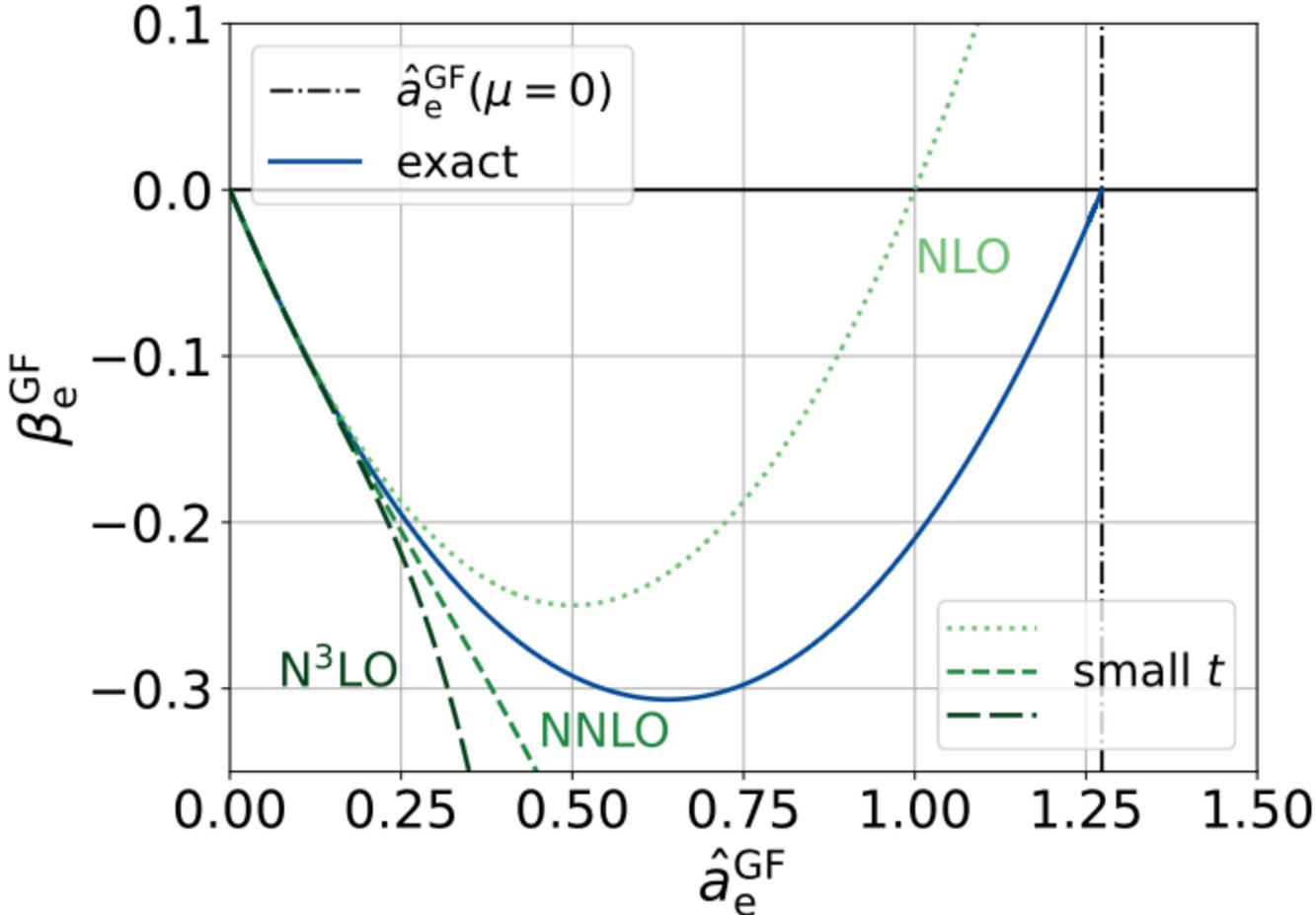
$$\equiv \sqrt{\frac{\pi}{8}} \hat{a}_e(p)$$

# Application to QED<sub>3</sub>



Georg, RH, Mason 2025

# Application to QED<sub>3</sub>



Georg, RH, Mason 2025

# Application to gravity

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$
$$B_\mu(t = 0, x) = A_\mu(x)$$

gravity  
→

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

... in preparation ...

RH, Kluth, Kohlen, Werthenbach

# Conclusions

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Gradient flow: interesting field theoretical concept

Established tool in lattice calculations

Full potential not yet fully explored (I think...)

Here: renormalization scheme  
for QCD parameters  
for composite operators

Application to gravity?

Many things not even discussed:  
Static QCD potential, quark masses, EDMs, ...