The Gradient Flow Formalism in Perturbation Theory

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DFG Deutsche Forschungsgemeinschaft

No motivation

Properties and uses of the Wilson flow in lattice QCD #8

Martin Lüscher (CERN and Geneva U.) (Jun 23, 2010)

Published in: *JHEP* 08 (2010) 071, *JHEP* 03 (2014) 092 (erratum) • e-Print: 1006.4518 [hep-lat]

Trivializing maps, the Wilson flow and the HMC algorithm #9

Martin Luscher (CERN) (2009)

Published in: *Commun.Math.Phys.* 293 (2010) 899-919 • e-Print: 0907.5491 [hep-lat]

Infinite N phase transitions in continuum Wilson loop operators #5

R. Narayanan (Florida Intl. U.), H. Neuberger (Rutgers U., Piscataway) (Jan, 2006)

Published in: *JHEP* 03 (2006) 064 • e-Print: hep-th/0601210 [hep-th]

$$\boxed{\frac{\partial}{\partial t}B_\mu(t) = \mathscr{D}_\nu(t)\, G_{\nu\mu}(t) \qquad\qquad B_\mu(t=0) = A_\mu}$$

$$G_{\mu\nu}(t) = \frac{i}{g_0}[\mathscr{D}_\mu(t), \mathscr{D}_\nu(t)] \qquad \sim \partial B(t) + g_0 B^2(t)$$

$$\mathscr{D}_\mu(t) = \partial_\mu - i g_0 T^a B^a_\mu(t)$$

$$\partial_t B \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$$

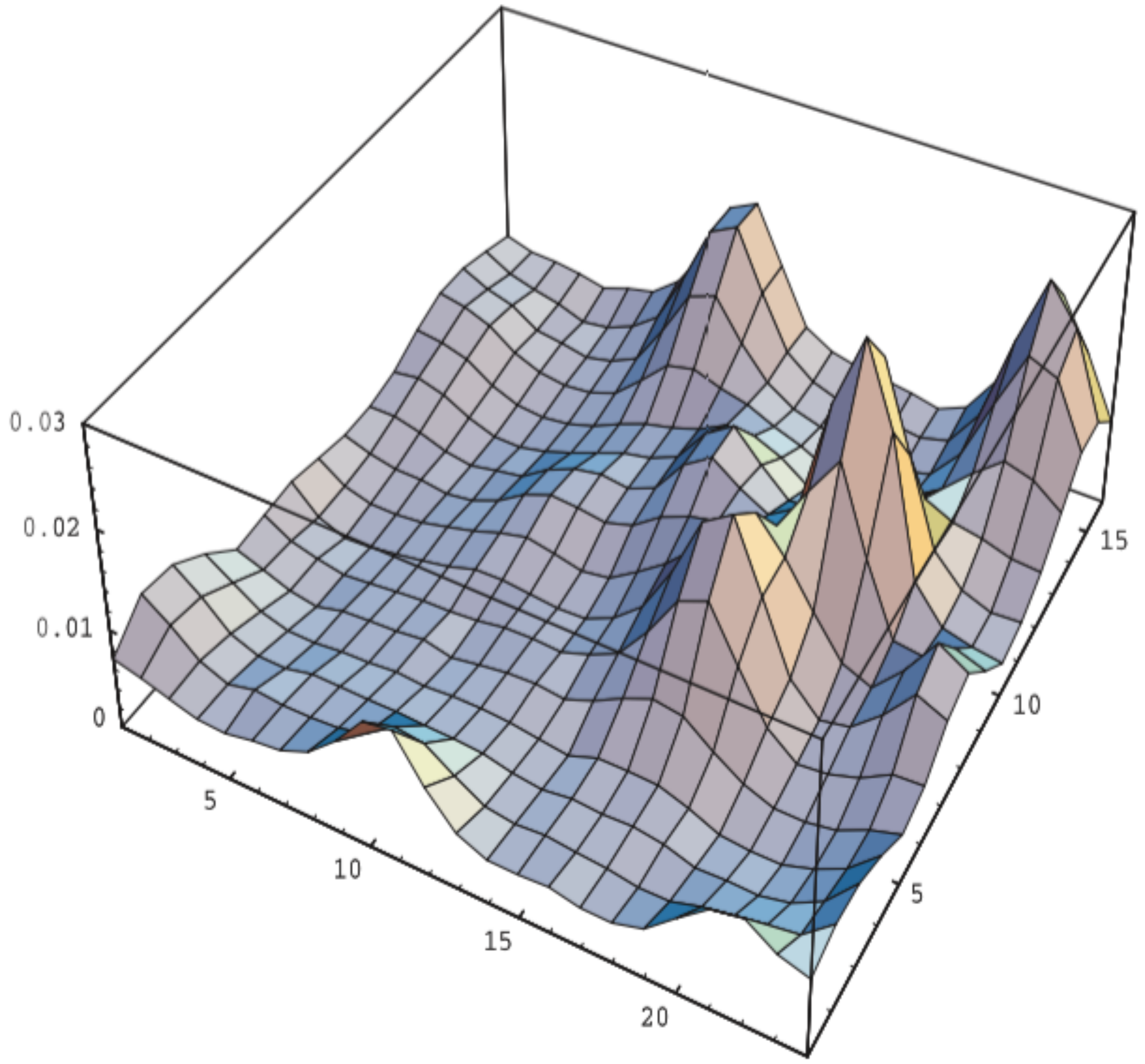
Perturbative solution

flow equation: $\partial_t B \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$

$$B_\mu(t = 0) = A_\mu$$

perturbative ansatz: $B = B_1 + g_0 B_2 + \ldots$

momentum space: $\tilde{B}_1(t, p) = e^{-tp^2} \tilde{A}(p)$



cf. heat equation: $\partial_t u(t) = \Delta u(t)$

Perturbative solution

flow equation: $\partial_t B \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$ $B_\mu(t=0) = A_\mu$

perturbative ansatz: $B = B_1 + g_0 B_2 + \ldots$

momentum space: $\tilde{B}_1(t,p) = e^{-tp^2}\tilde{A}(p)$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\,\tilde{A}(p)\,\tilde{A}(p-q)$$

$$K(t,s,p,q) \sim \exp\left[-tp^2 - 2sq(q-p)\right]$$

etc.

Exponential damping in momentum integrals!

Quantum field theory

$$\mathscr{L} = \mathscr{L}_{\rm QCD} + \mathscr{L}_B + \mathscr{L}_\chi$$

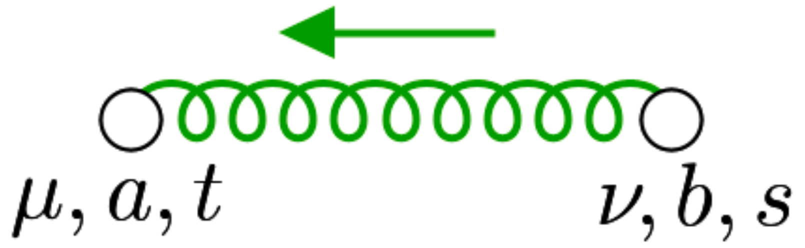
$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

L_μ Lagrange multiplier field

Lüscher, Weisz 2011

$\sim \langle 0 | T B_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$



$\sim \langle 0 | T L_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$

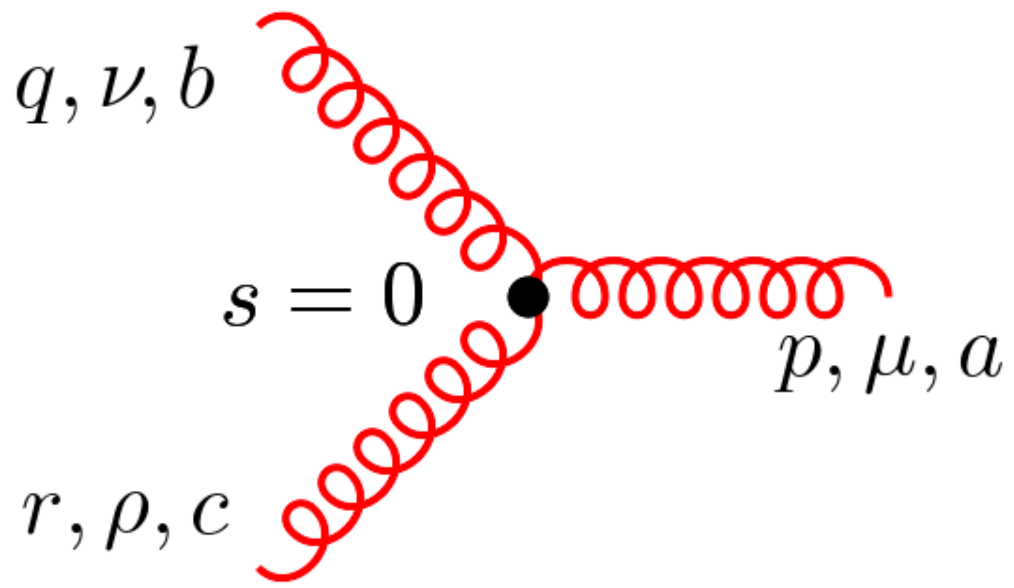
$$\delta_{ab}\delta_{\mu\nu}\, \theta(t-s)\, e^{-(t-s)p^2}$$

“gluon flow line”

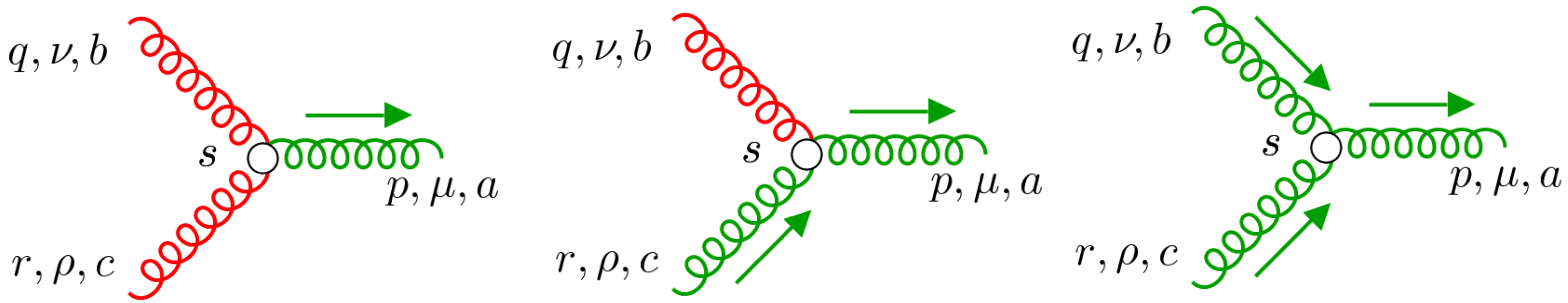
analogously for quarks: Lüscher 2013

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \Delta \right) \chi + \mathrm{h.c.}$$

Vertices



regular 3-gluon vertex



$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu \\ + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

analogously for 4-gluon vertex and quarks

Quantum field theory

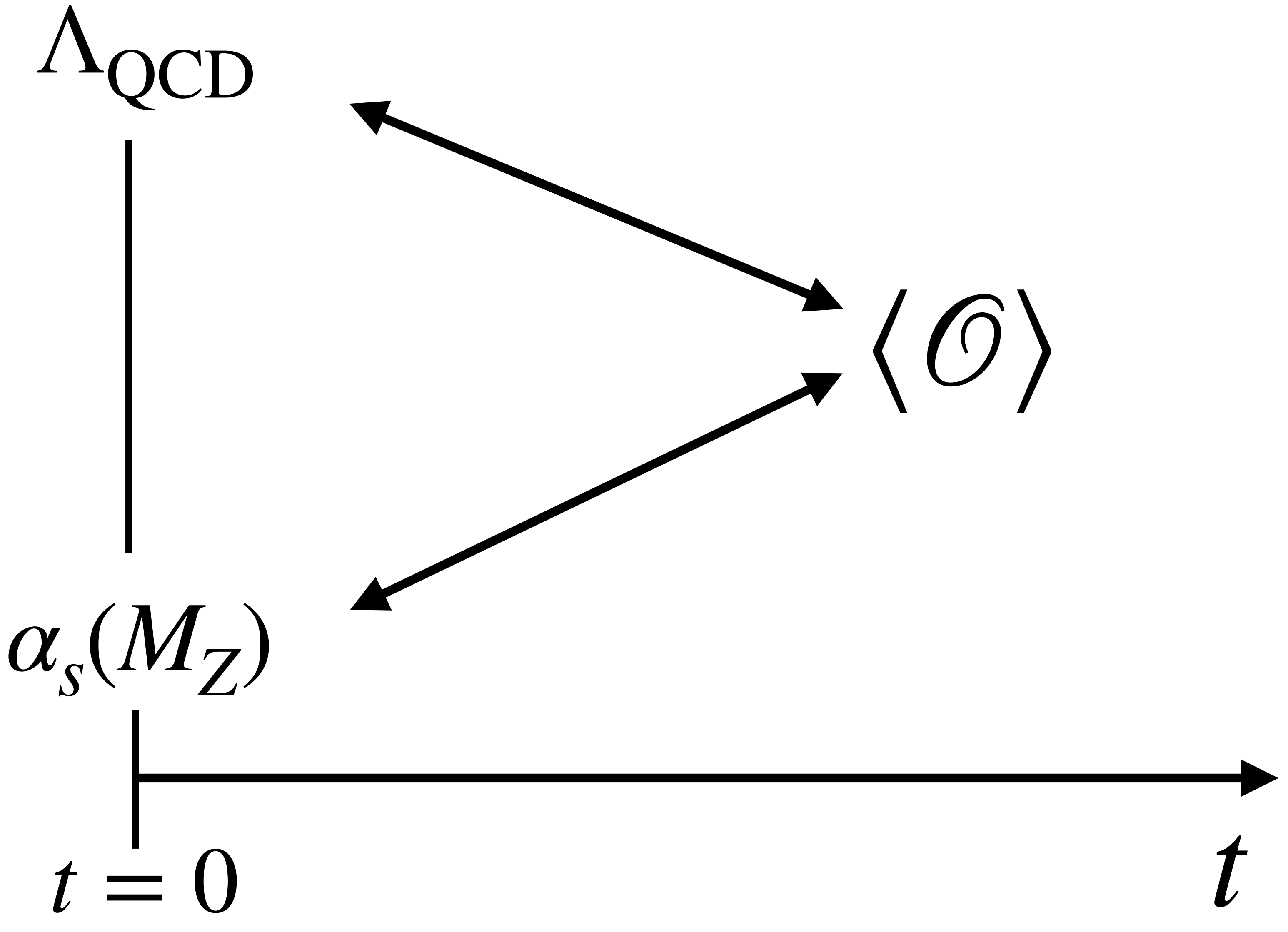
$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda}\left(\partial_t - \Delta\right)\chi + \mathrm{h.c.}$$

"Bulk" is UV regulated

⇒ renormalization unaffected!

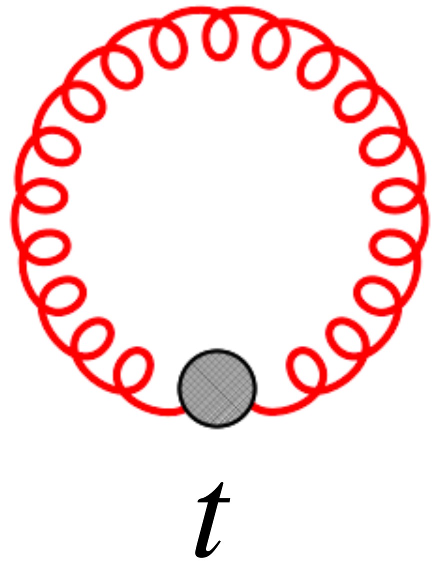


Let’s calculate

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

LO:



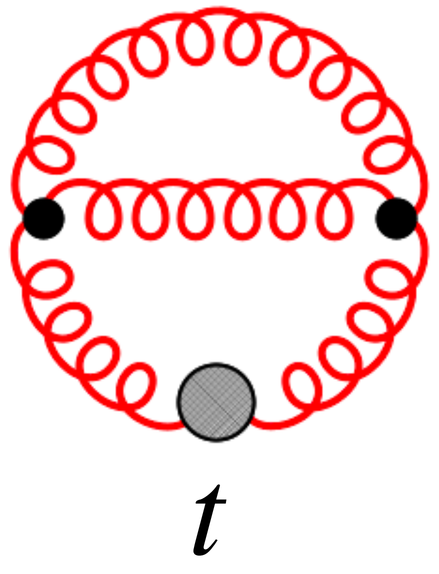
$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

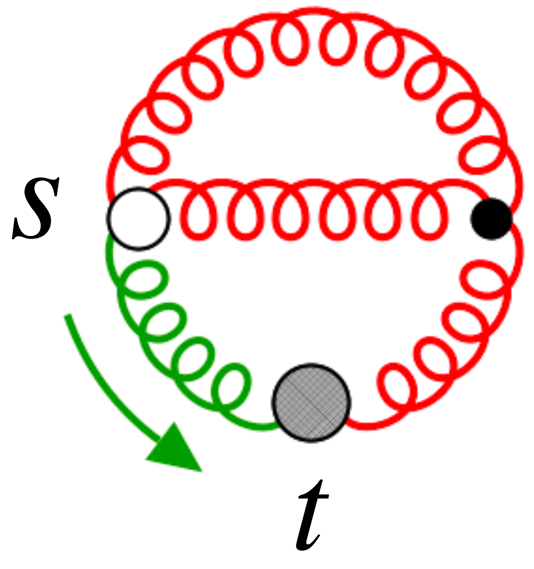
explicitly: $$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

→ measure αs on the lattice?

$$\alpha_s = \alpha_s(\mu)$$

Higher orders





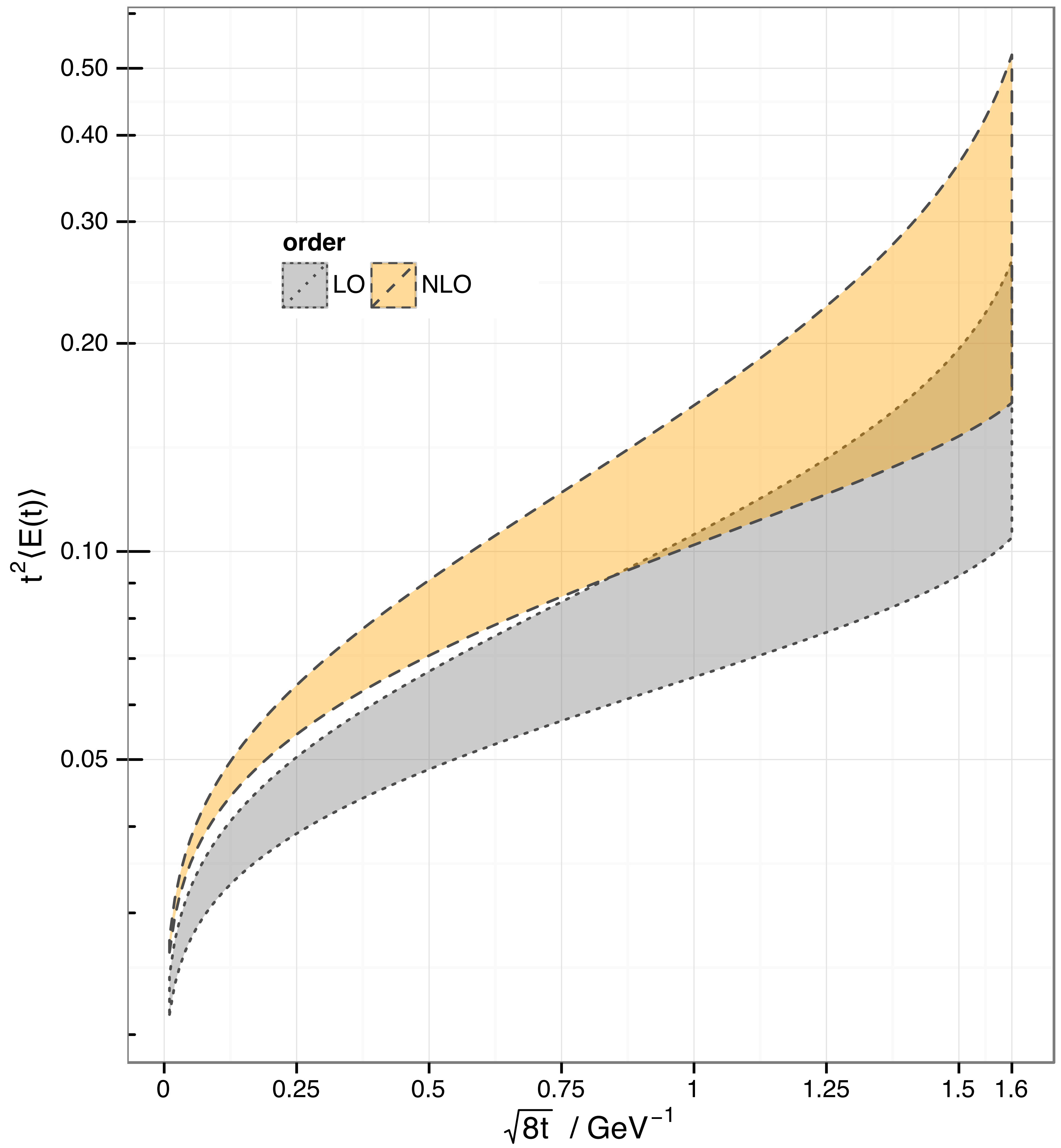
$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4\,k^2\,(p-k)^2}$$

$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0\, L_{t\mu}$$

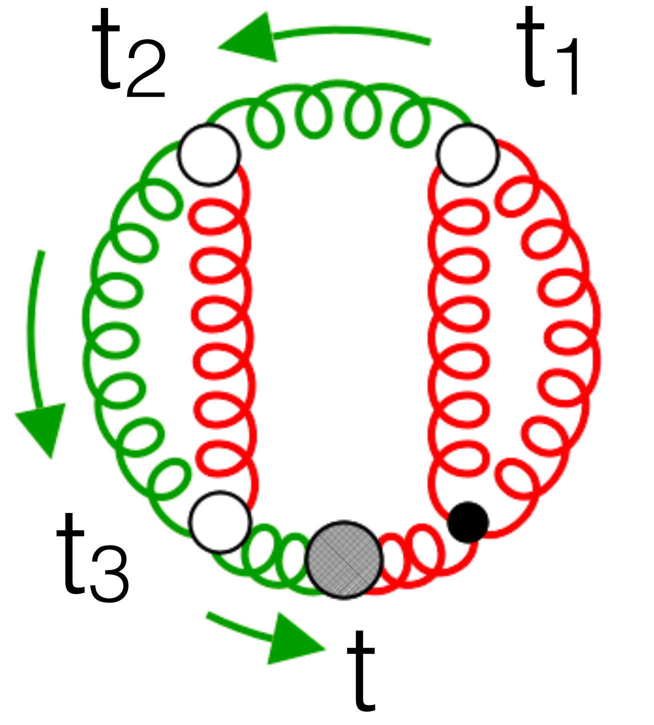
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_{\mathrm{E}}$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative accuracy on αs: ± 3-5%

PDG: ± 1%

Three-loop calculation



$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\}) =$$

$$= \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2)^{b_1}\,\cdots\,(p_6^2)^{b_6}}$$

$$u_i = t_i/t$$

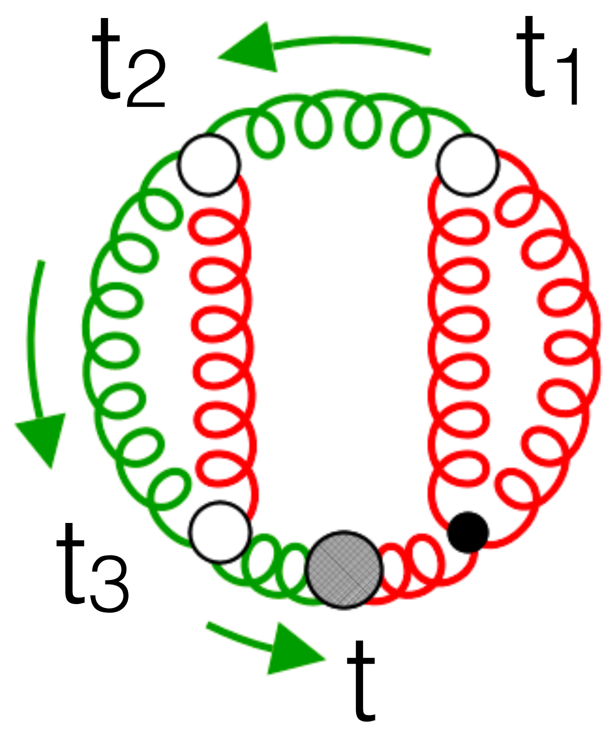
IbP identities: $\frac{\partial}{\partial p_i}\cdot p_j\;\; I(\cdots)$ → modifies c_k and b_k

$$\frac{\partial}{\partial u_i}\;\; I(\cdots) \;= I(\cdots)\Big|_{u_i=1} - I(\cdots)\Big|_{u_i=0}$$ → modifies c_k, b_k and a_k

Artz, RH, Lange, Neumann, Prausa ’19

Numerical evaluation

RH, Neumann (2016)



$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\}) =$$

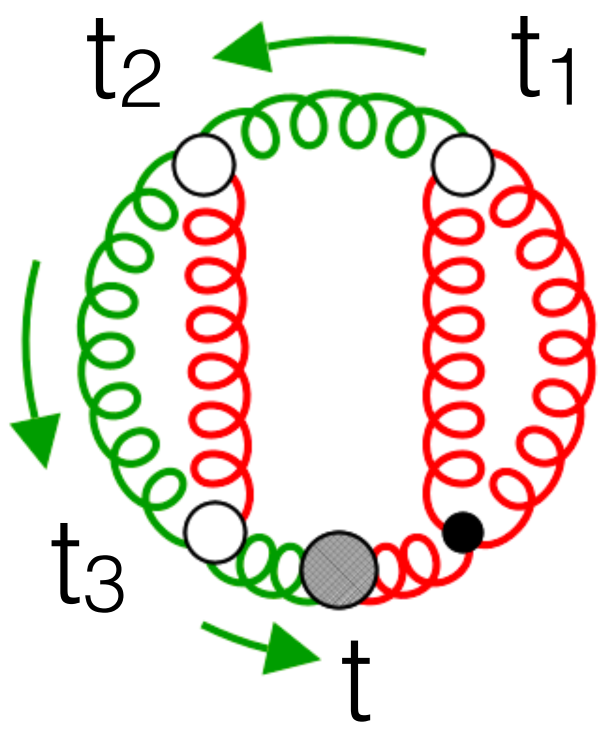
$$= \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2)^{b_1}\,\cdots\,(p_6^2)^{b_6}}$$

Schwinger parameters: $$\frac{1}{(p^2)^b} \sim \int_0^\infty dx\, x^{b-1}\, e^{-xp^2} \qquad \left(\xrightarrow{\text{map}} \int_0^1 dx\, \cdots\right)$$

$$\sim \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\left(\prod_{j=1}^{6}\int_0^\infty dx_j\, x_j^{b_j-1}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \exp\left[-t\,\mathbf{p}^T A(x,u)\,\mathbf{p}\right]$$

Numerical evaluation

RH, Neumann (2016)



$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\}) =$$

$$= \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2)^{b_1}\,\cdots\,(p_6^2)^{b_6}}$$

Schwinger parameters: $\frac{1}{(p^2)^b} \sim \int_0^\infty dx\, x^{b-1}\, e^{-xp^2}$ $\left(\overset{\text{map}}{\rightarrow}\int_0^1 dx\,\cdots\right)$

$$\sim \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\left(\prod_{j=1}^{6}\int_0^1 dx_j\, x_i^{b_j-1}\right)\left[\det A(x,u)\right]^{-D/2}$$

→ sector decomposition

Binoth, Heinrich (2000)

Implementation

$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\})$$

$$c_1 = c_2 = 0$$

$$a_1 = u_1u_2\,,\quad a_2 = u_2\,,\quad a_3 = u_2 - u_1u_2$$

$$a_4 = 1\,,\quad a_5 = 1 + u_1u_2\,,\quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

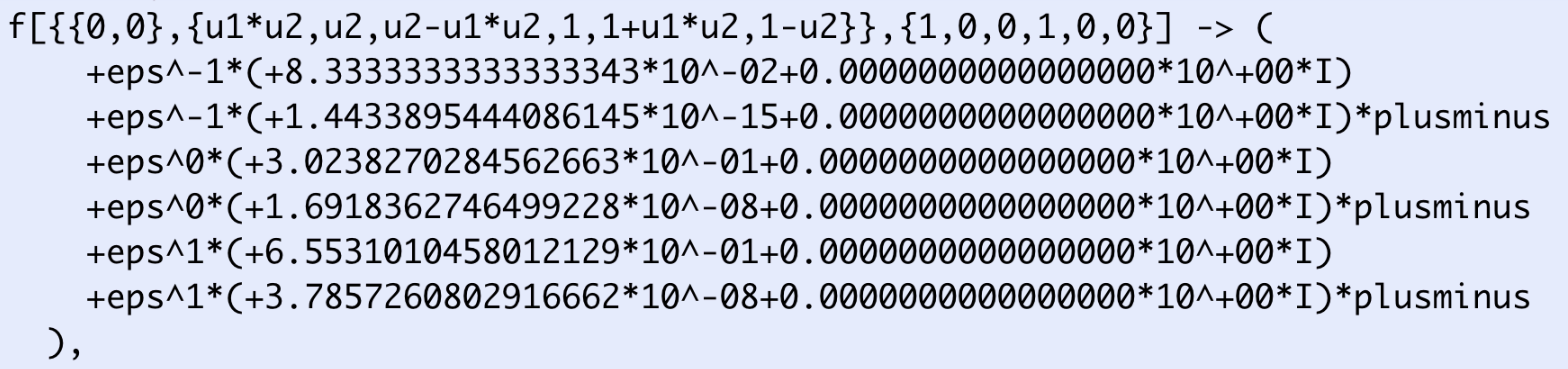
$$b_2 = b_3 = b_5 = b_6 = 0$$

$\downarrow$

ftint RH, Nellopoulos, Olsson (in prep)

(based on pySecDec)

Heinrich, Magerya, Kerner, Jones, …



```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}},{1,0,0,1,0,0}] -> (
    +eps^-1*(+8.333333333333343*10^-02+0.0000000000000000*10^+00*I)
    +eps^-1*(+1.443389544408614 5*10^-15+0.0000000000000000*10^+00*I)*plusminus
    +eps^0*(+3.023827028456266 3*10^-01+0.0000000000000000*10^+00*I)
    +eps^0*(+1.691836274649922 8*10^-08+0.0000000000000000*10^+00*I)*plusminus
    +eps^1*(+6.553101045801212 9*10^-01+0.0000000000000000*10^+00*I)
    +eps^1*(+3.785726080291666 2*10^-08+0.0000000000000000*10^+00*I)*plusminus
  ),
```

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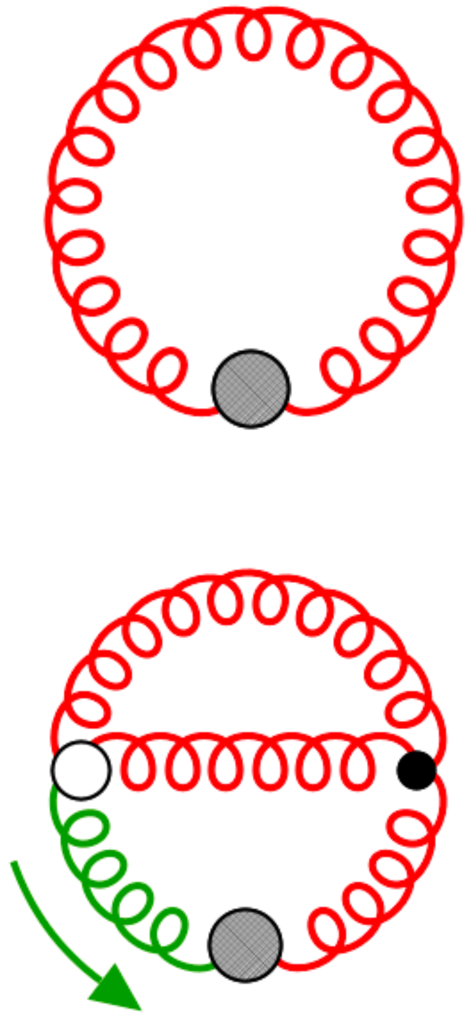
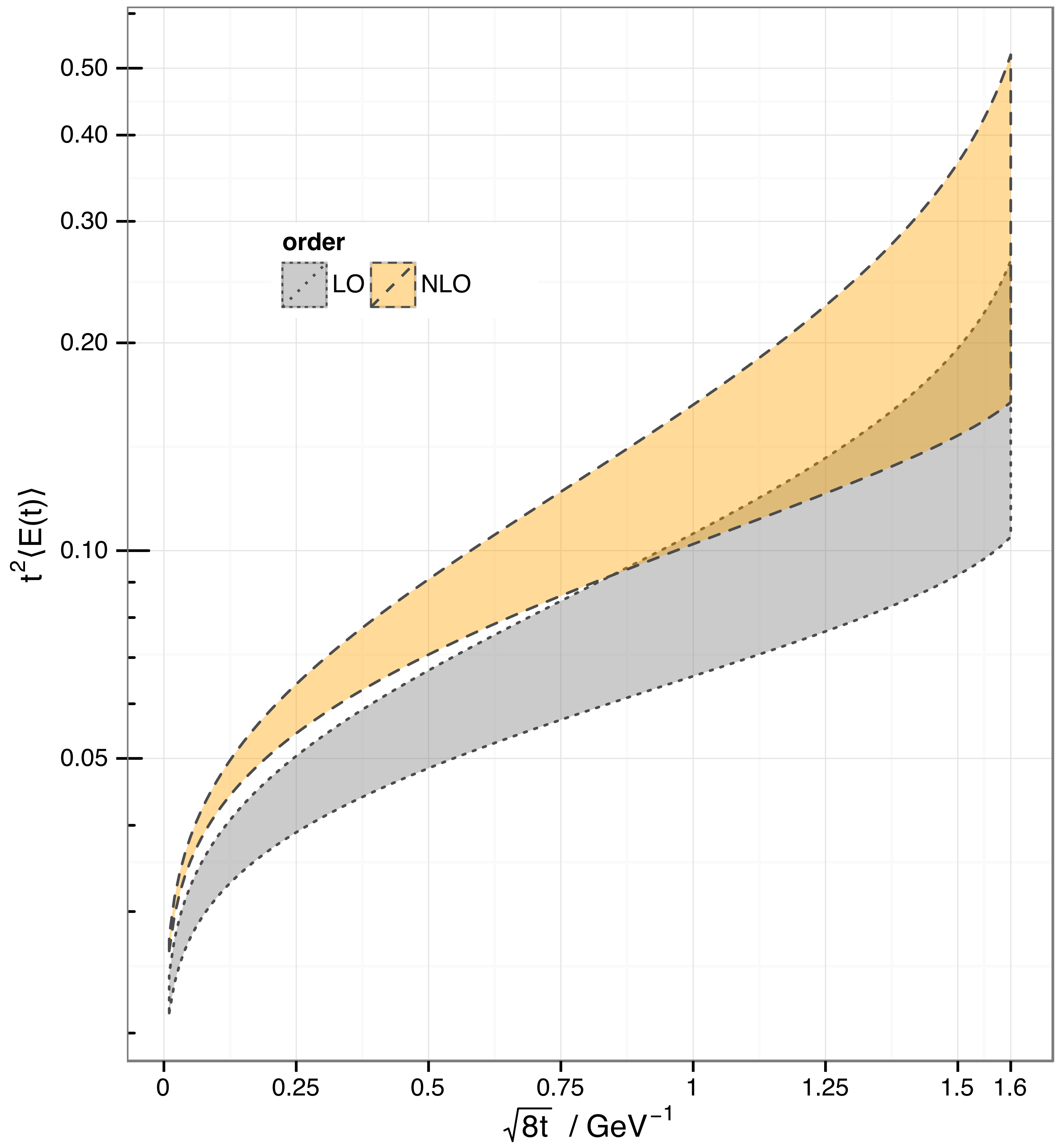
Lüscher 2010





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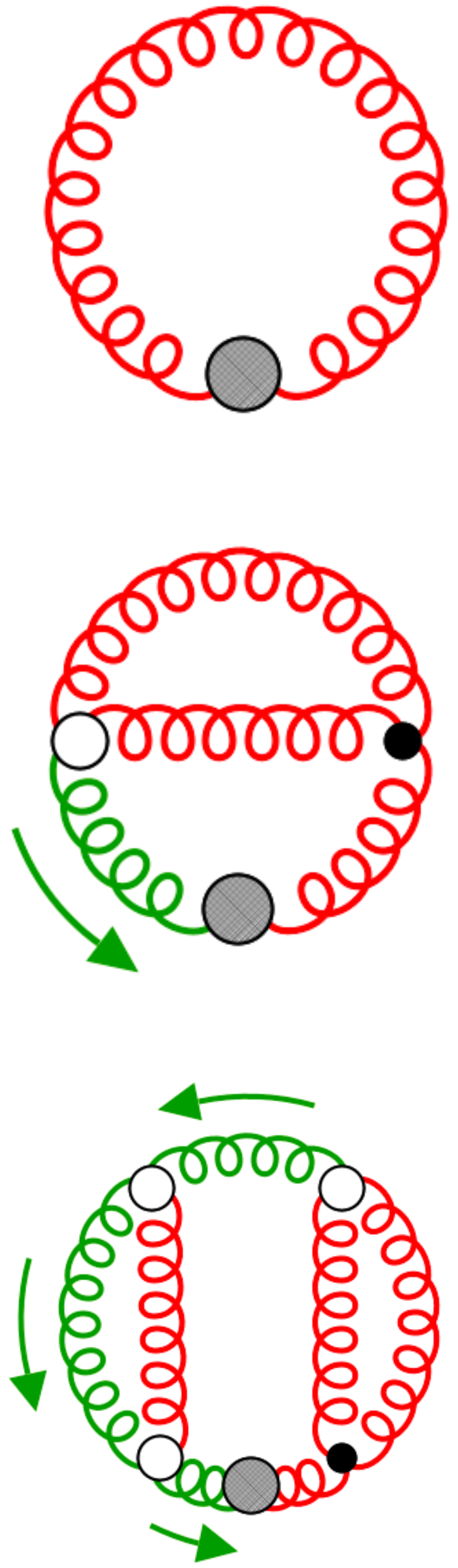
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_{\rm E}$$

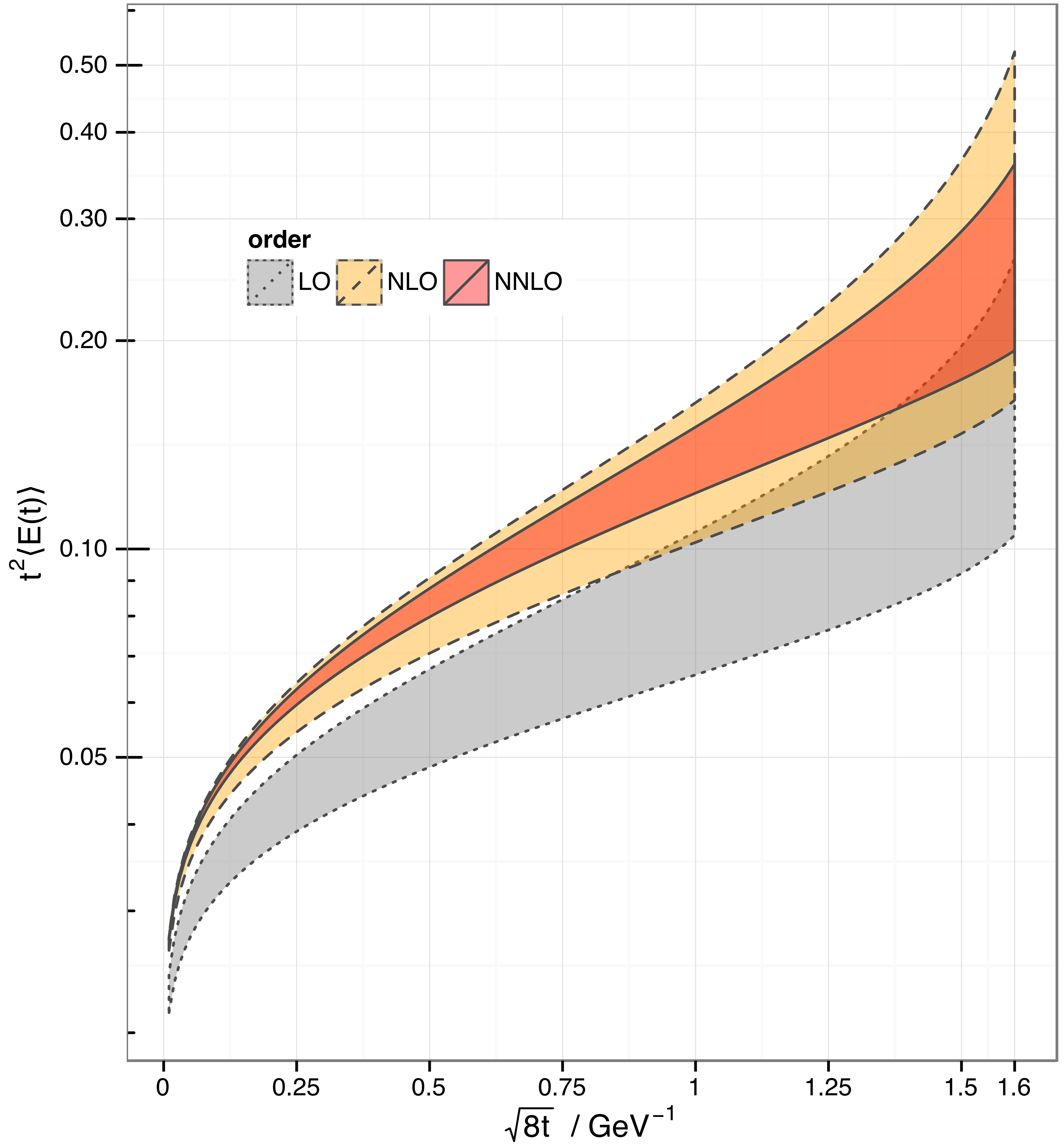
resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann 2016

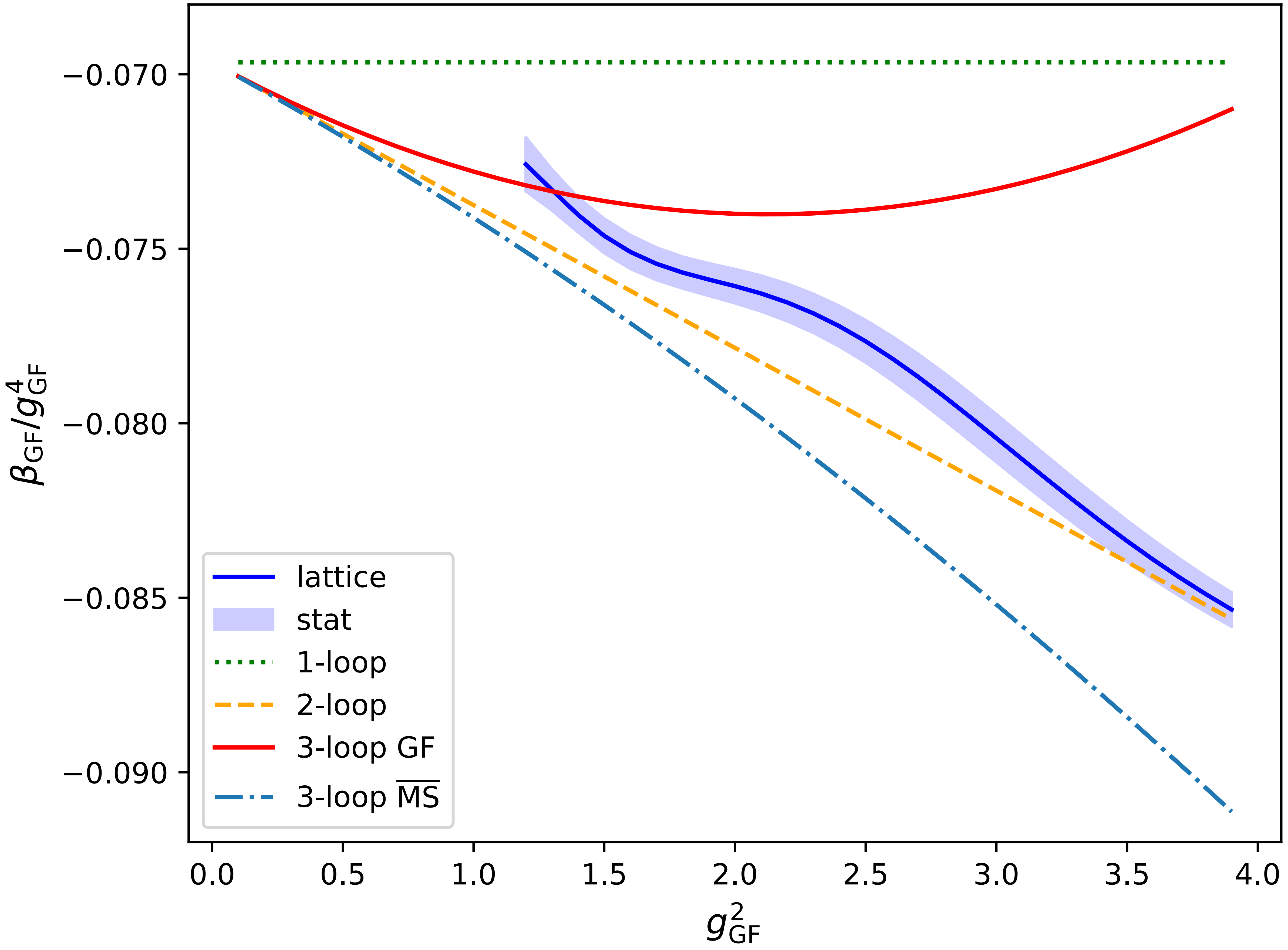




resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \quad \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$

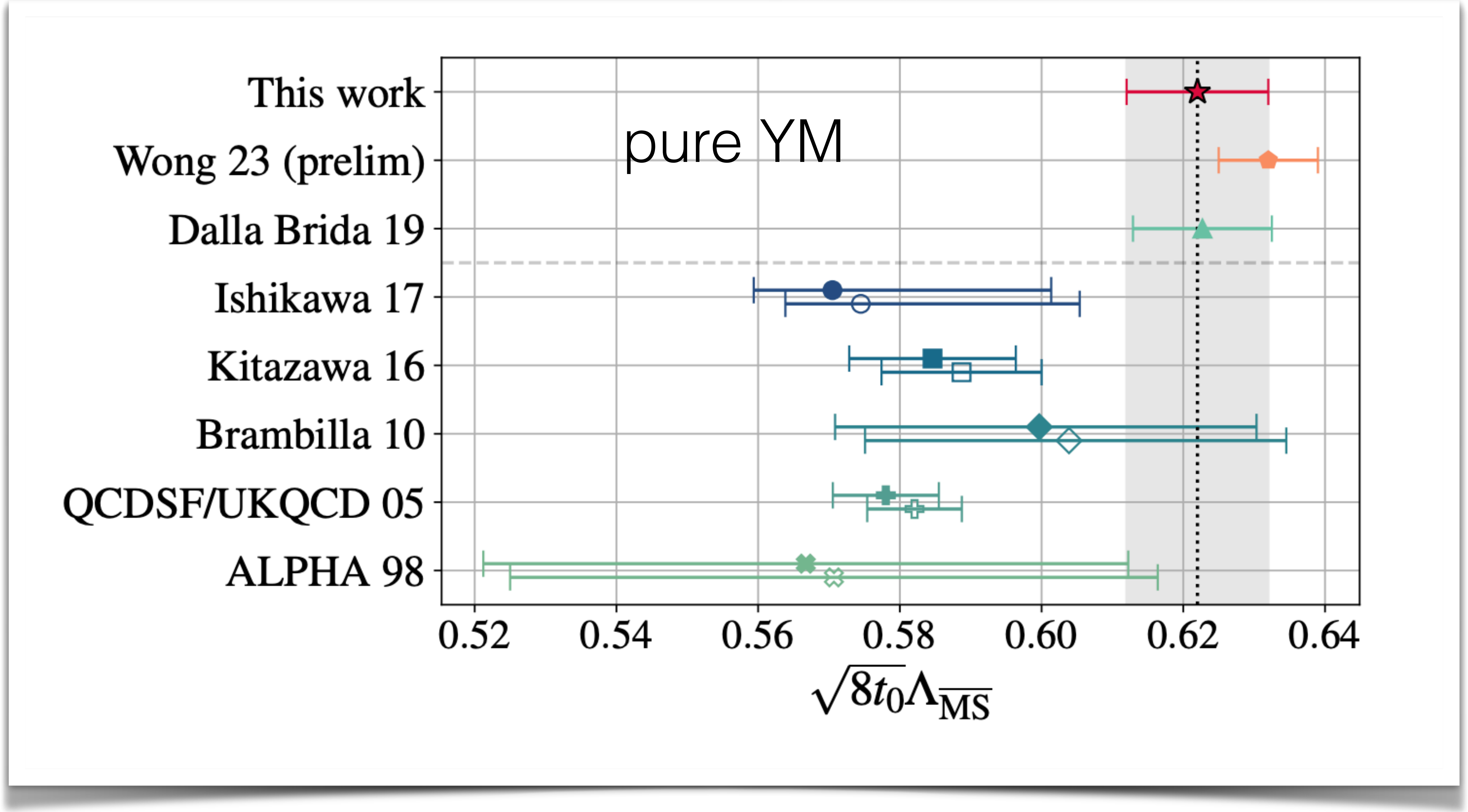


$$t\frac{d}{dt}\hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

$$= \hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$

universal

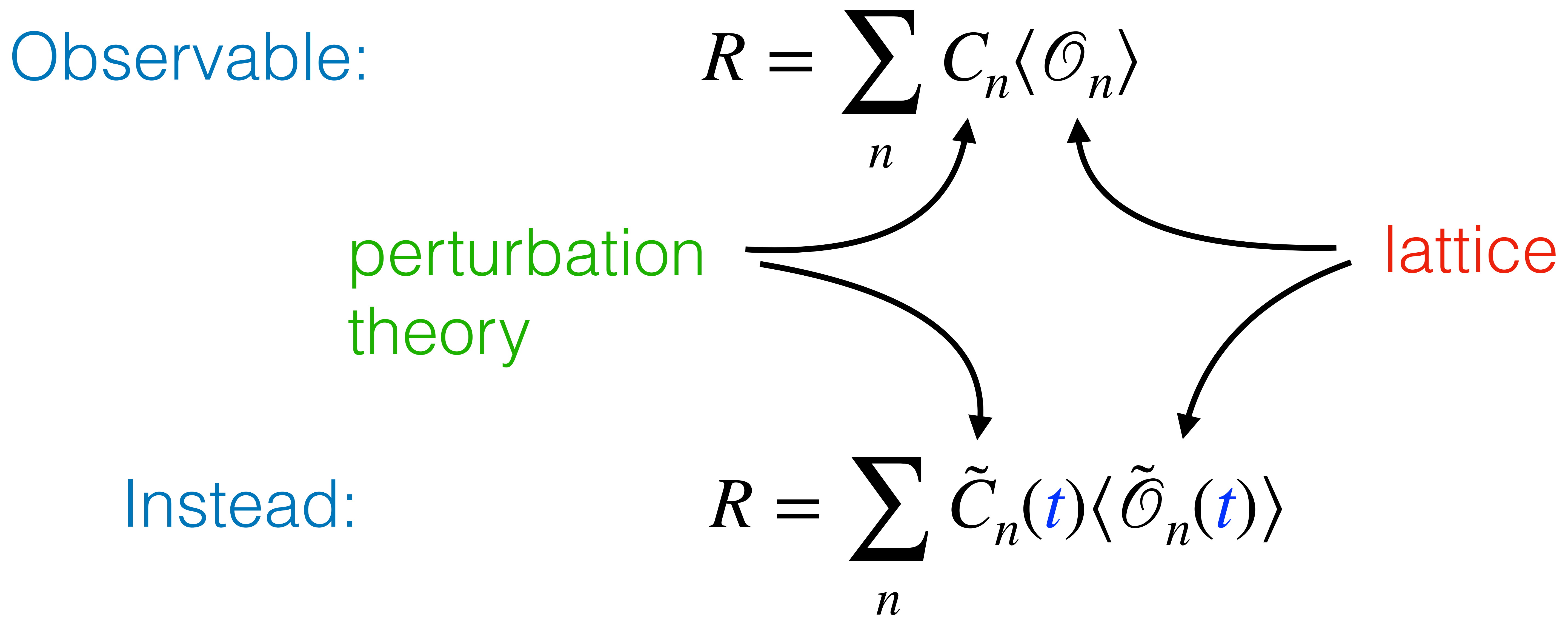
GF specific
depends on k_2



A. Hasenfratz, Peterson, Sickle, Witzel (2023)

see also C.H. Wong et al.

Application to effective field theories



Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

→ need $\tilde{C}(t)$

Small flow-time expansion

Observable:
$$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

small flow-time expansion:

Lüscher, Weisz '11
Suzuki '13
Lüscher '13

$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

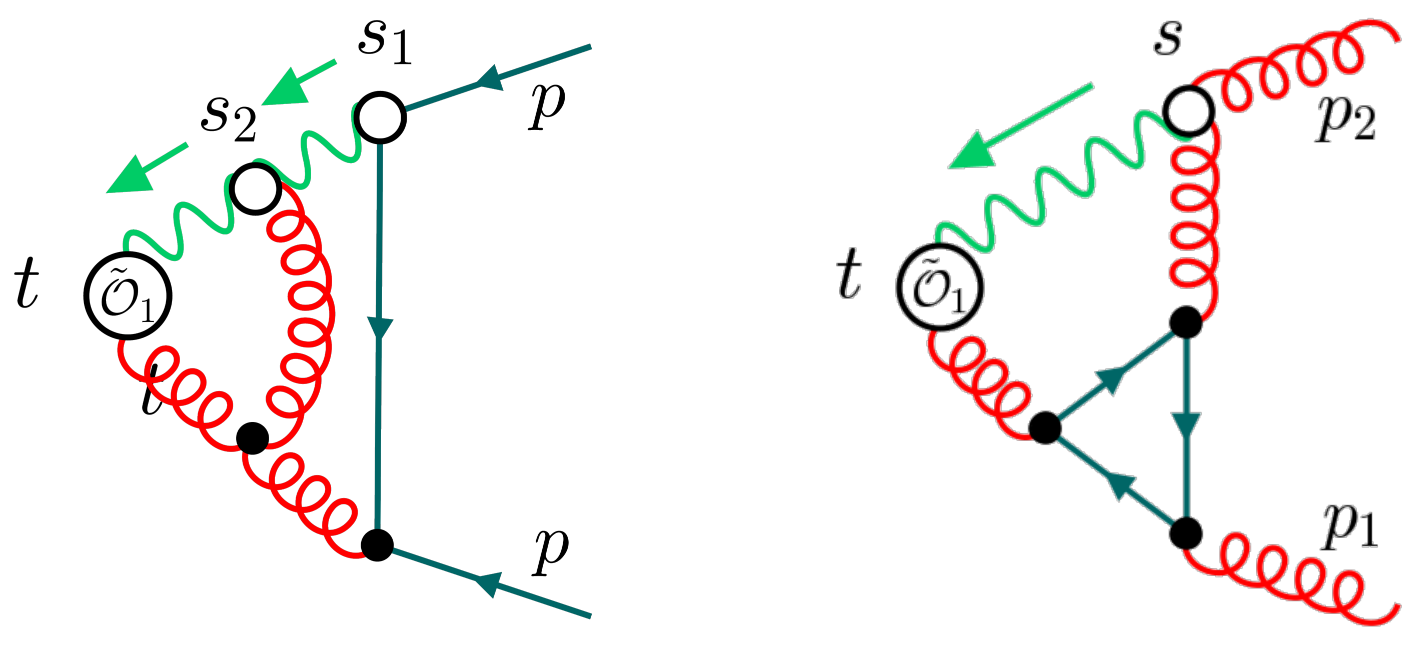
$$\tilde{C}_n(t) \overset{t\to 0}{\to} \sum_m C_m\, \zeta^{-1}_{mn}(t)$$

⇒ need $\zeta_{nm}(t)$ for small t ⇒ perturbation theory

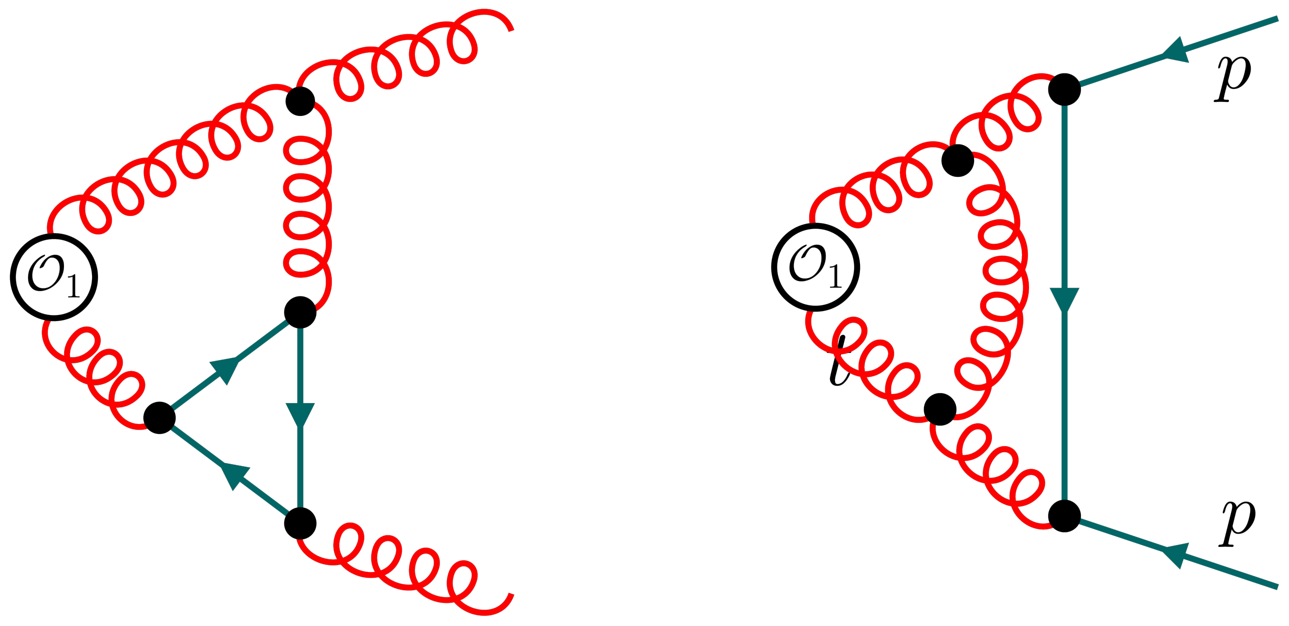
Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \langle \mathcal{O}_m \rangle$$



$\Big|_{p=m=0} \to$

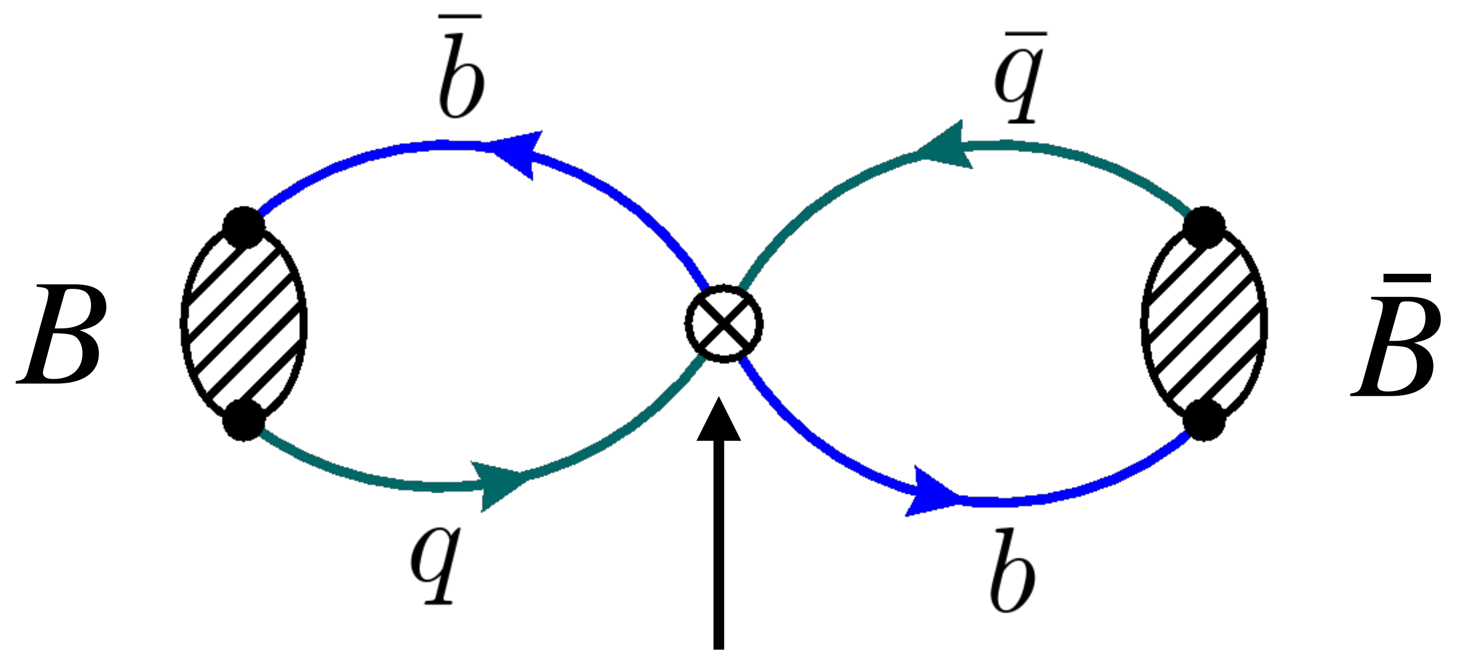


$\Big|_{p=m=0}$

only tree-level diagrams survive on r.h.s.

Gorishnii, Larin, Tkachov '83

Example: Meson mixing



$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}_n \equiv \sum_n \tilde{C}(t)_n \tilde{\mathcal{O}}(t)_n$$

$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\text{L}} q)(\bar{b}\gamma_\mu^{\text{L}} q) \longrightarrow B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$$

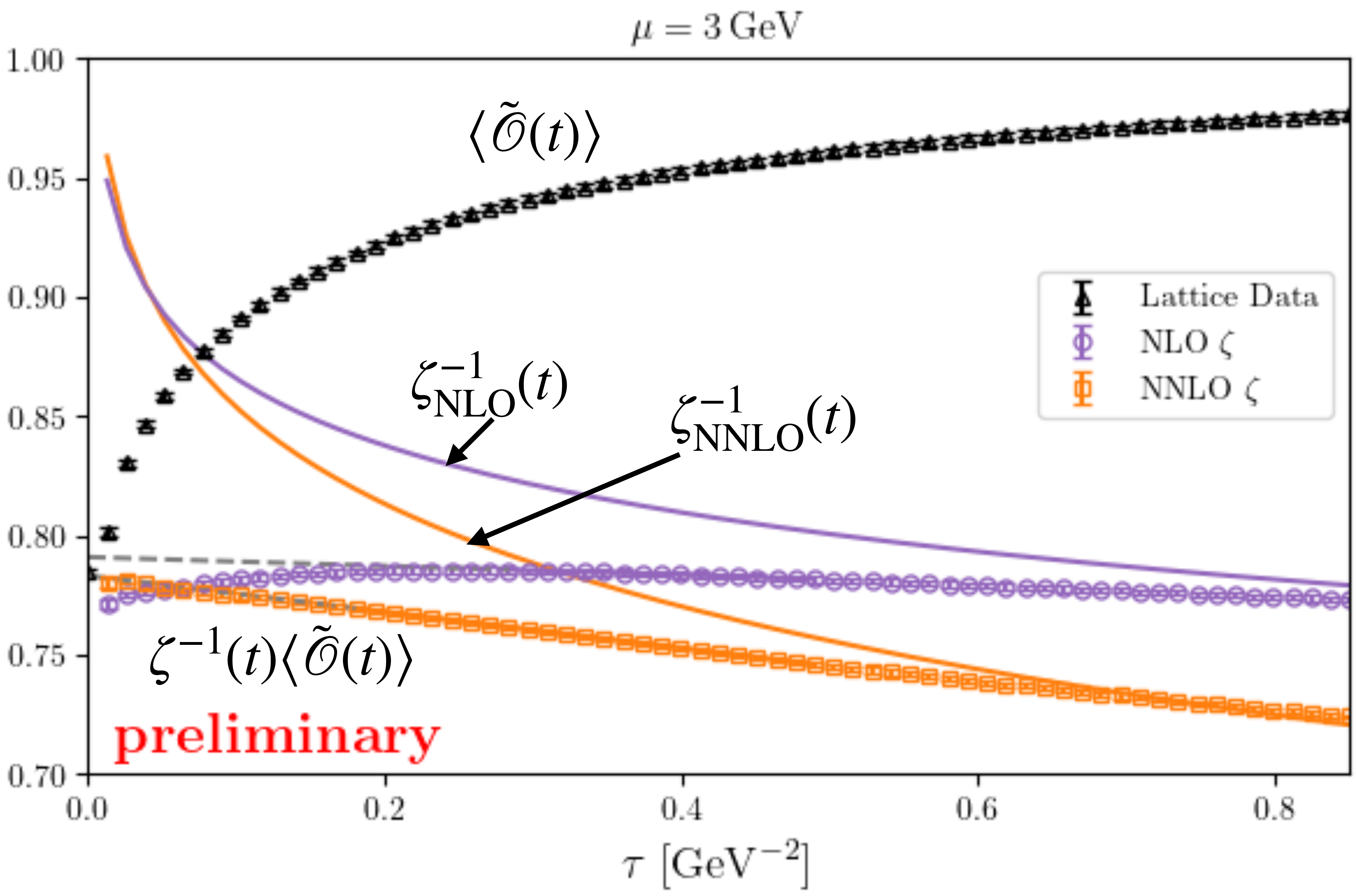
bag parameter

$$= \zeta^{-1}(t) \langle B | \tilde{\mathcal{O}}_1(t) | B \rangle$$

perturbative

lattice

Bag parameter



Black, RH, Lange, Rago, Shindler, Witzel (2023)

CRC TRR 257

Other applications

electric dipole operators

NLO: Mereghetti, Monahan, Rizik, Shindler, Stoffer (2022)
Crosas, Monahan, Rizik, Shindler, Stoffer (2023)
Bühler, Stoffer (2023)
NNLO: Borgulat, RH, Rizik, Shindler (2022)

hadronic vacuum polarization

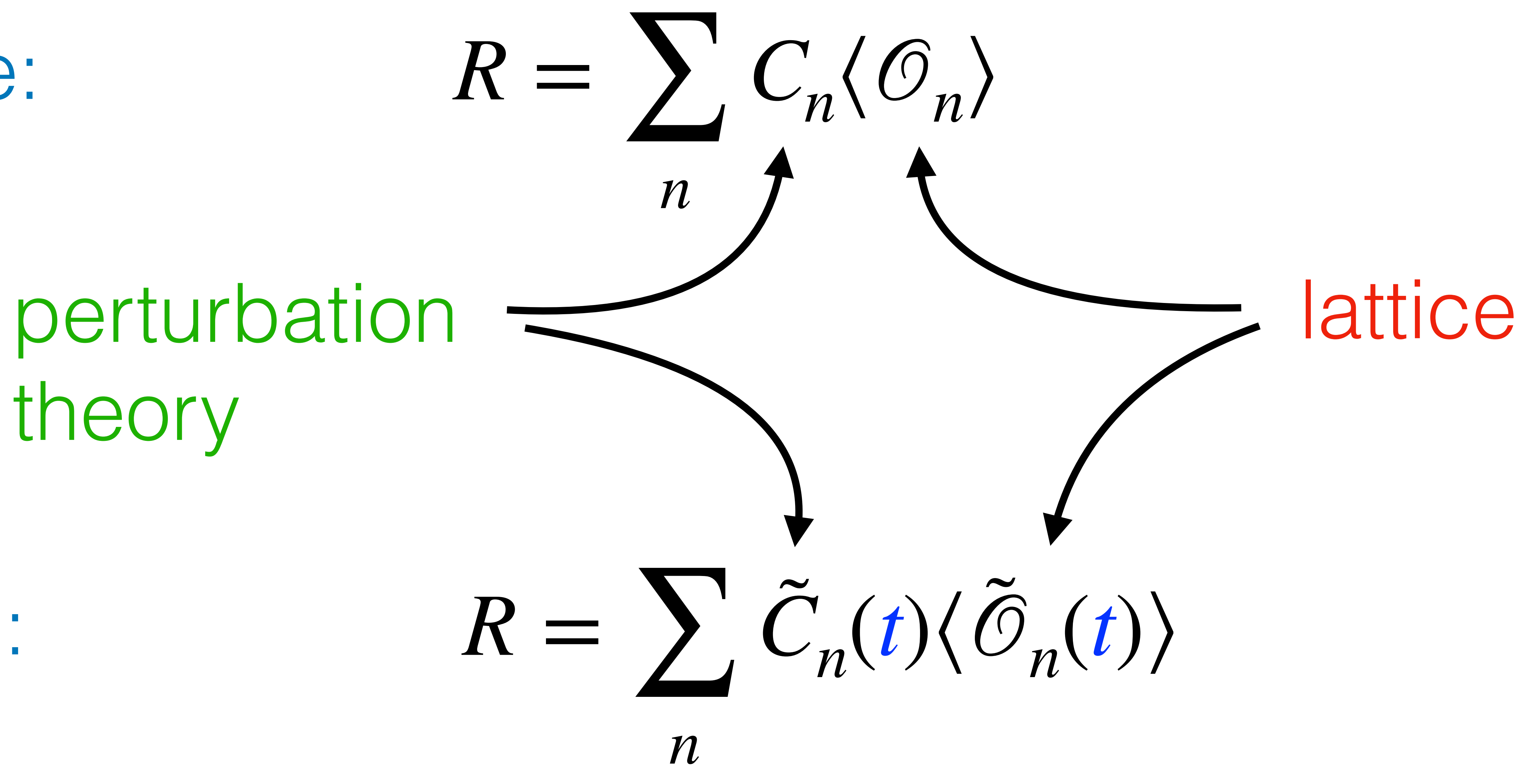
NNLO: RH, Lange, Neumann (2020)

quark bilinears

NLO: Hieda, Suzuki (2016)
NNLO: Borgulat, RH, Kohnen, Lange (2023)

…

Application to effective field theories



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perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

⇒ Lorentz invariance preserved!

application: energy-momentum tensor Suzuki '13

parton density functions Shindler '24

Energy momentum tensor

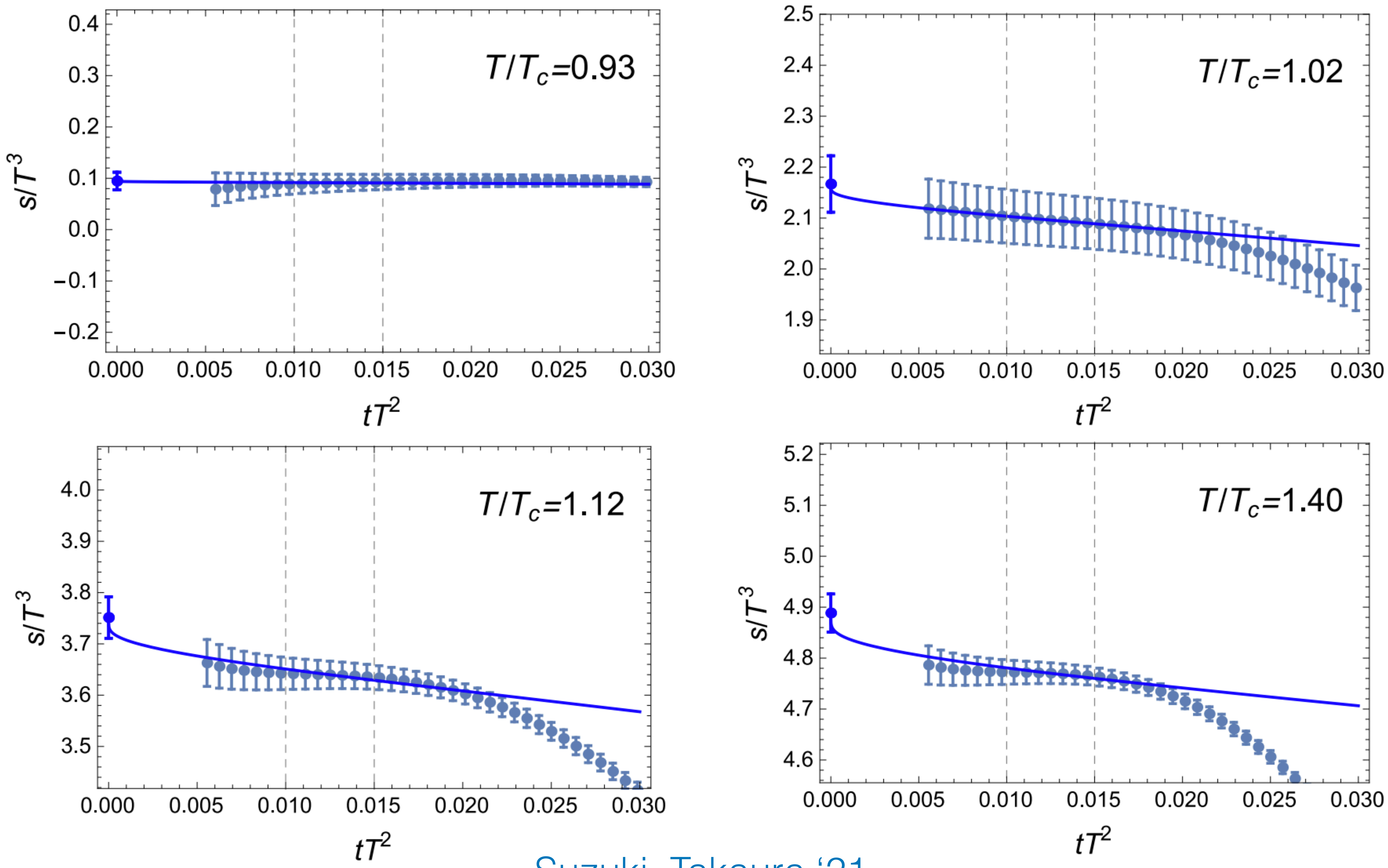
Suzuki '13
Suzuki, Makino '14

Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

$$T_{\mu\nu} = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

NLO



Suzuki, Takaura '21

Energy momentum tensor

Suzuki ‘13
Suzuki, Makino ‘14

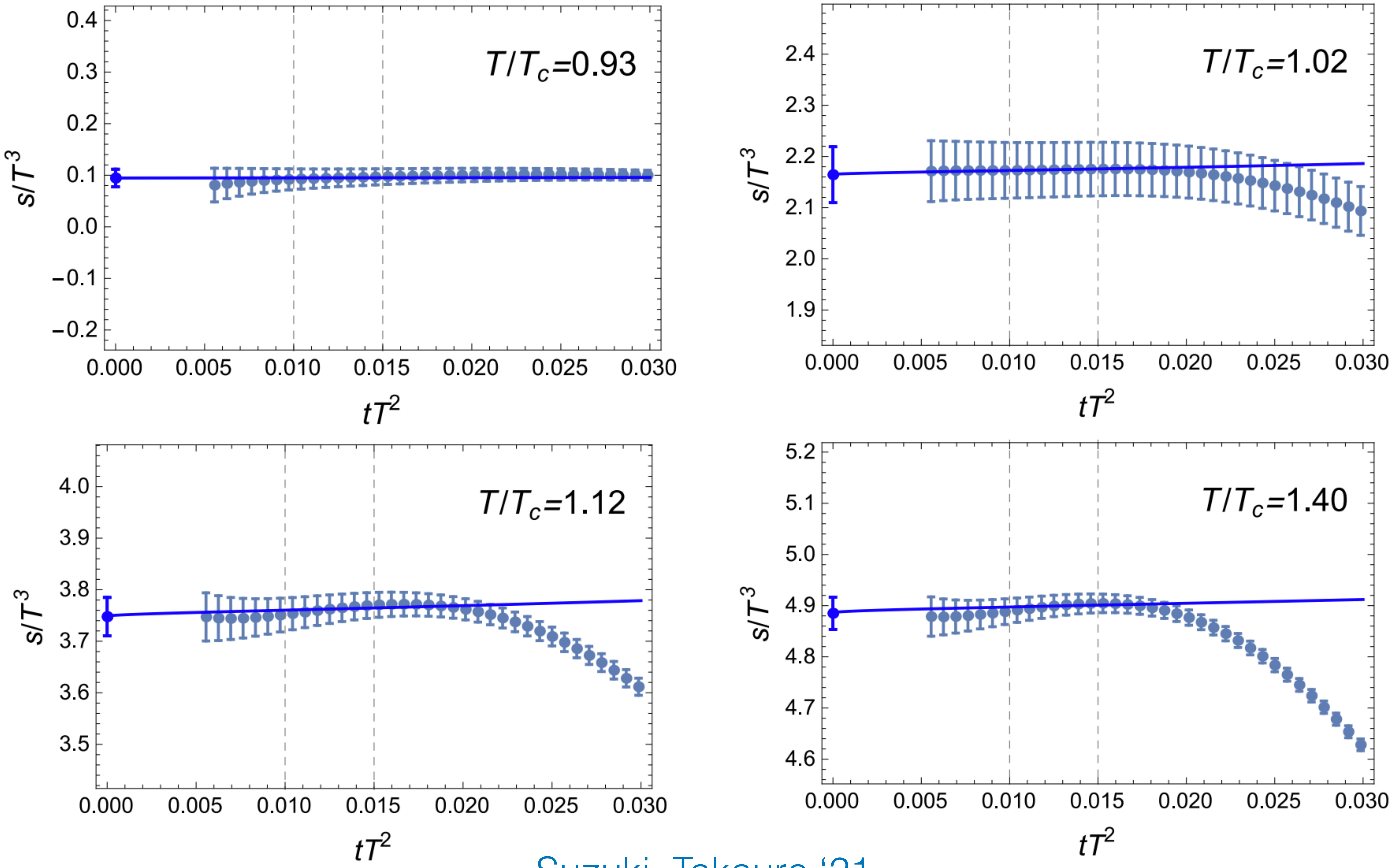
Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

$$T_{\mu\nu} = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

NNLO

RH, Kluth, Lange ‘18



Suzuki, Takaura ‘21

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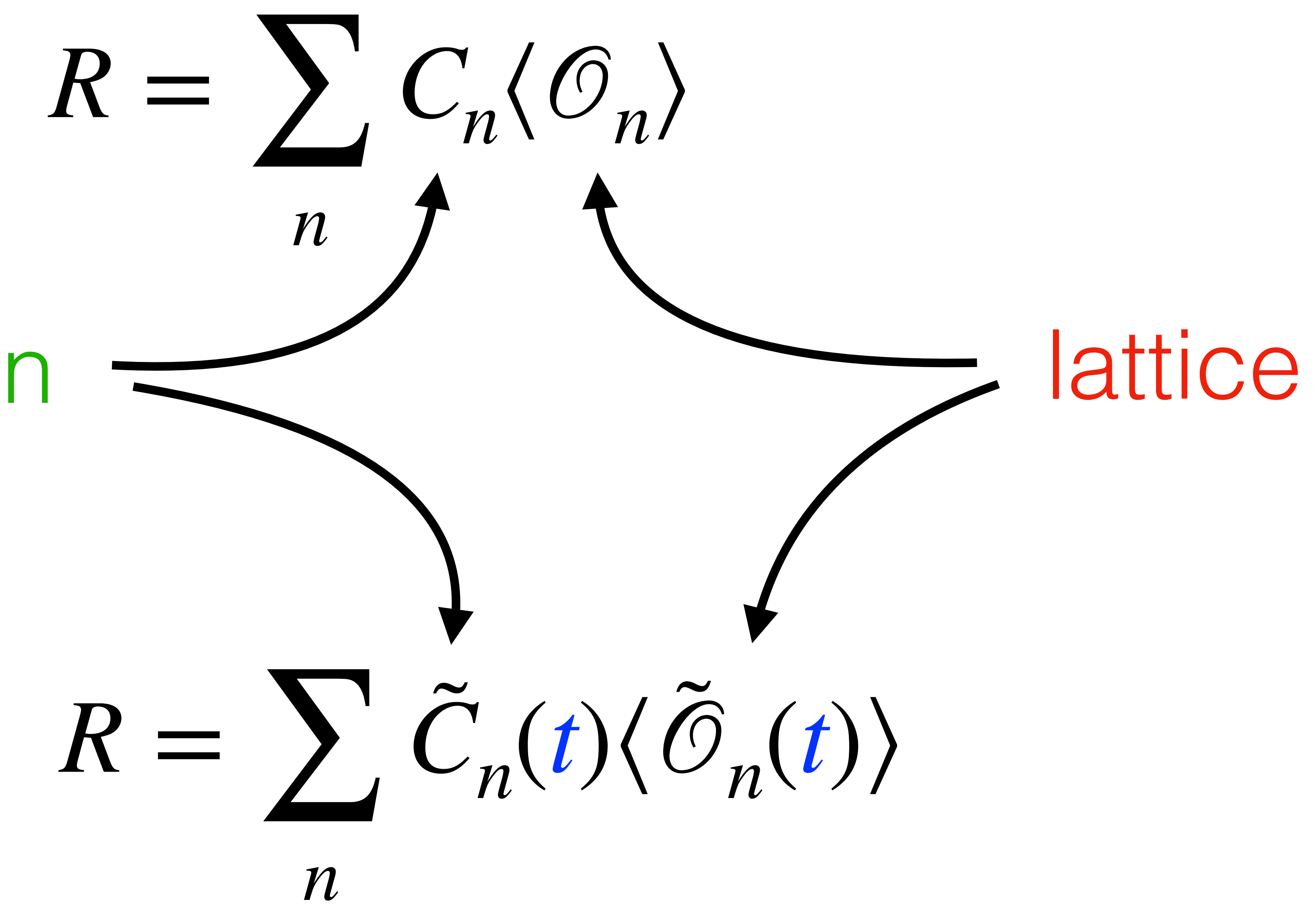

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match renormalization schemes?

gradient flow renormalization

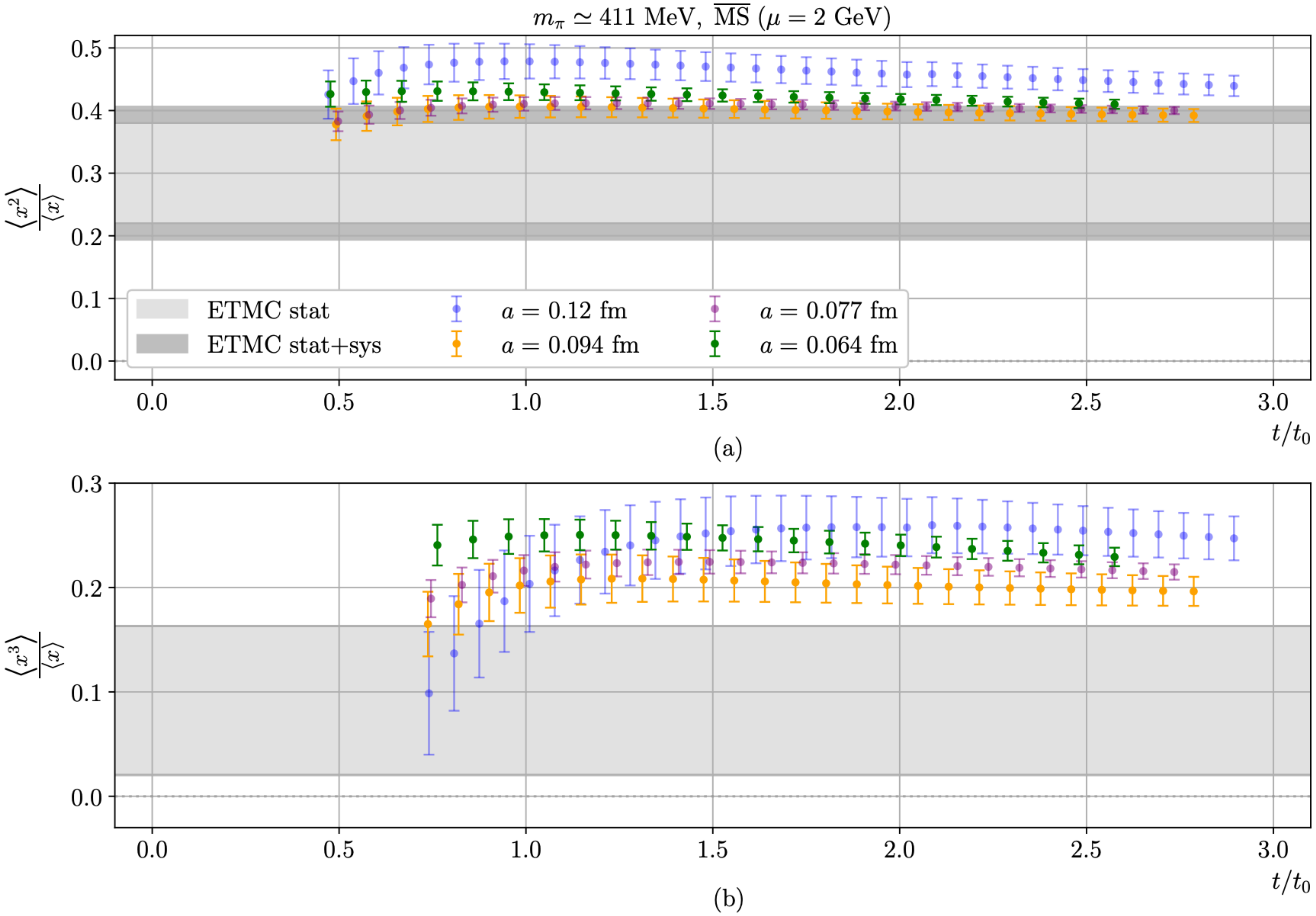
$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a \to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

$\Rightarrow$ Lorentz invariance preserved!

application:	energy-momentum tensor	Suzuki '13
	parton density functions	Shindler '24

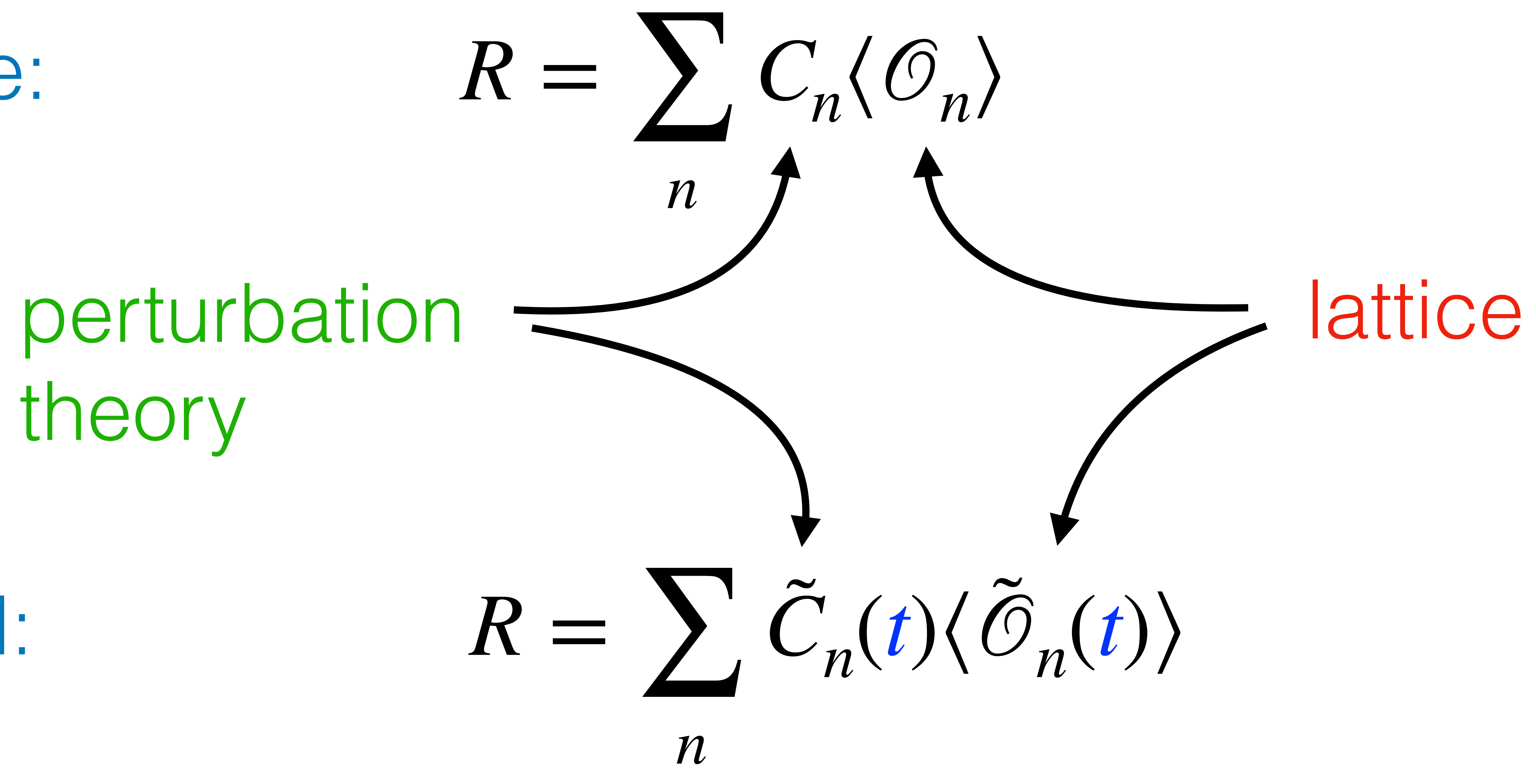
Parton densities

Shindler '24



Francis et al. '24

Application to effective field theories



Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

$\overline{\text{MS}}$ renormalization of composite operators

$\overline{\text{MS}}$ renormalization of composite operators

$$\mathscr{L}_{\rm eff} = \sum_n C_n \mathscr{O}_n$$

needs renormalization:
$$= \sum_n (CZ)_n (Z^{-1}\mathscr{O})_n = \sum_n C_n^{\rm R} \mathscr{O}_n^{\rm R}$$

gradient-flow scheme:
$$= \sum_n \tilde{C}_n(t)\, \tilde{\mathscr{O}}_n(t) = \sum_n (C\zeta^{-1}(t))_n\, (\zeta(t)\mathscr{O})_n(t)$$

small flow-time expansion:
(SFTX)
$$\tilde{\mathscr{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathscr{O}_m = \sum_m (\zeta(t) Z^{-1})_{nm} \mathscr{O}_m^{\rm R}$$

UV finite

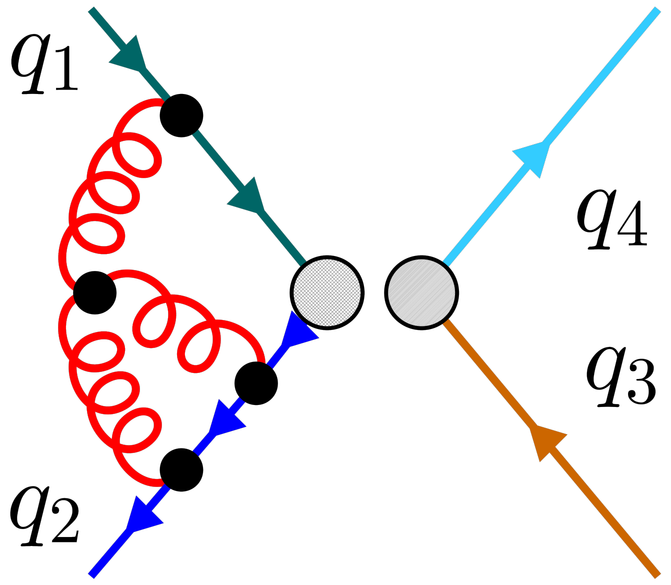
⇒ calculation of $\zeta(t)$ also determines Z in $\overline{\text{MS}}$ scheme!

$\overline{\text{MS}}$ renormalization of composite operators

application: check for four-quark operators

RH, Lange (2022)
RH, Kohnen, Lange (in prep)

Buras, Gorbahn, Haisch, Nierste (2006)
Aebischer, Pesut, Virto (2024)



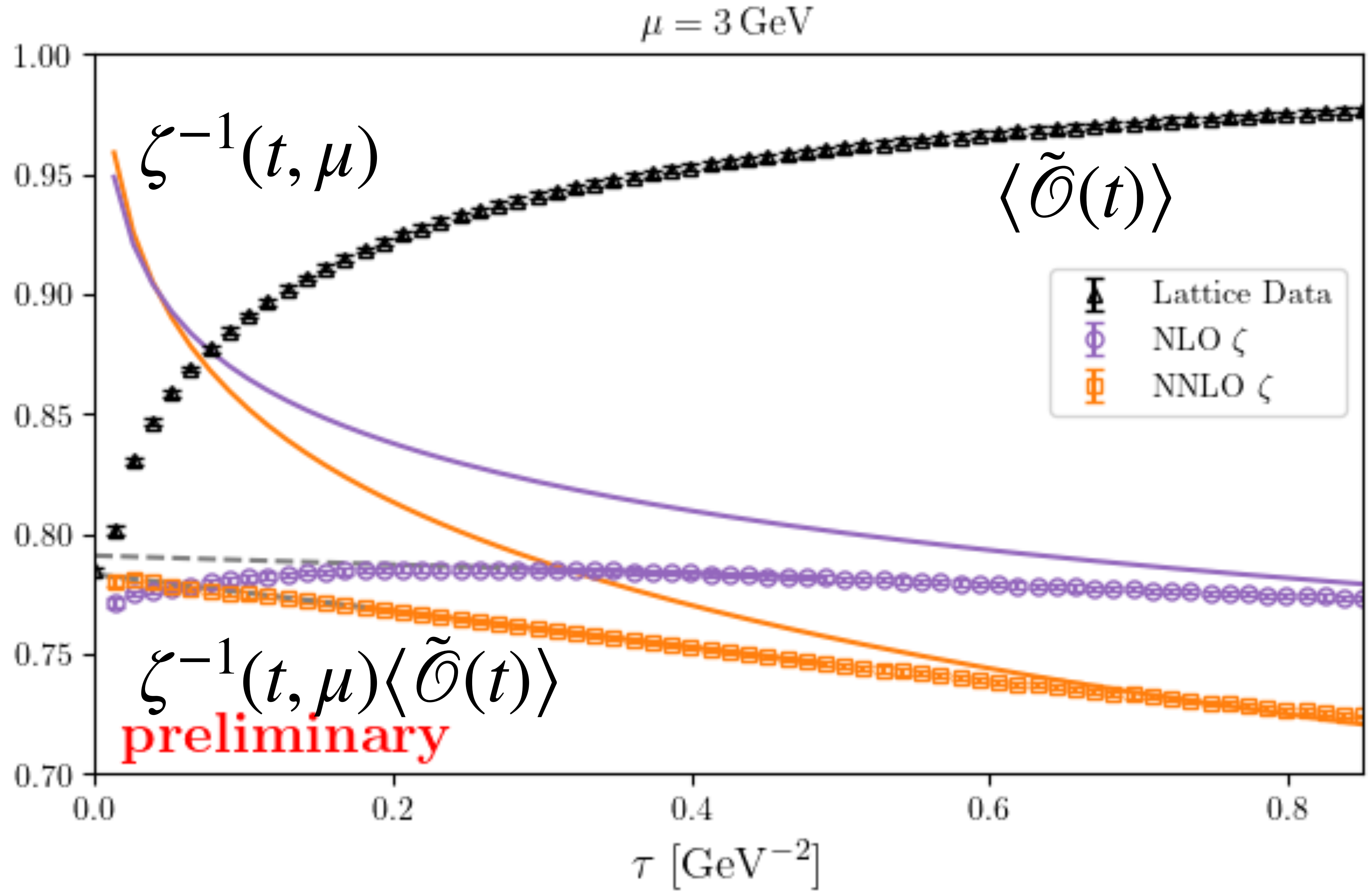
The gradient flow scheme

$$\Delta\Gamma \sim \sum_n C_n^{\mathrm{R}}(\mu)\,\langle \mathcal{O}_n^{\mathrm{R}}\rangle(\mu)$$

$$= \sum_n C_n^{\mathrm{R}}(\mu)\left[\zeta^{-1}(t,\mu)\langle\tilde{\mathcal{O}}_n\rangle(t)\right]$$

$$= \sum_n \left[C_n^{\mathrm{R}}(\mu)\,\zeta^{-1}(t,\mu)\right]\langle\tilde{\mathcal{O}}_n\rangle(t)$$

$$= \sum_n \tilde{C}_n(t)\langle\tilde{\mathcal{O}}_n\rangle(t)$$



[…] Hence, no conversion to the MS scheme is needed any more, the advantage being that the renormalization scheme employed is well-defined beyond perturbation theory.

Ammer, Dürr '24

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n\,\langle \zeta(t)\,\mathcal{O}\rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O}\rangle_n$
$= \sum_n \tilde{C}_n(t)\,\langle \tilde{\mathcal{O}}_n(t)\rangle$	$= \sum_n C^{\mathrm{R}}_n(\mu)\,\langle \mathcal{O}^{\mathrm{R}}_n\rangle(\mu)$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\mathrm{R}}(\mu) = C^{\mathrm{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$

RH, Lange, Neumann '20

Borgulat, Felten, RH, Kohnen '25

The GF scheme

GF	$\overline{\text{MS}}$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\mathrm{R}}(\mu) = C^{\mathrm{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\mu\frac{d}{d\mu}\ln Z$
$\tilde{C}(t) = \tilde{C}(t_0)\,\exp\left[\int_{\tilde{\alpha}_s(t_0)}^{\tilde{\alpha}_s(t)}\frac{dx}{x}\frac{\tilde{\gamma}(x)}{\tilde{\beta}(x)}\right]$	$C^{\mathrm{R}}(\mu) = C^{\mathrm{R}}(\mu_0)\,\exp\left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\mu}{\mu}\frac{\gamma(x)}{\beta(x)}\right]$
$t\frac{d}{dt}\tilde{\alpha}_s(t) = \tilde{\beta}\,\tilde{\alpha}_s(t)$	$\mu\frac{d}{d\mu}\alpha_s(\mu) = \beta\,\alpha_s(\mu)$

Conclusions

- Gradient flow provides a bridge between lattice and perturbation theory
- Perturbative calculations very close to standard QCD
- Still challenges on the lattice
- Proofs of principle exist
- Option for otherwise inaccessible problems (mixing of different mass dimensions)

- Plenty of opportunities!