

Gradient Flow in Perturbation Theory

Robert Harlander

RWTH Aachen University

13 May 2026

Standard Model parameters and observables from gradient flow

Edinburgh, 12-15 May 2026

P  **H**
CRC TRR 257

supported by:

DFG Deutsche
Forschungsgemeinschaft



Man 1: A man with long hair and sunglasses, wearing a dark blue jacket and light blue jeans.

Man 2: A man wearing a dark cap, a dark jacket over a black t-shirt with white text, and light blue jeans. The t-shirt text includes "IGSND", "4BILT", "404P7 7C", and "THINGS".

Man 3: A man wearing a dark cap, a white t-shirt with a graphic, and blue jeans.

Man 4: A man wearing a dark cap, a dark jacket, and blue jeans.

Man 5: A man wearing a dark jacket and light-colored pants.

Man 6: A man wearing glasses, a plaid shirt, and dark pants, with a camera hanging from his neck.

	Greetings	<i>Matthew Black et al.</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	08:45 - 09:00
09:00	Gradient flow in perturbation theory	<i>Robert Harlander</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	09:00 - 10:00
10:00	Gradient-flowed operator product expansion of the Adler function without IR renormalons	<i>Martin Beneke</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	10:00 - 10:30
	Coffee Break	
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	10:30 - 11:00
11:00	Quark mass effects in flowed action density	<i>Robert Mason</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	11:00 - 11:30
	The SFTX of LEFT	<i>Oscar Lara Crosas</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	11:30 - 12:00
12:00	The perturbative matching of four-quark operators between gradient flow and \overline{MS}	<i>Jonas Kohnen</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	12:00 - 12:30
	Lattice determination of bag parameters using gradient flow	<i>Antonio Rago</i>
	<i>Lecture theatre B, 3rd Floor, James Clerk Maxwell Building</i>	12:30 - 13:00
13:00	Lunch	



Motivation

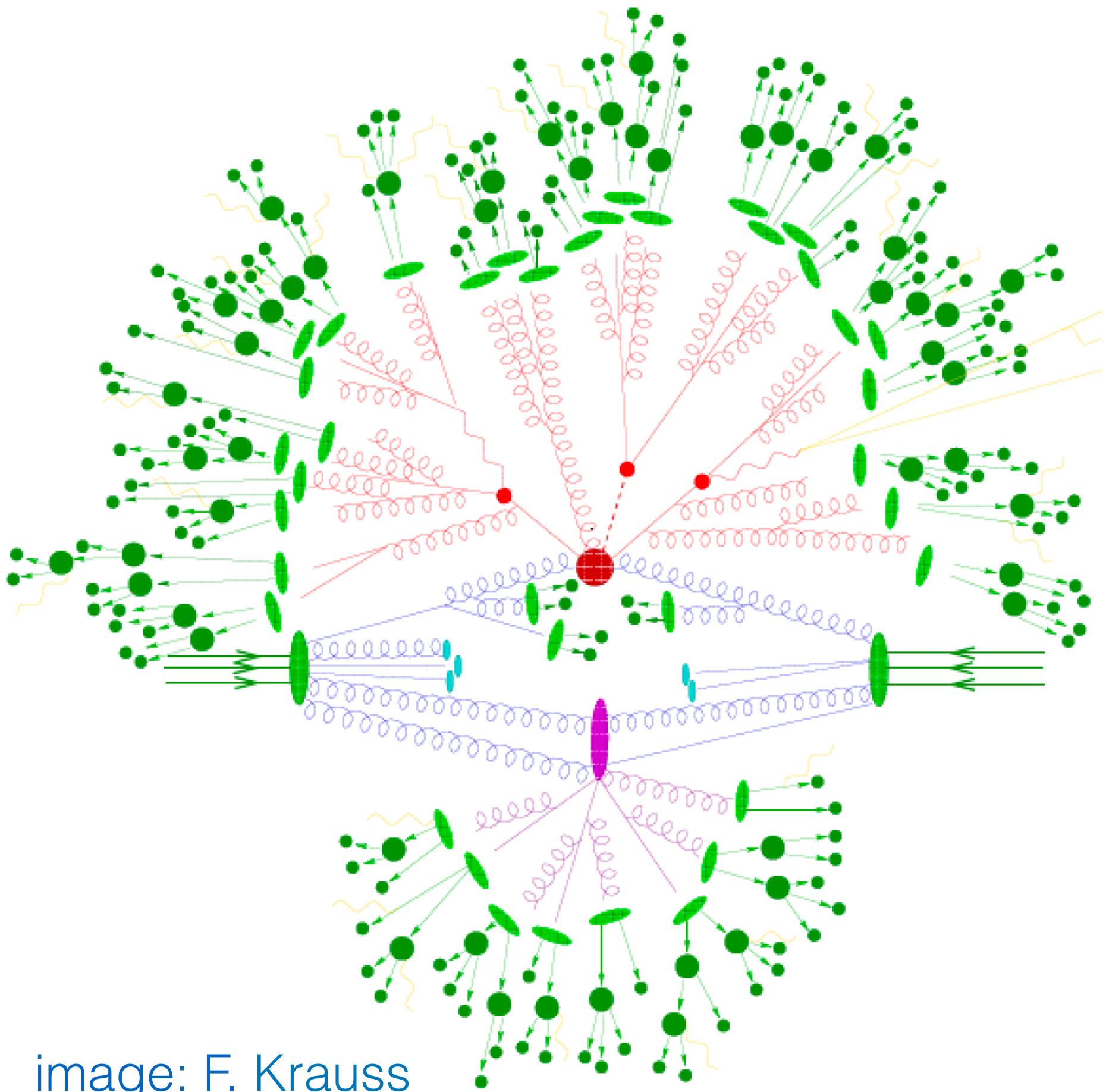


image: F. Krauss

Motivation

perturbative contributions:
first principles

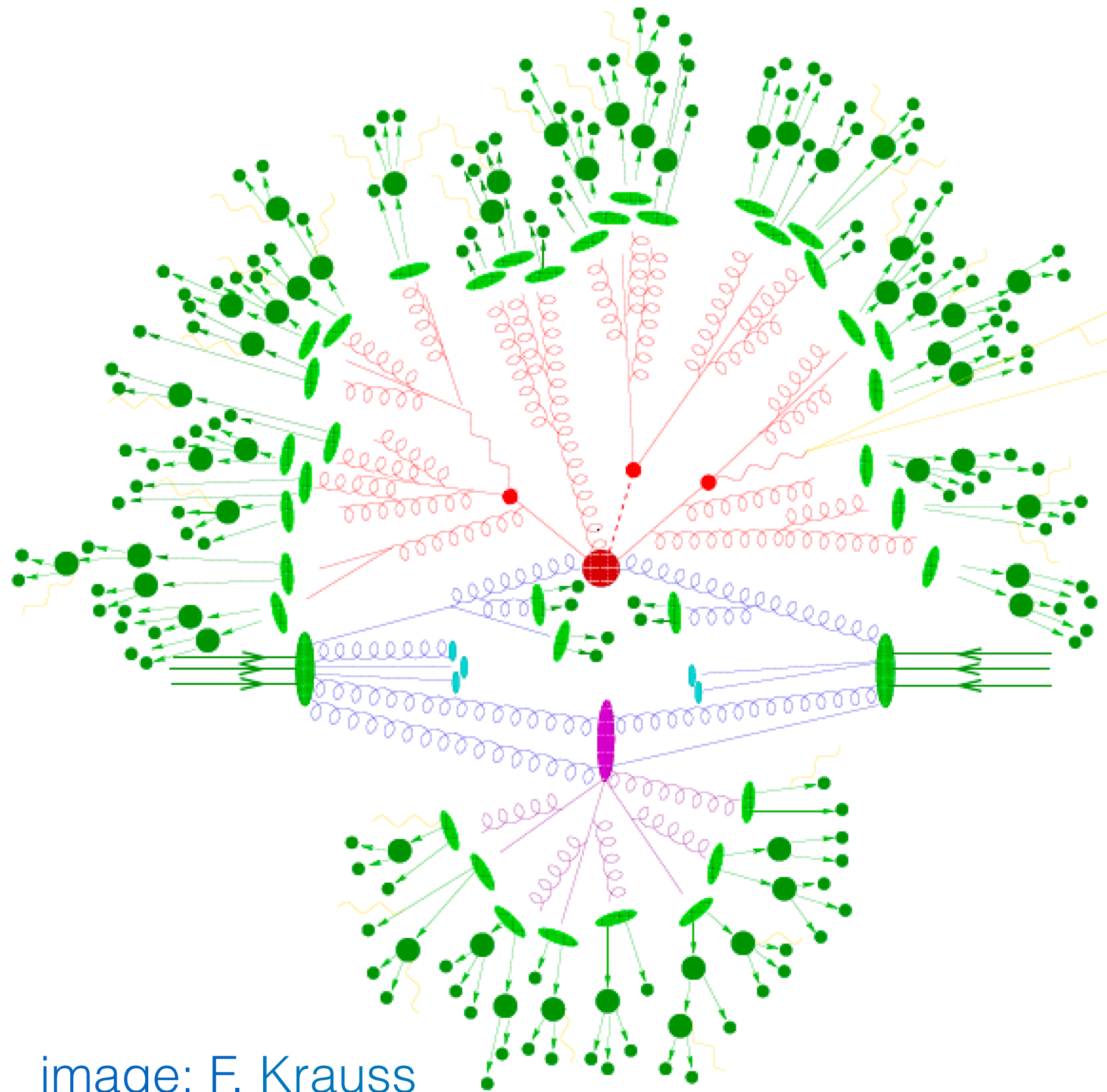


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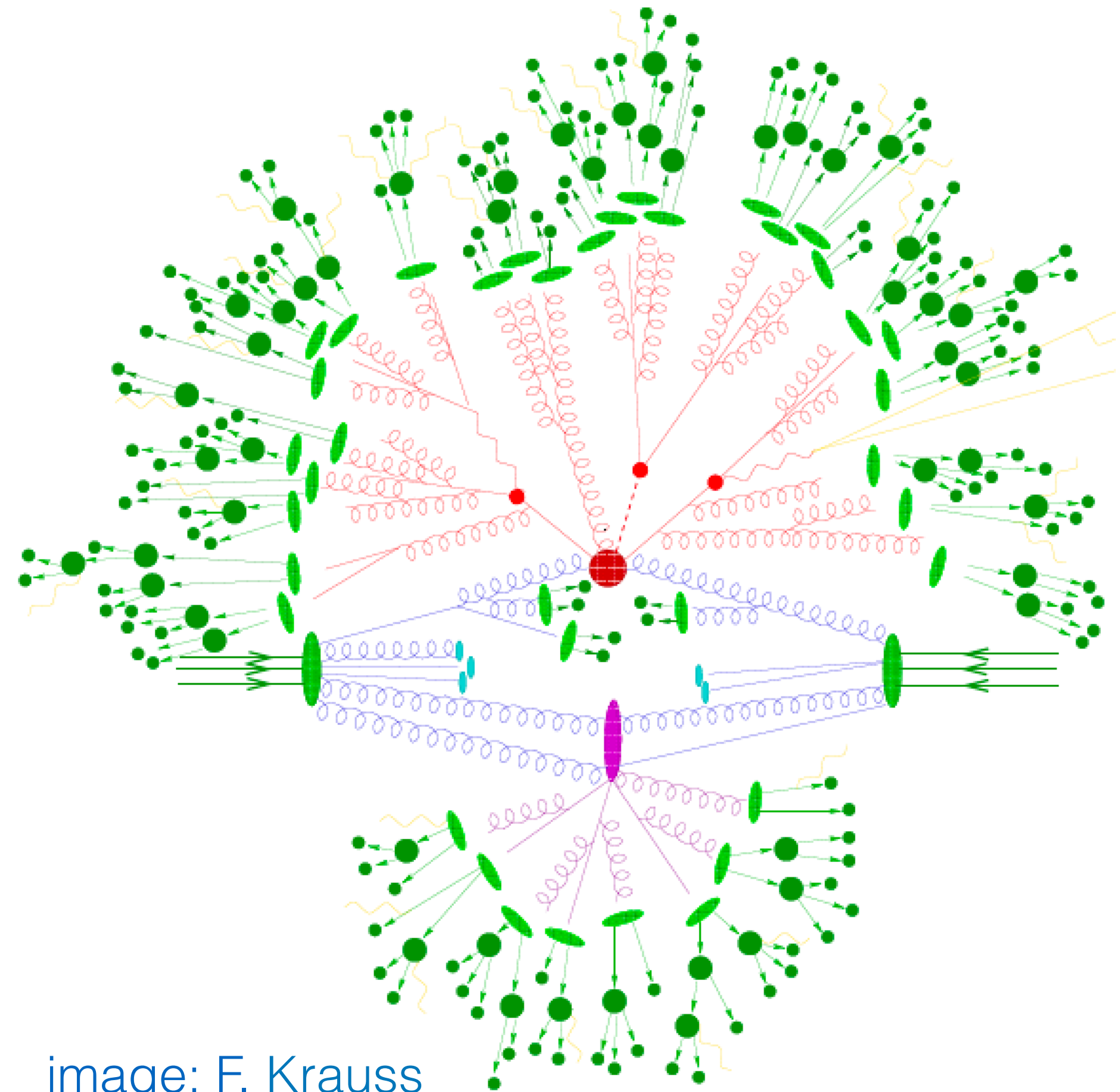


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perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

Motivation

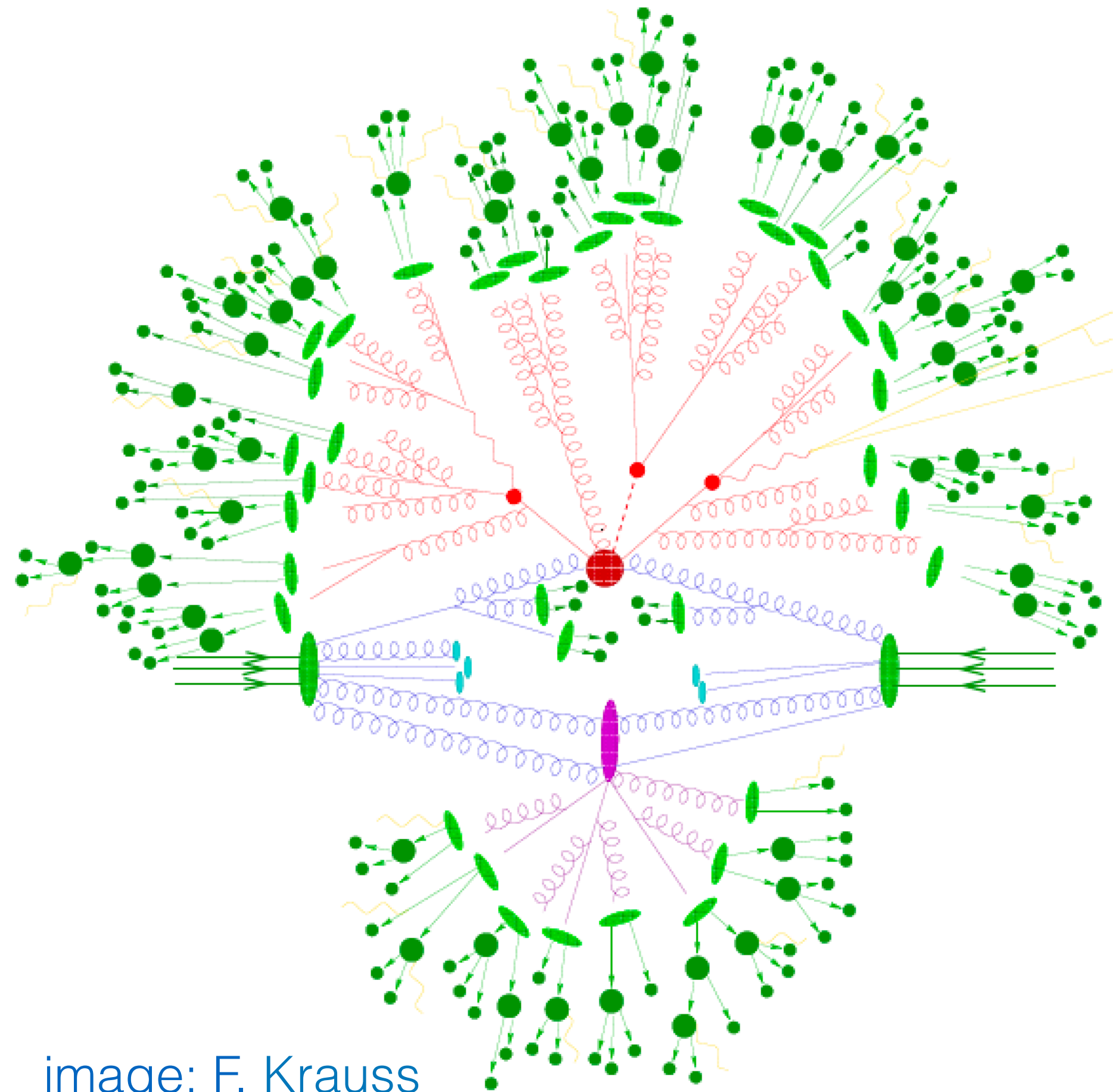


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lattice?

Motivation

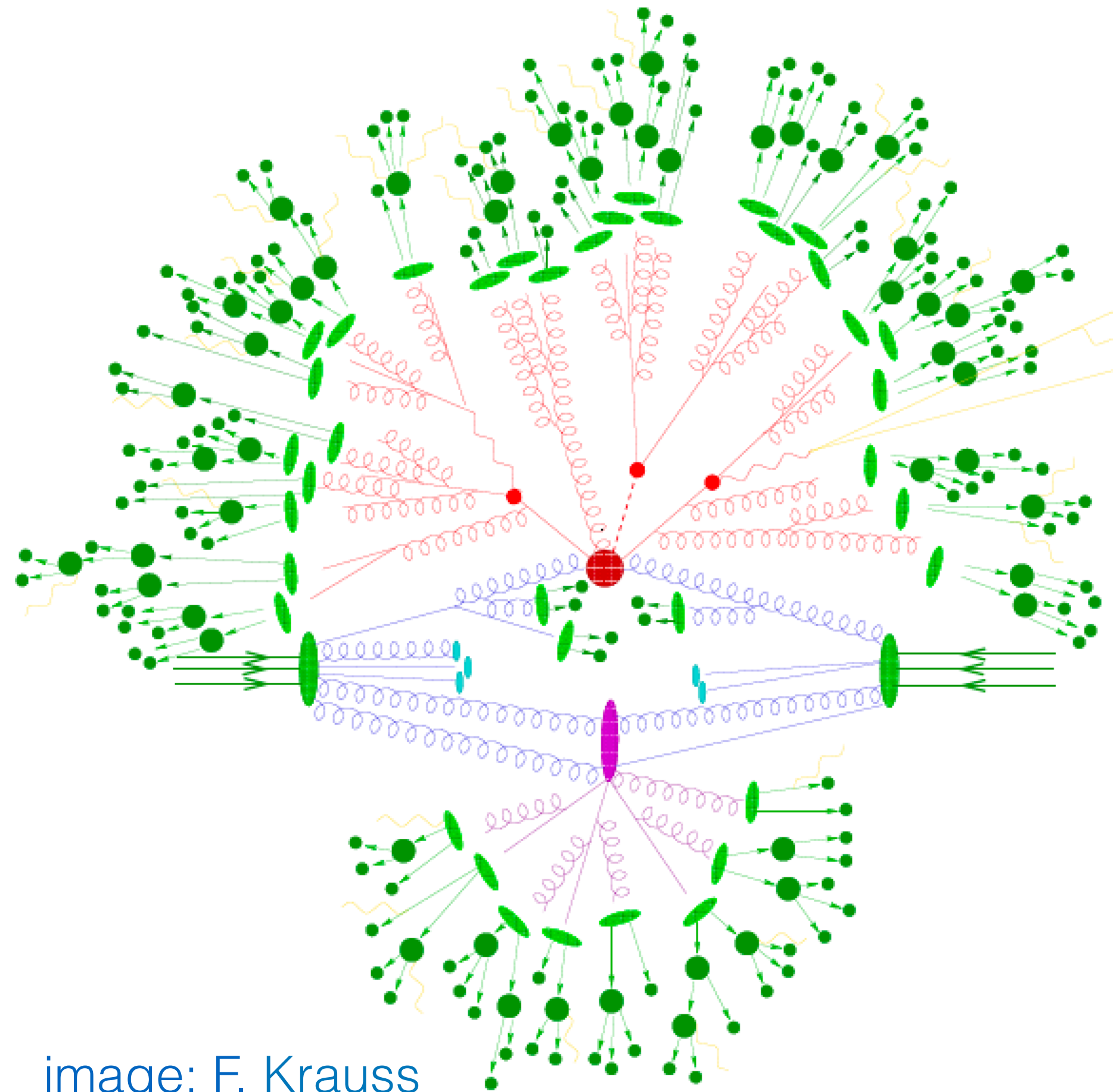


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lattice?

PDFs defined via light-cone correlators,
lattice formulated in Euclidean space

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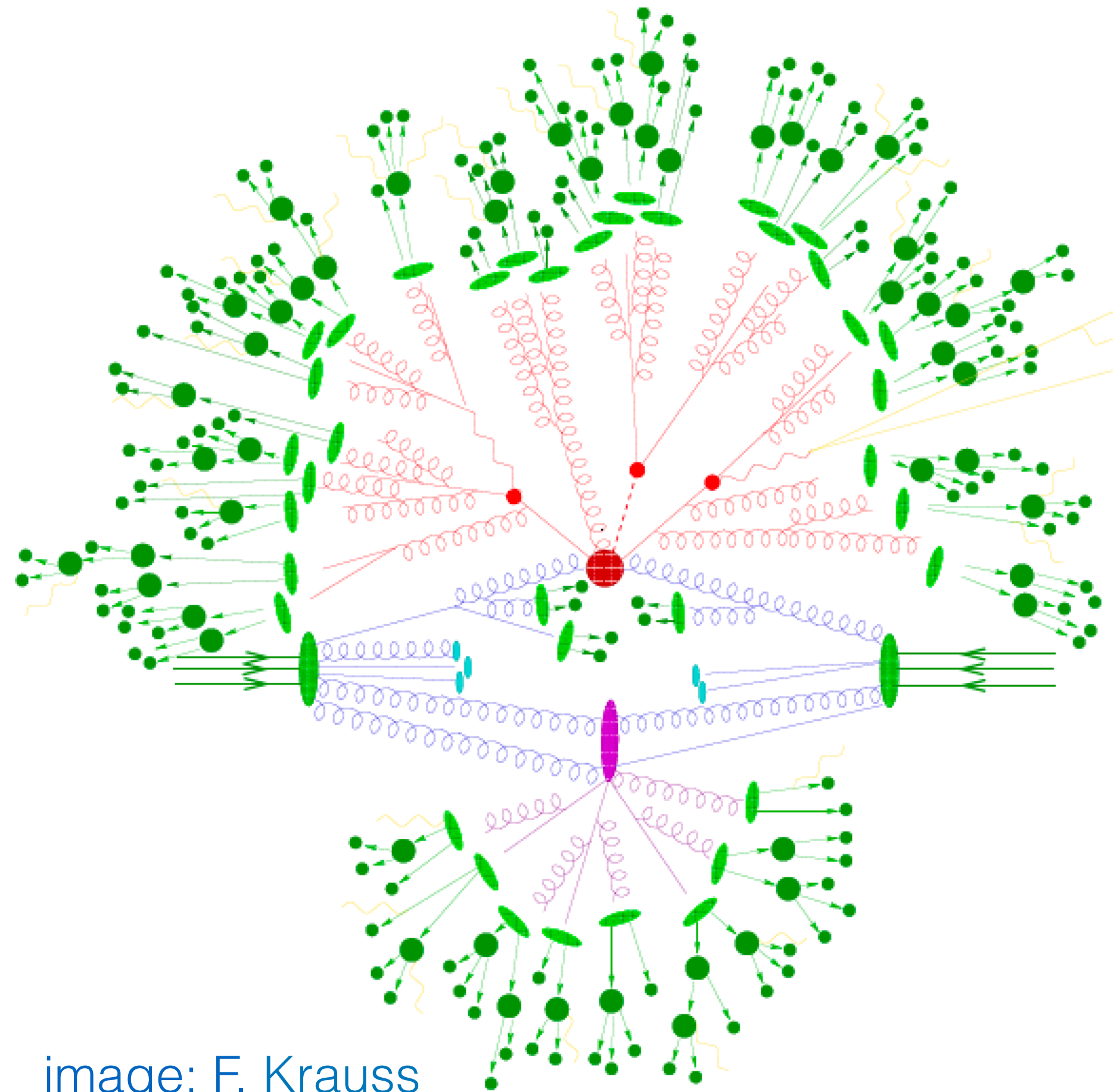


image: F. Krauss

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alternative: moments

$$\langle x^n \rangle_q = \int_0^1 dx x^n q(x)$$

Moments of parton densities

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \dots p_{\mu_n} \langle x^{n-1} \rangle$$

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$$\mathcal{O}_{\mu_1 \dots \mu_n} \sim \int d^4x \bar{\psi}(x) \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \dots \overleftrightarrow{D}_{\mu_n} \psi(x)$$

$$D_{\mu} = \partial_{\mu} + igA_{\mu}$$

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problem: Lattice breaks Lorentz symmetry

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problem: Lattice breaks Lorentz symmetry

→ mixing with lower-dimensional operators

→ $a \rightarrow 0$ limit challenging

→ only results up to $\langle x^3 \rangle$ available

Martinelli, Sachrajda '87, '88

Alexandrou et al. '20, '21

Gradient flow as cutoff

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In $\mathcal{O}_{\mu_1 \dots \mu_n}$, remove high-momentum modes $p^2 \gtrsim 1/t$

$$\psi(x) \rightarrow \chi(x, t)$$

$$A_{\mu}(x) \rightarrow B_{\mu}(x, t)$$

in a gauge and Lorentz invariant way

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→ continuum limit can be taken, result depends on UV cutoff t

Gradient flow as cutoff

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$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k e^{ikx} \tilde{A}_\mu(k)$$

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$$G_{\mu\nu}(x, t) \sim [D_\mu(t), D_\nu(t)]$$

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analogously:

$$\partial_t \chi(x, t) = D^2 \chi(x, t)$$

$$\chi(x, 0) = \psi(x)$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x, t), \cdot]$$

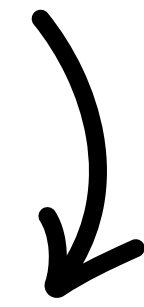
$$G_{\mu\nu}(x, t) \sim [D_\mu(t), D_\nu(t)]$$

Moments of parton densities

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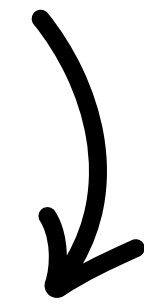


Shindler '24

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}}(t) | h(p) \rangle$$

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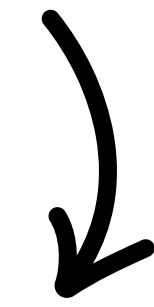
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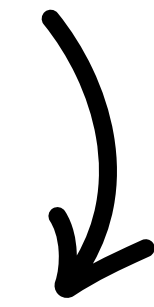
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Andrea Shindler (Fri 09:30)

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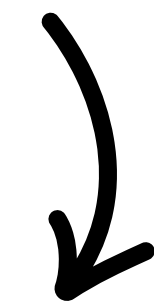
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Continuum limit exists.

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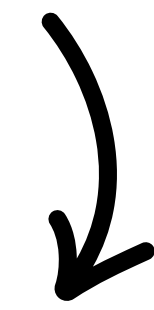
Andrea Shindler (Fri 09:30)

Continuum limit exists.

Lorentz and gauge invariance preserved.

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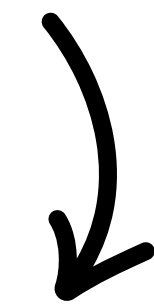
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$t \rightarrow 0$?

Short flow time expansion

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$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t) \mathcal{O}_m + t(\dots)$$

Lüscher '14
Suzuki '15

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Lüscher '14

Suzuki '15

higher-dimensional operators → neglect

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Lüscher '14
Suzuki '15

higher-dimensional operators \rightarrow neglect

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}} | h(p) \rangle = \sum_m \zeta_{mn}^{-1}(t) \langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}}(t) | h(p) \rangle$$

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lattice

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perturbation theory

lattice

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perturbation theory

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Lüscher '14
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perturbation theory

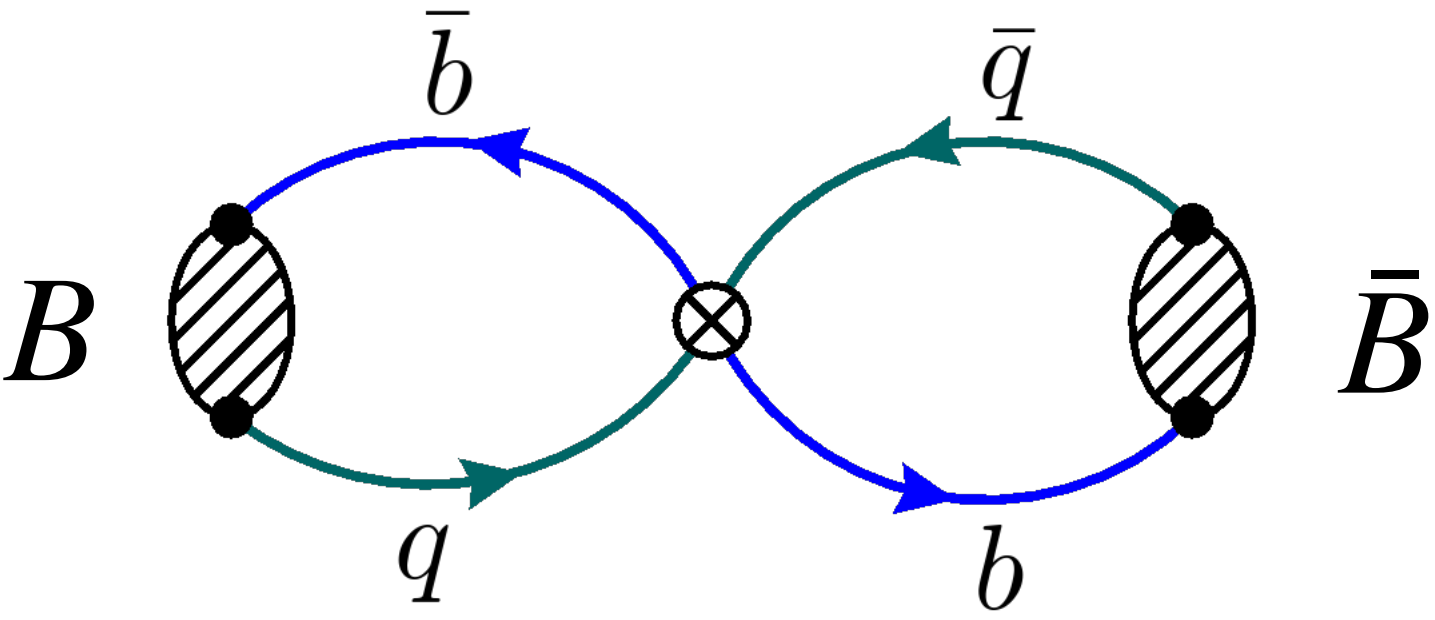
lattice

actually:

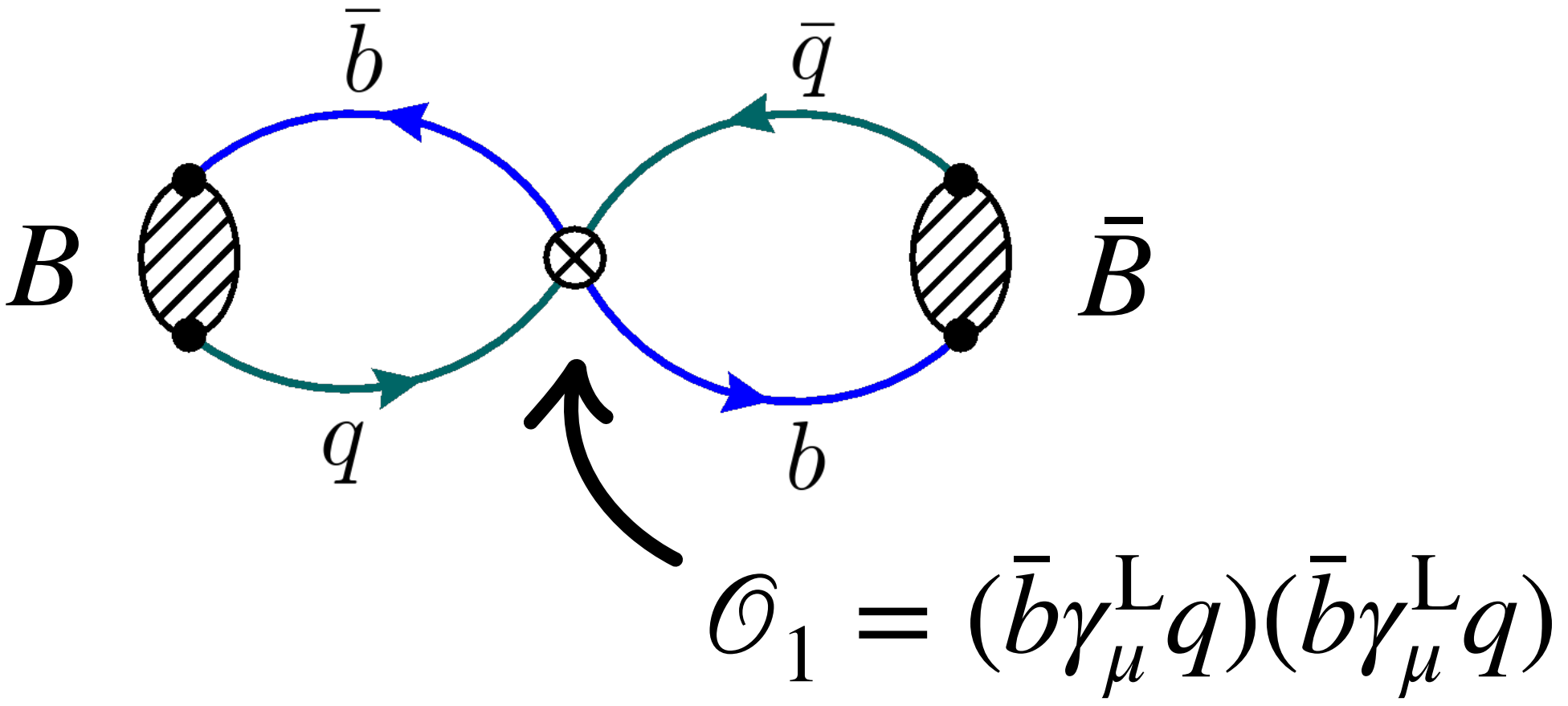
$$\zeta_{nm}^R(t, \mu) = \sum_k \zeta_{nk}(t) Z_{km}^{-1}$$



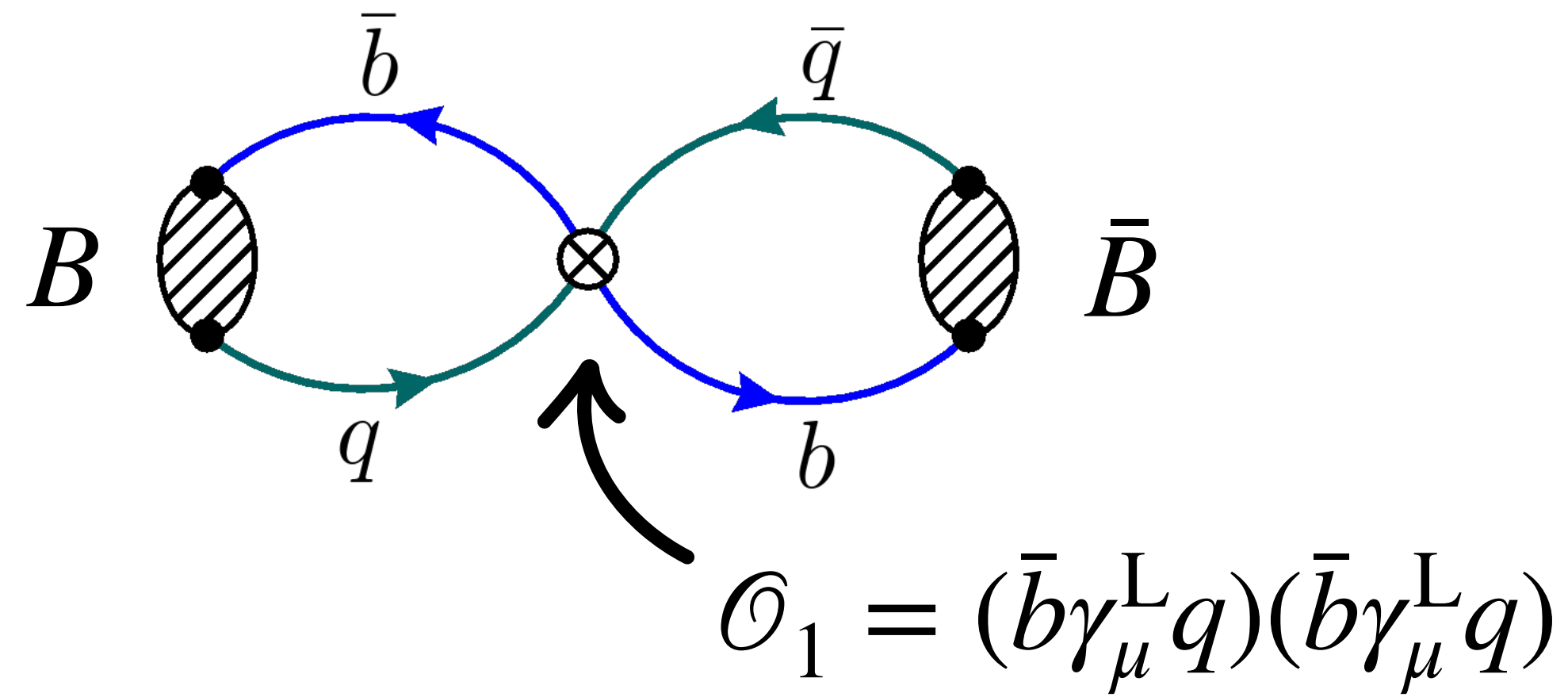
Meson mixing



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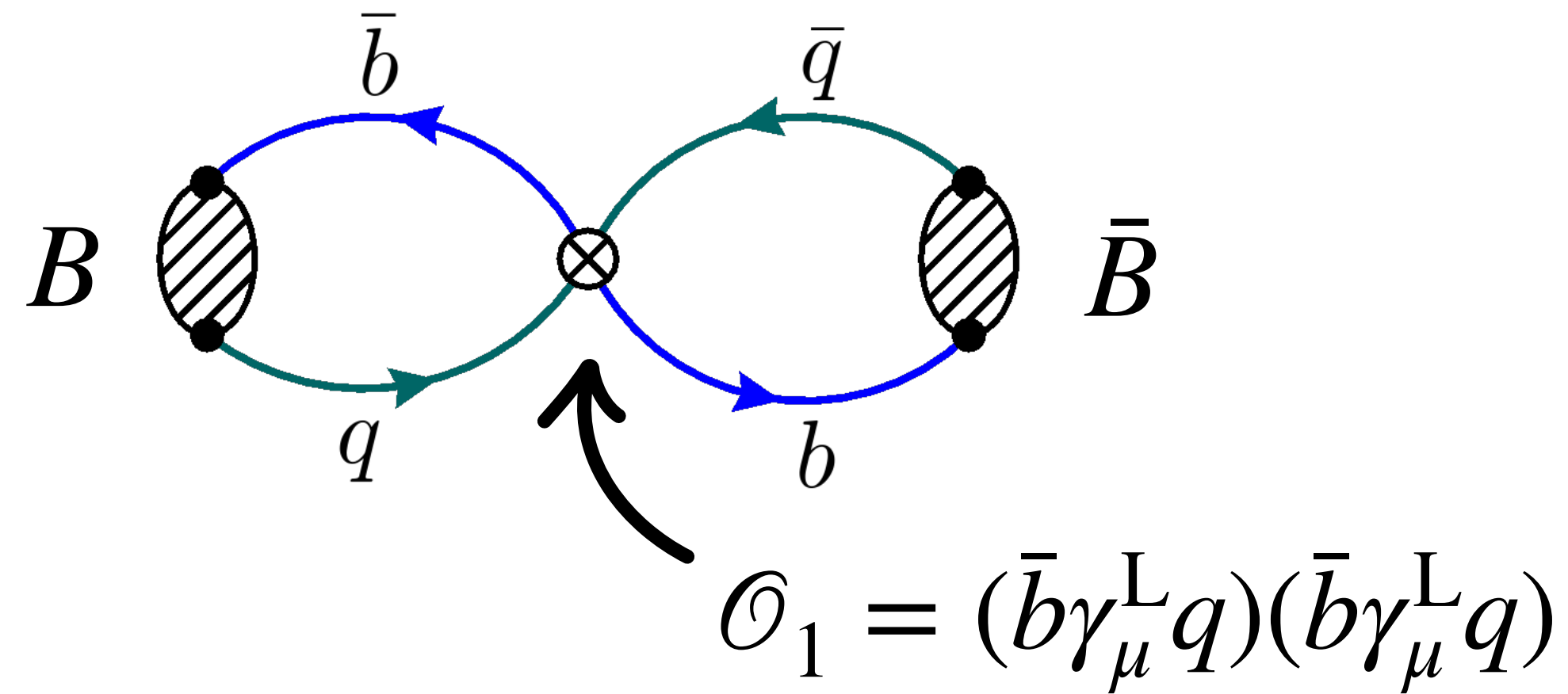


Meson mixing



$$\rightarrow B_1 \sim \langle B | \mathcal{O}_1 | B \rangle \quad \text{bag parameter}$$

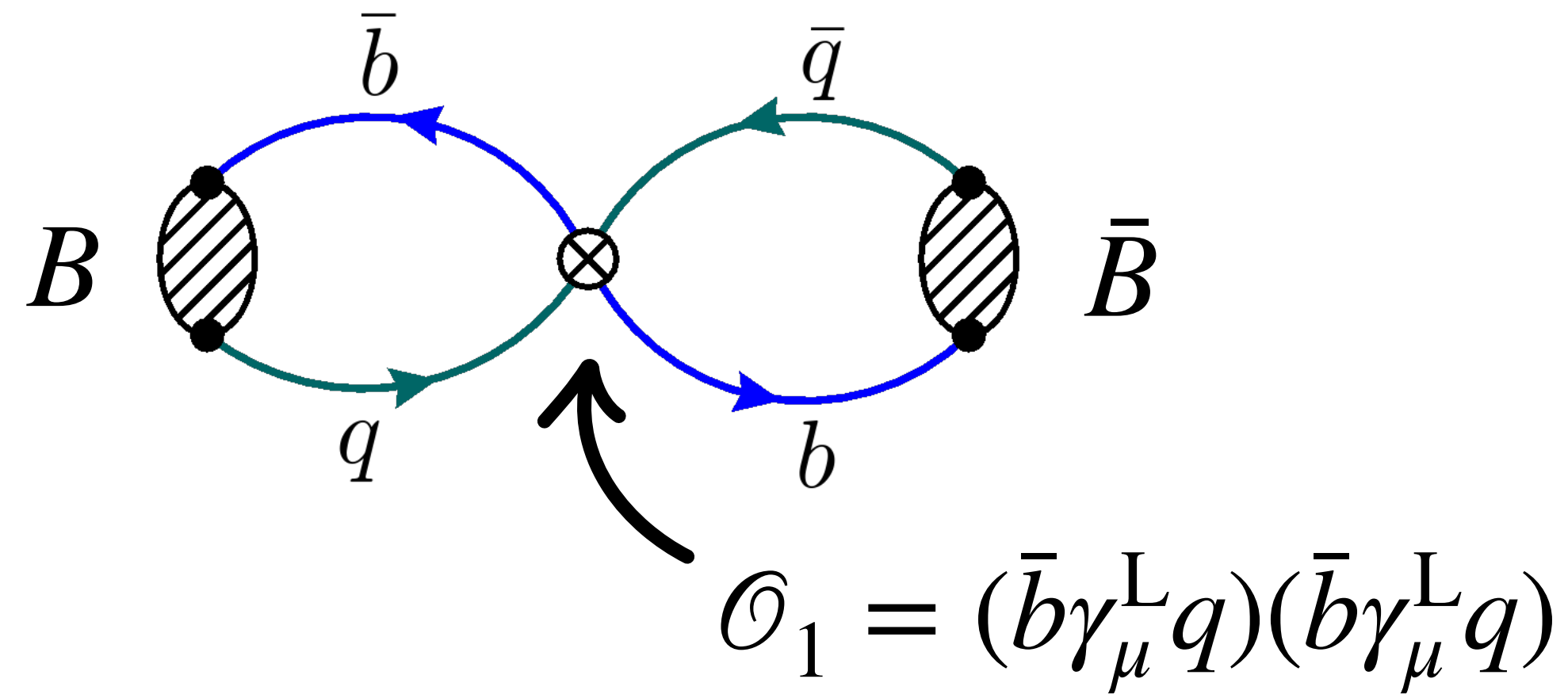
Meson mixing



$$\rightarrow B_1 \sim \langle B | \mathcal{O}_1 | B \rangle \quad \text{bag parameter}$$

$$= \zeta^{-1}(t) \langle B | \mathcal{O}_1(t) | B \rangle$$

Meson mixing

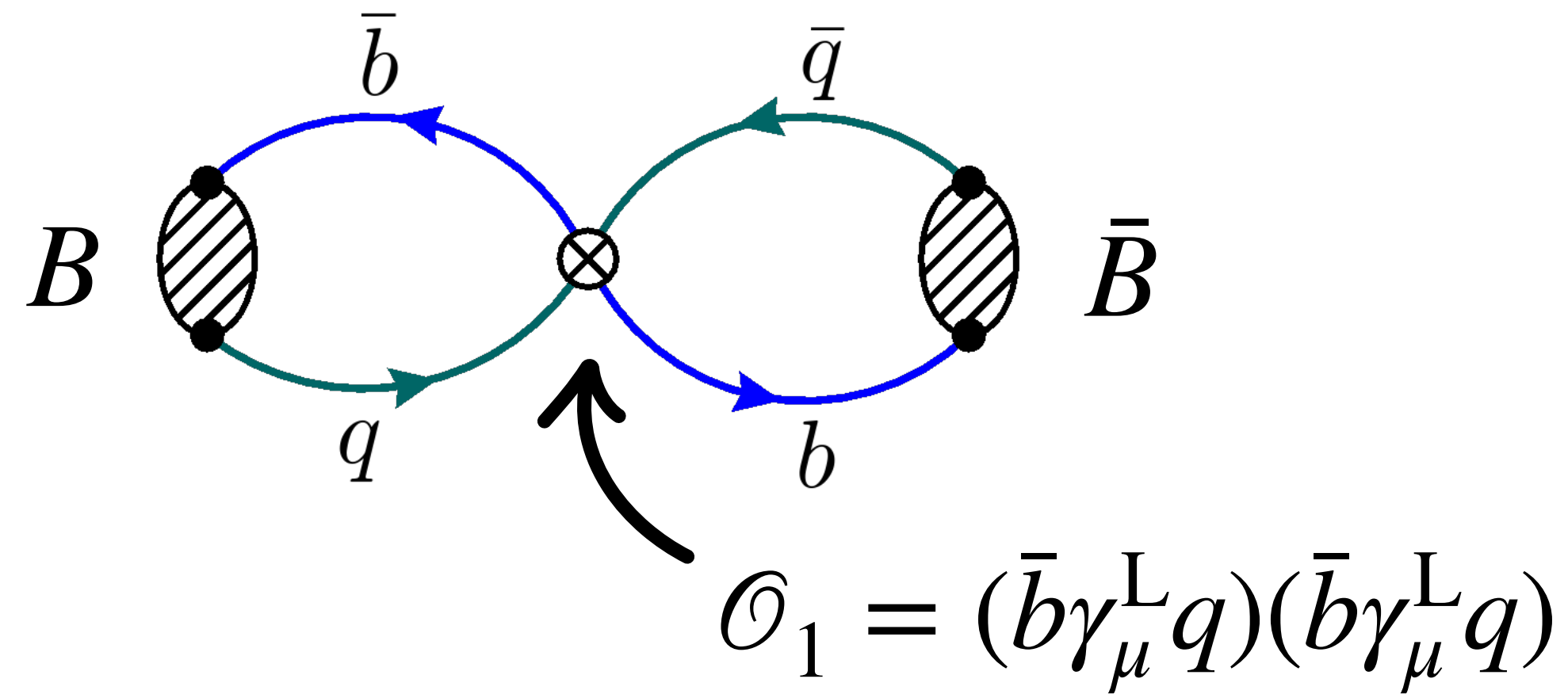


$$\rightarrow B_1 \sim \langle B | \mathcal{O}_1 | B \rangle \quad \text{bag parameter}$$

$$= \zeta^{-1}(t) \langle B | \mathcal{O}_1(t) | B \rangle$$

perturbative \nearrow

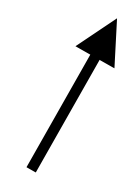
Meson mixing



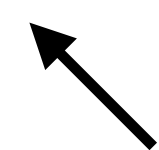
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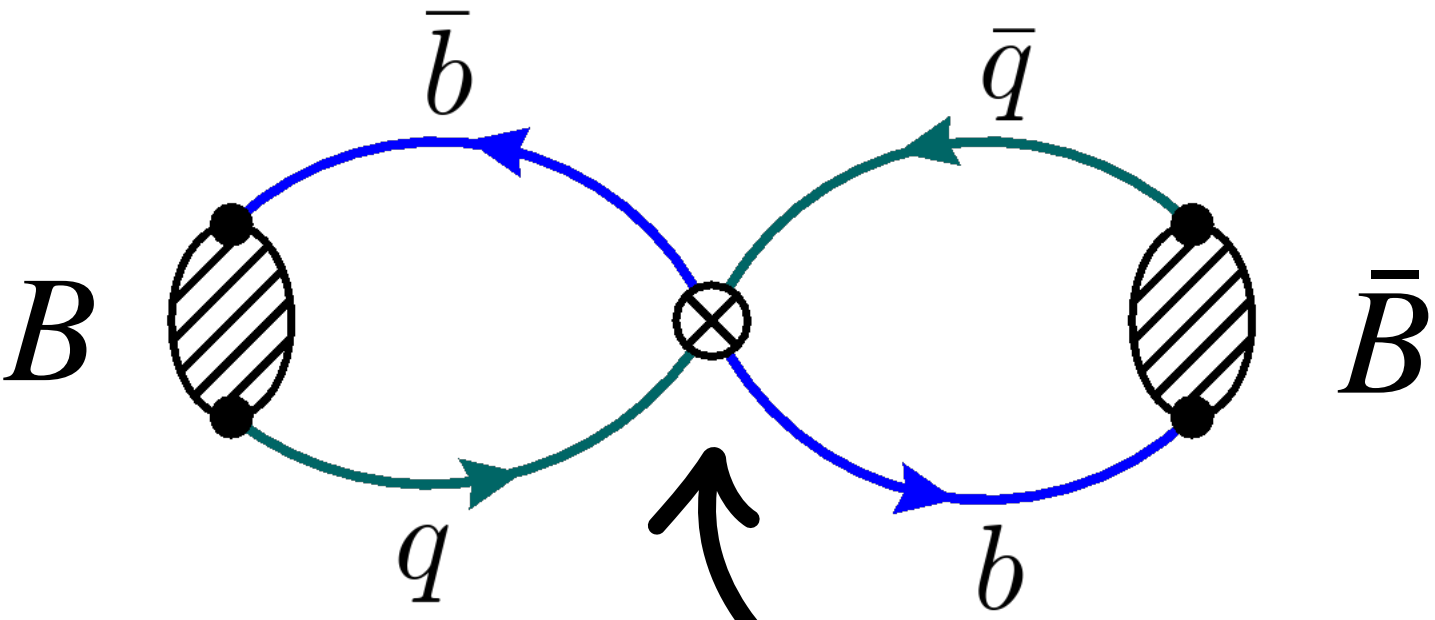
perturbative



lattice



Meson mixing

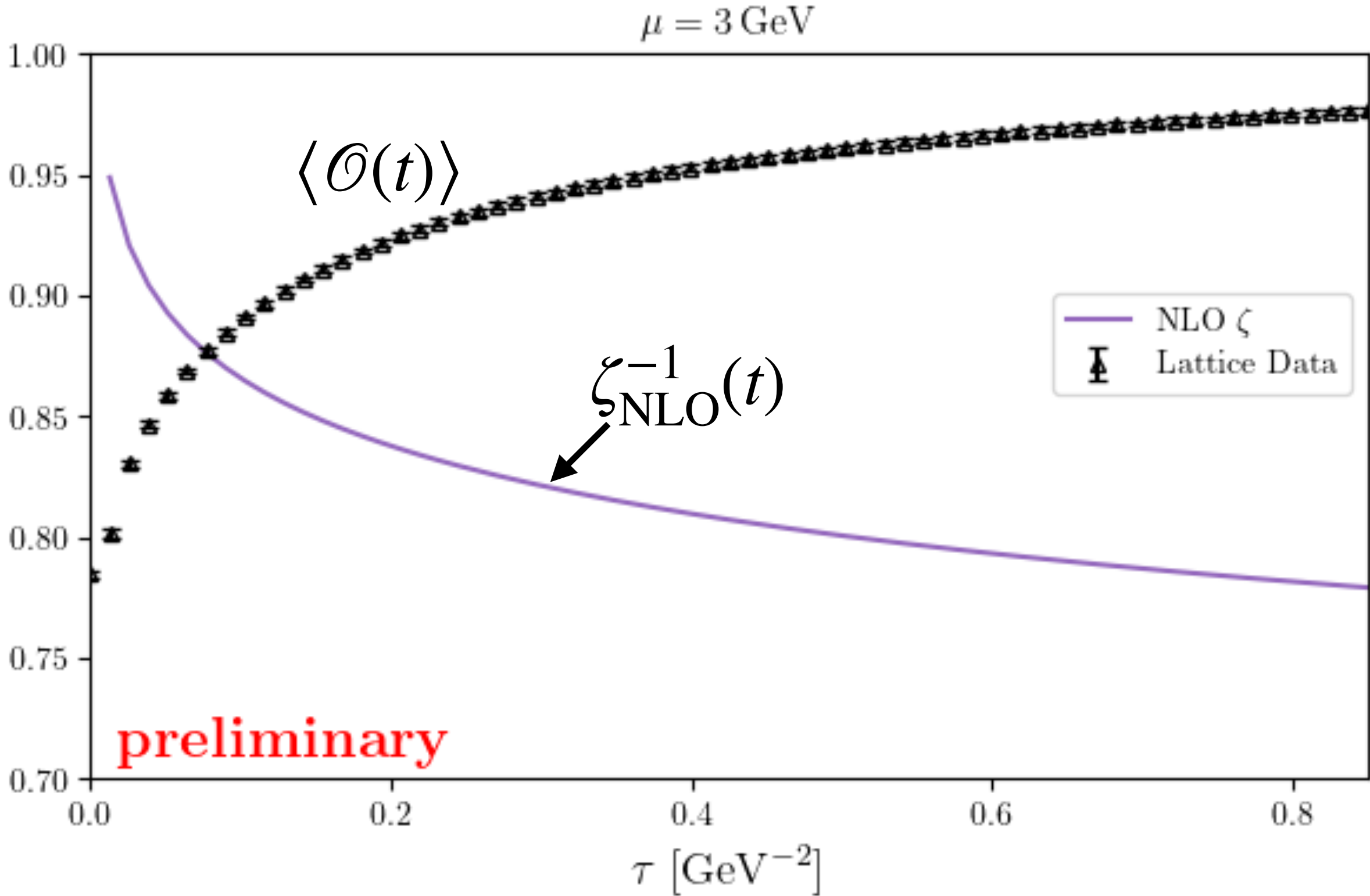


$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^L q)(\bar{b}\gamma_\mu^L q)$$

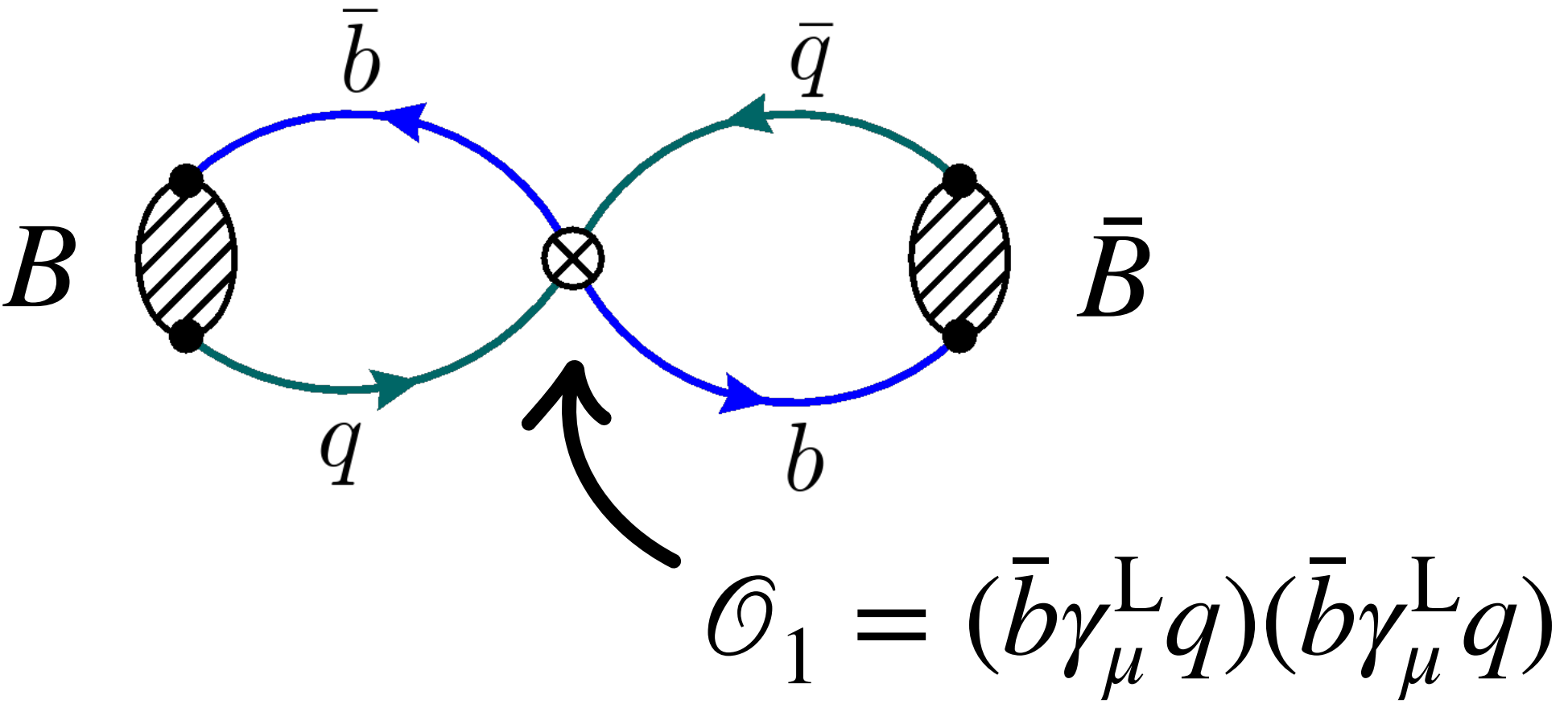
→ $B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$ bag parameter

$$= \zeta^{-1}(t) \langle B | \mathcal{O}_1(t) | B \rangle$$

perturbative \nearrow lattice



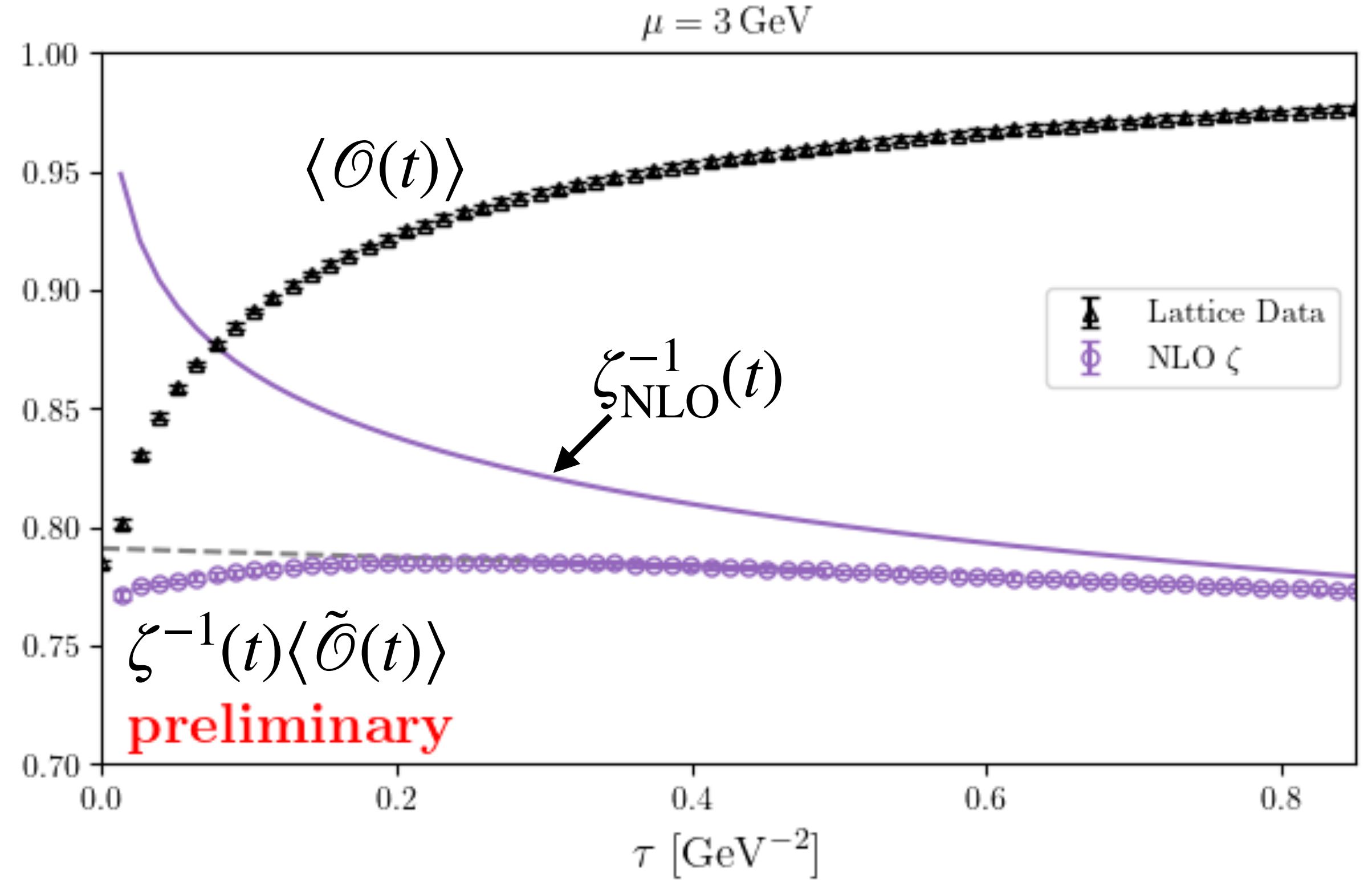
Meson mixing



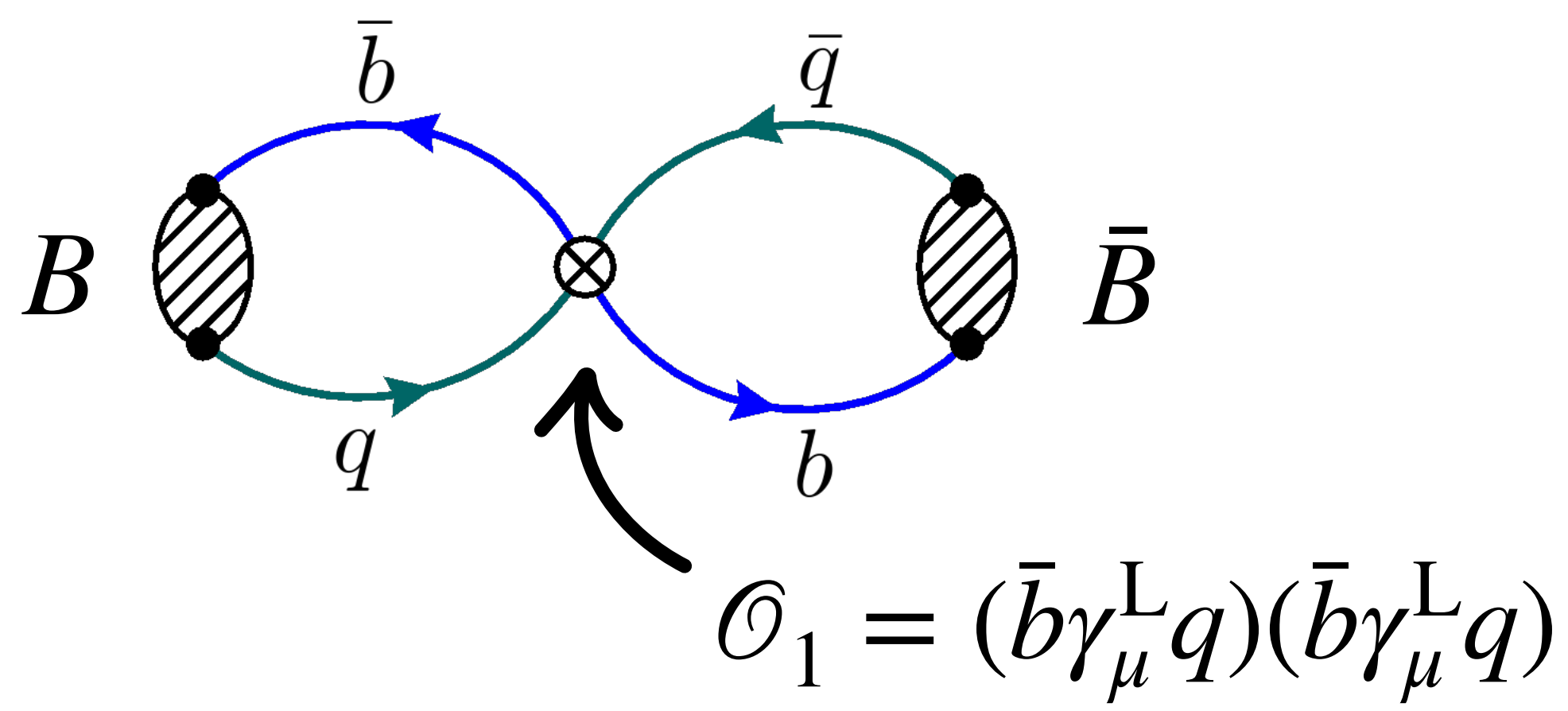
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$= \zeta^{-1}(t) \langle B | \mathcal{O}_1(t) | B \rangle$

perturbative ↗ ↖ lattice



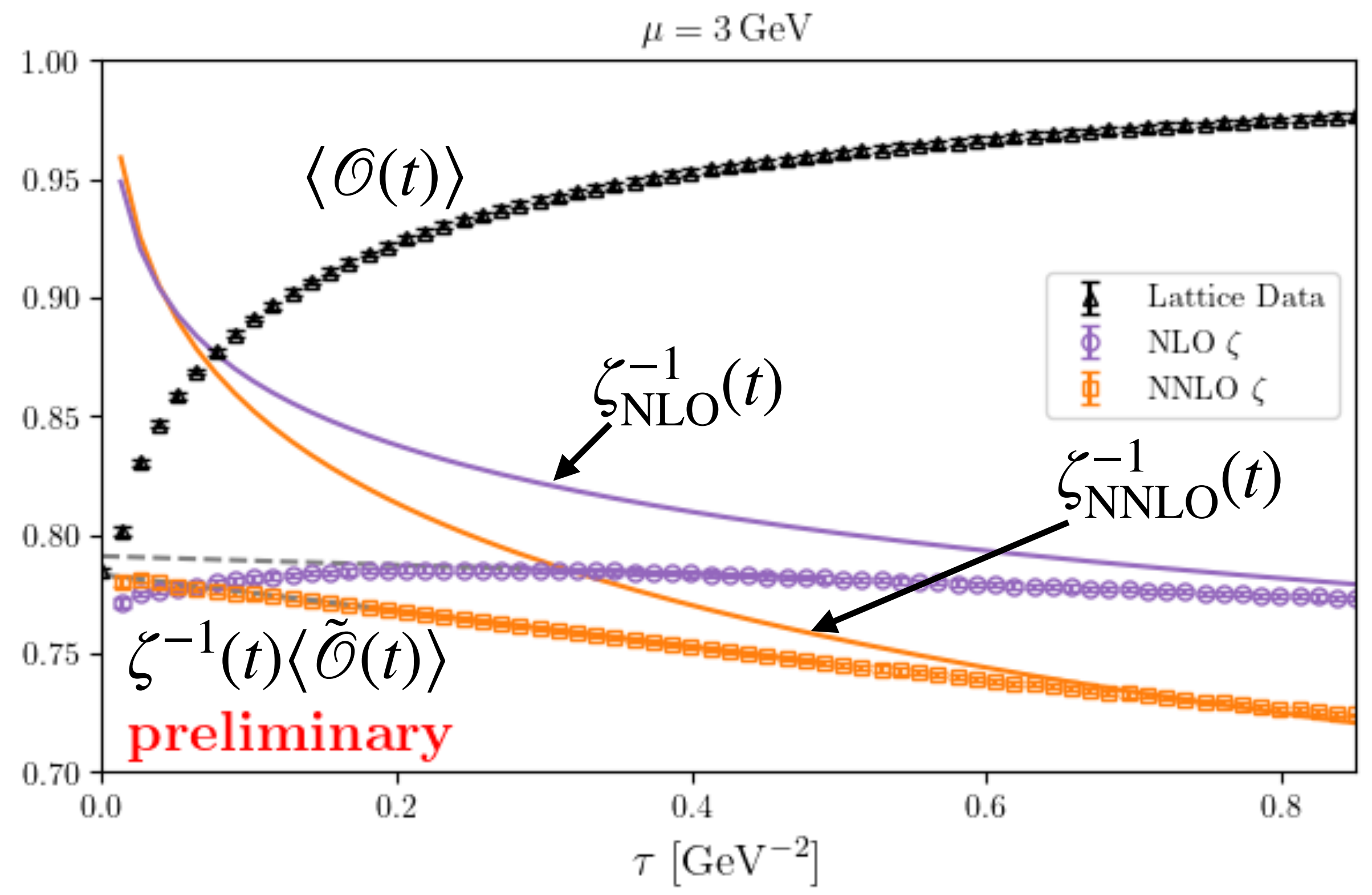
Meson mixing



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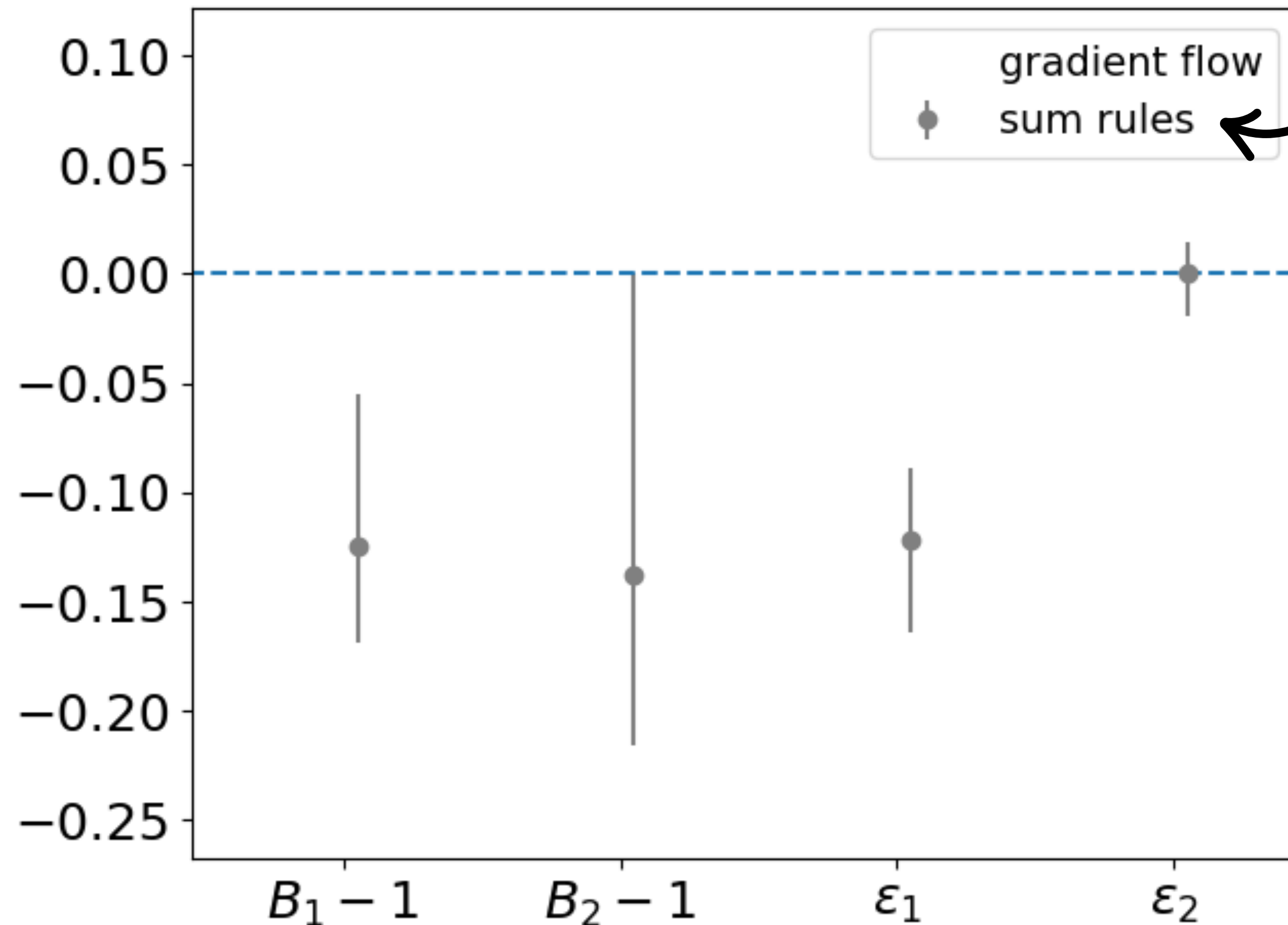
perturbative \nearrow lattice



Black, RH, Lange, Rago, Shindler, Witzel '26



D meson lifetime bag parameters



Black, Lang, Lenz, Wüthrich '25

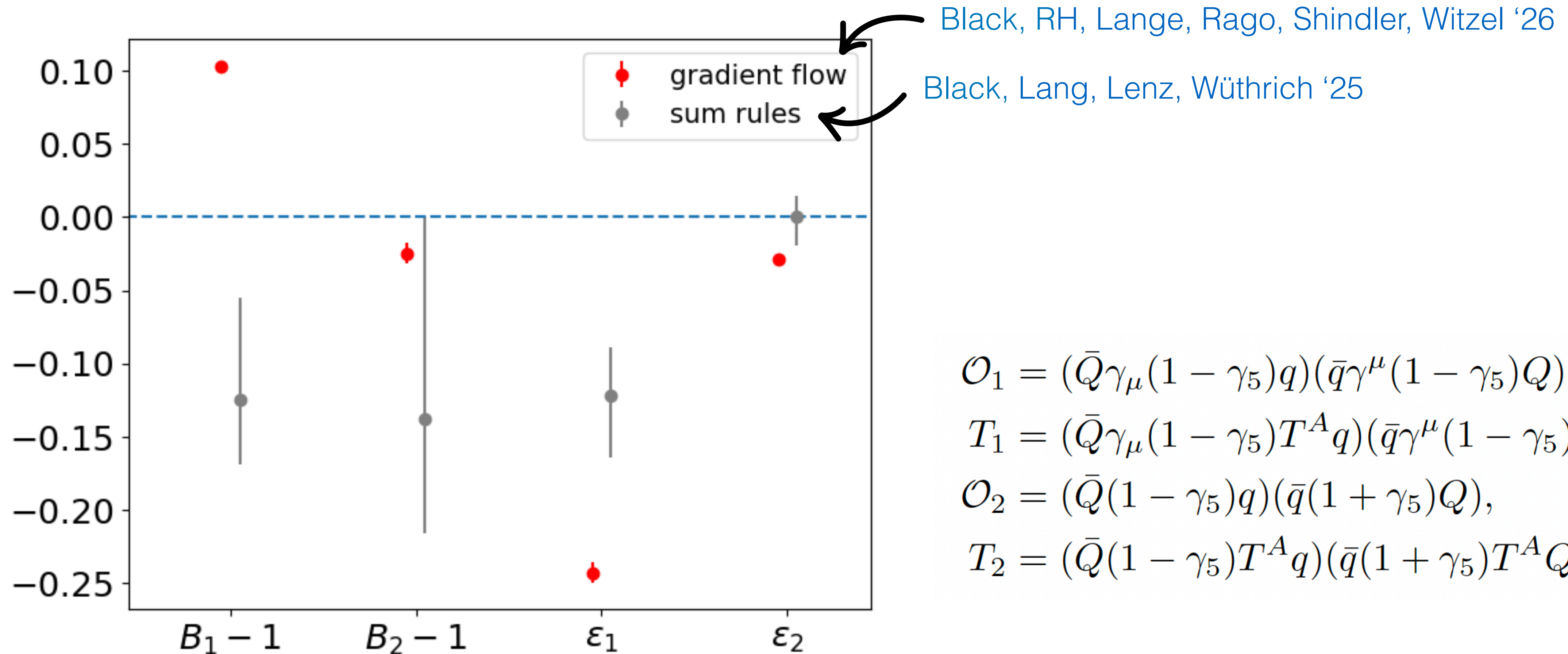
$$\mathcal{O}_1 = (\bar{Q}\gamma_\mu(1 - \gamma_5)q)(\bar{q}\gamma^\mu(1 - \gamma_5)Q),$$

$$T_1 = (\bar{Q}\gamma_\mu(1 - \gamma_5)T^A q)(\bar{q}\gamma^\mu(1 - \gamma_5)T^A Q),$$

$$\mathcal{O}_2 = (\bar{Q}(1 - \gamma_5)q)(\bar{q}(1 + \gamma_5)Q),$$

$$T_2 = (\bar{Q}(1 - \gamma_5)T^A q)(\bar{q}(1 + \gamma_5)T^A Q),$$

D meson lifetime bag parameters



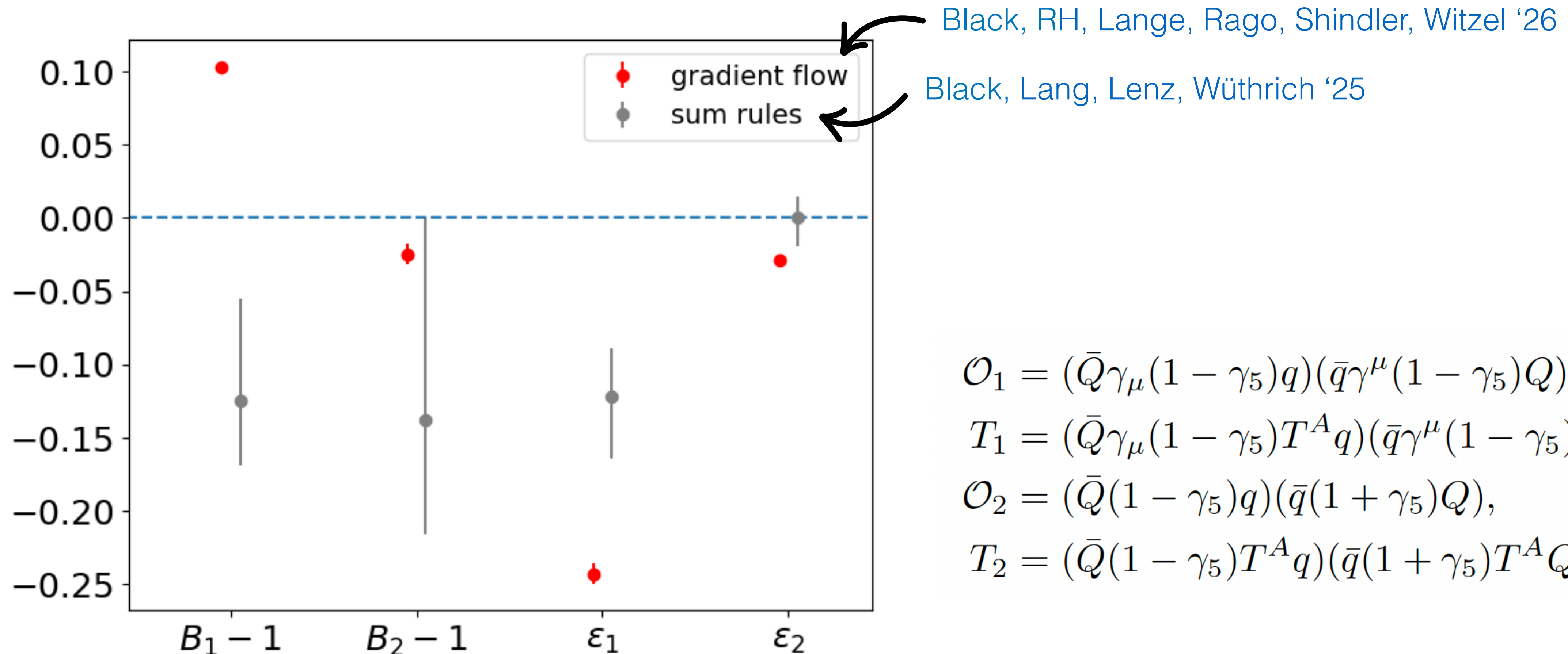
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D meson lifetime bag parameters



Black, RH, Lange, Rago, Shindler, Witzel '26

Black, Lang, Lenz, Wüthrich '25

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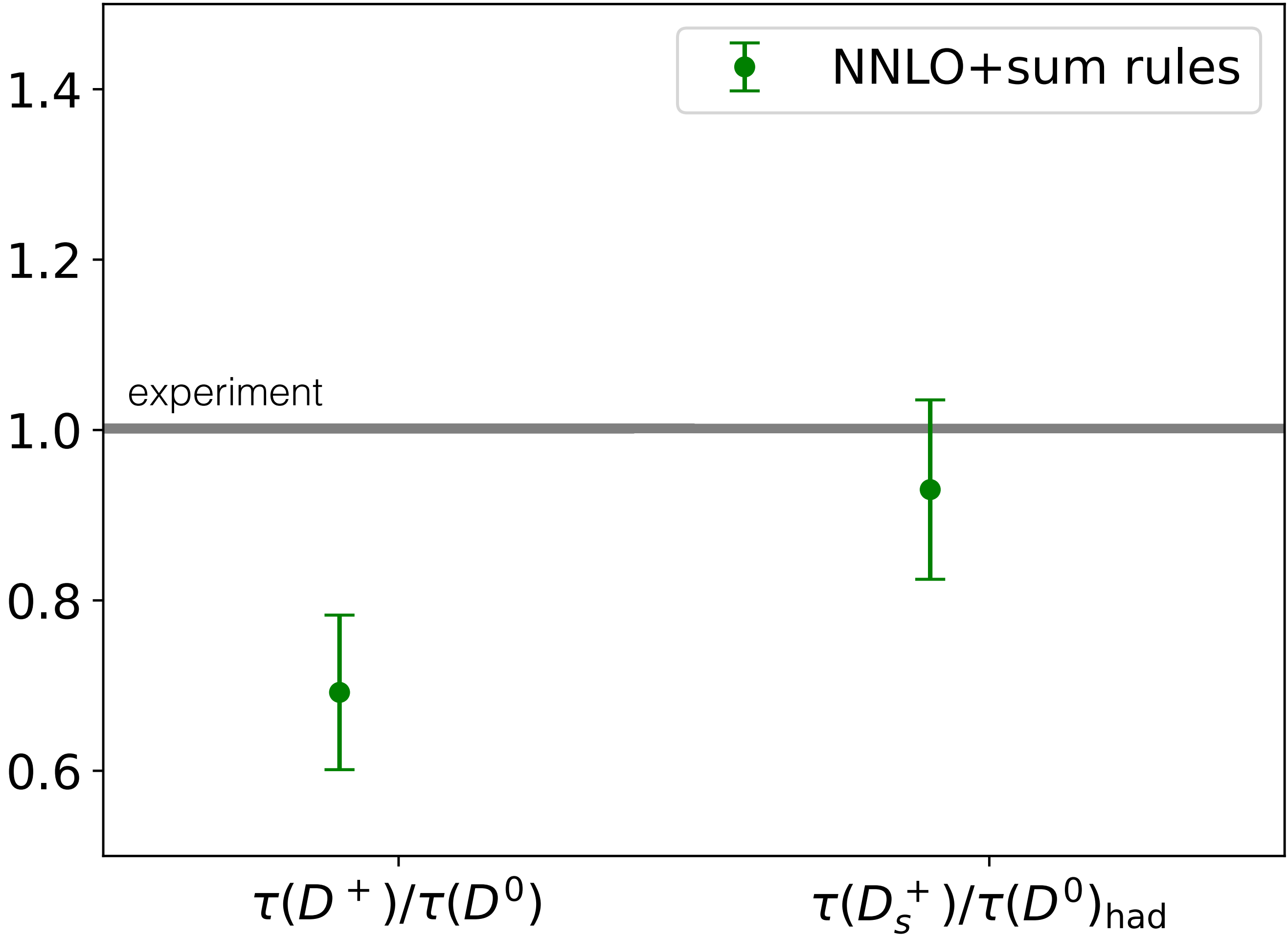
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but: comparison not fully consistent

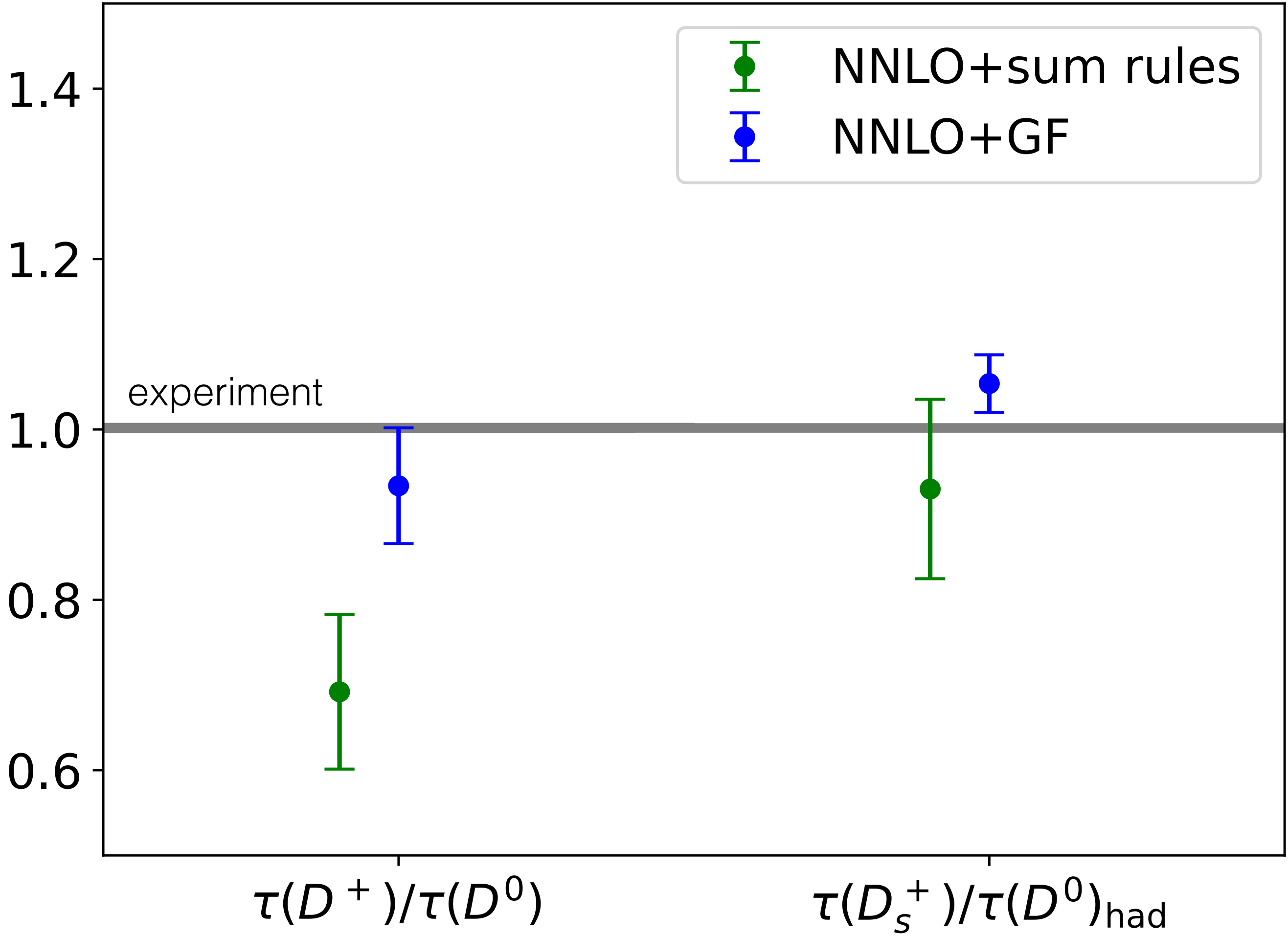
Lifetime ratios



Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser '26

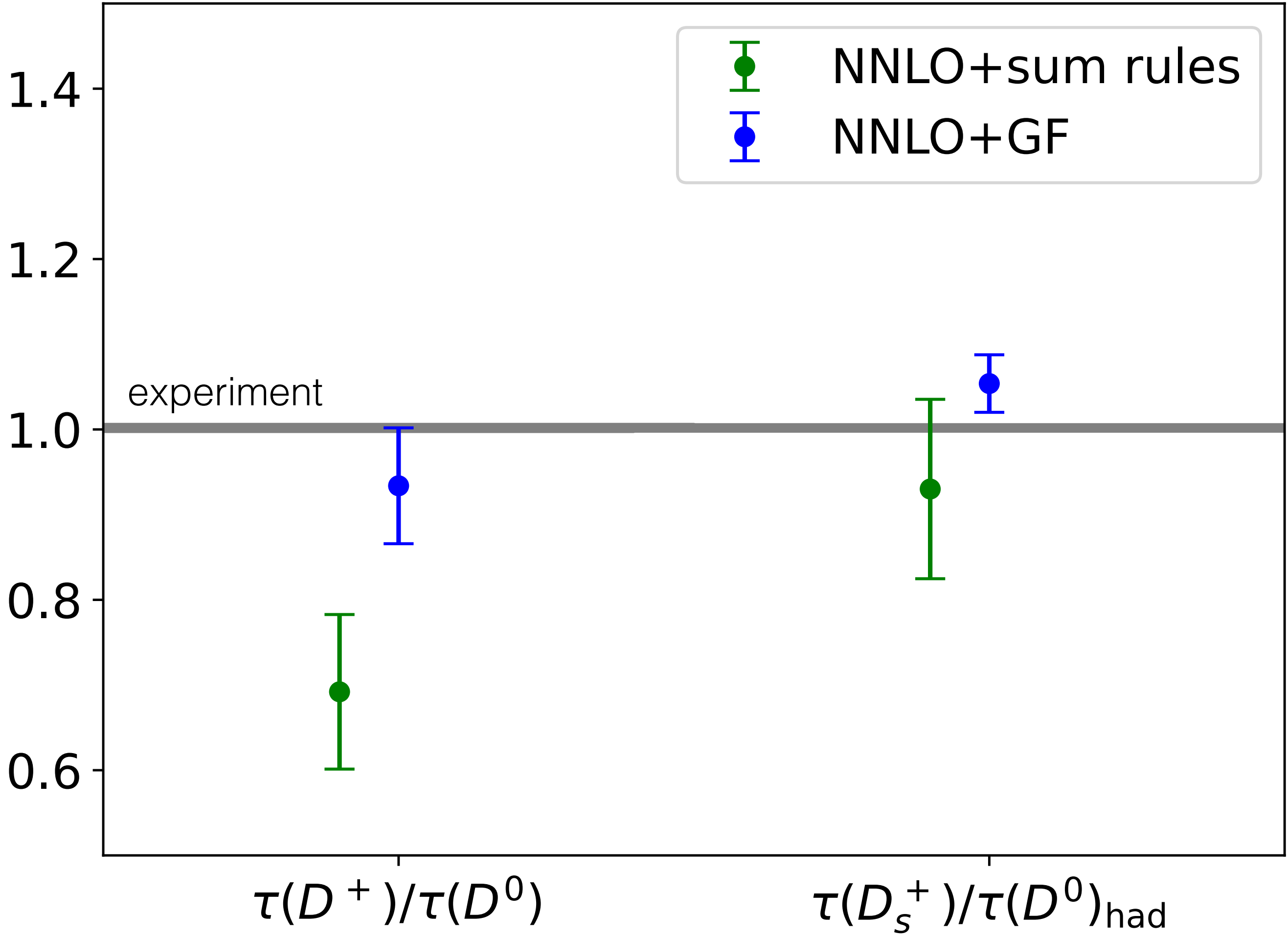
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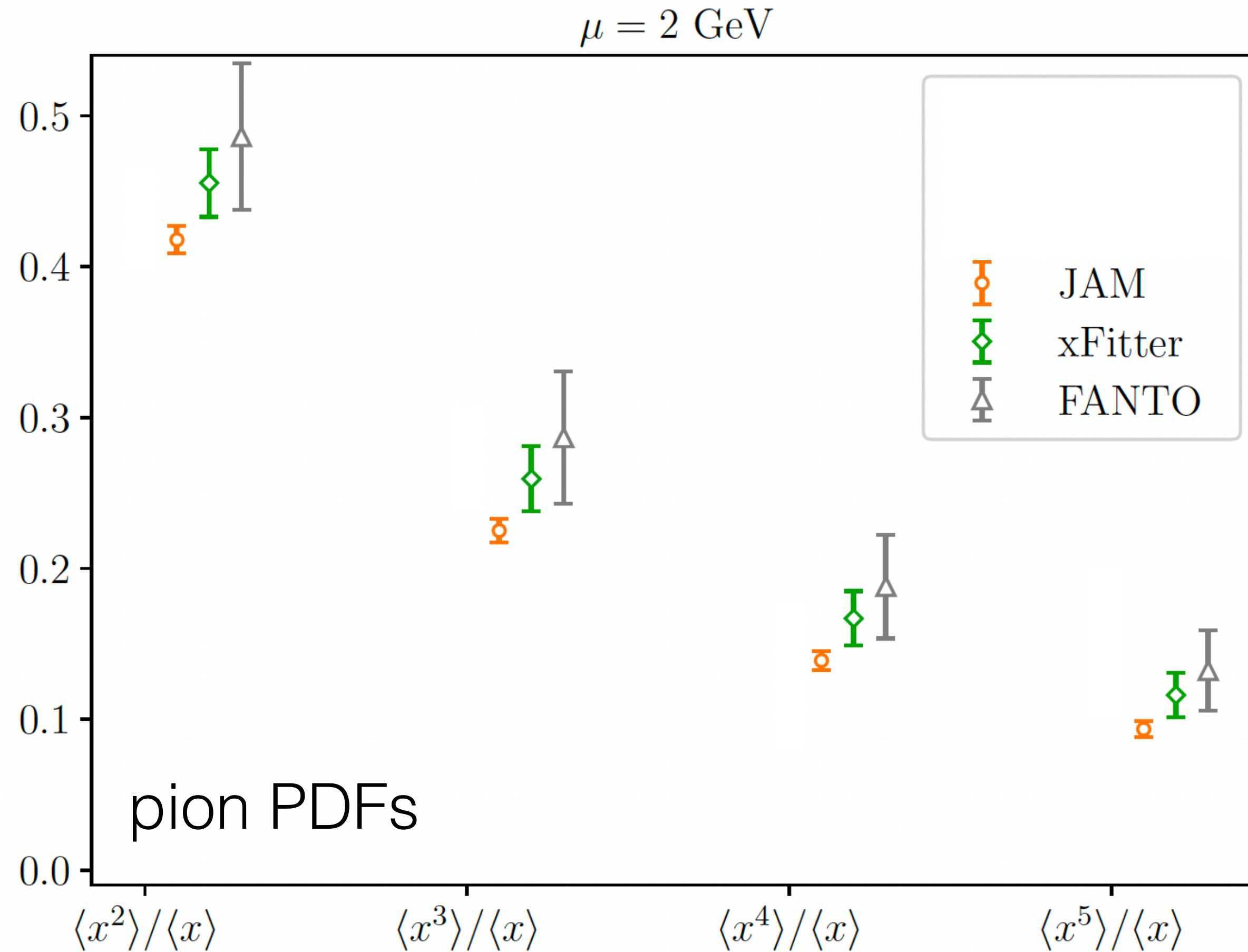


Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser '26

Jonas Kohnen (Tue 12:00)
Antonio Rago (Tue 12:30)

Moments of parton densities



$$\begin{aligned} \langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}} | h(p) \rangle &= \\ &= \sum_m \zeta_{mn}^{-1}(t) \langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}}(t) | h(p) \rangle \end{aligned}$$

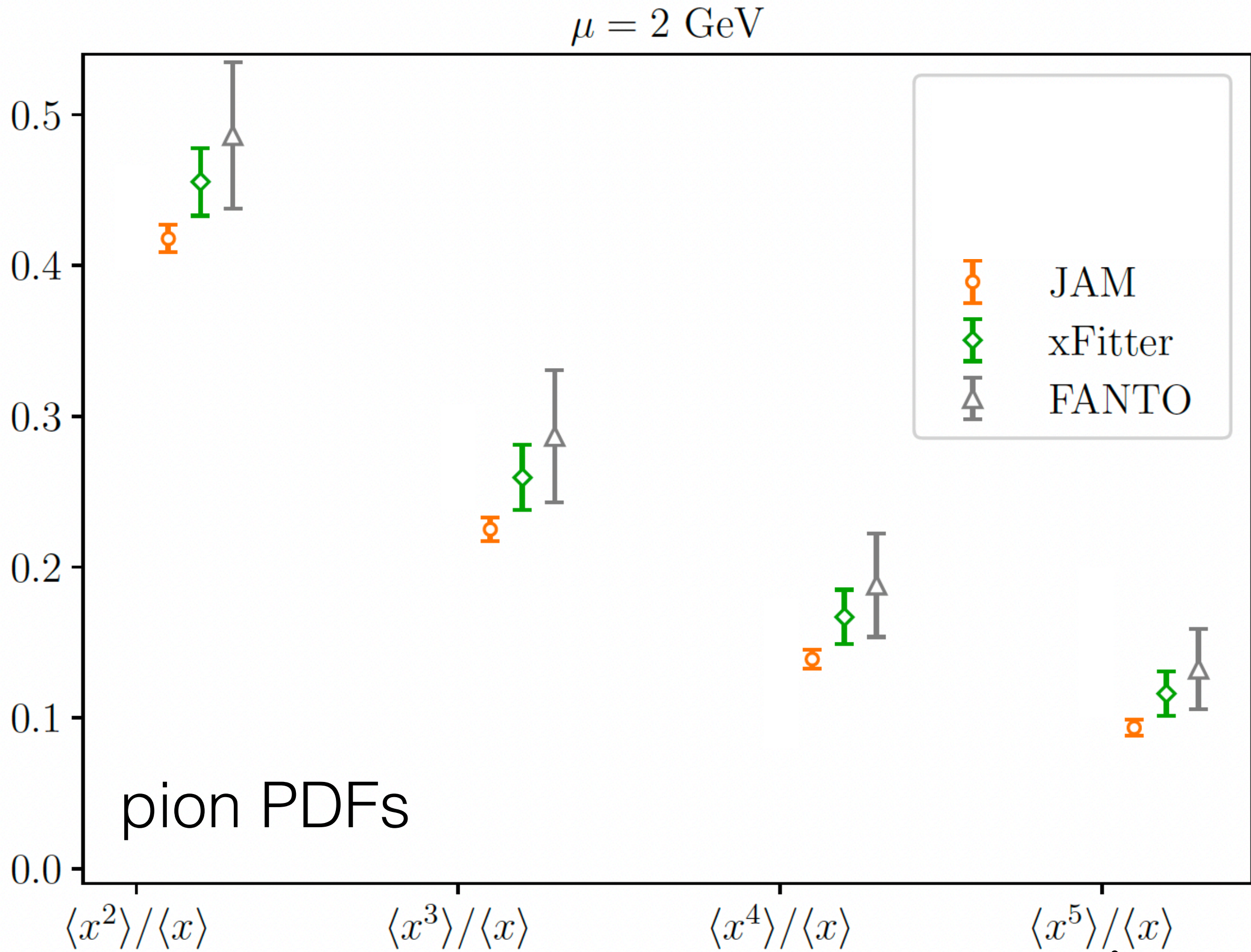
Francis, ..., Shindler, ... '25

Francis, ..., RH, Kohnen, Shindler, ... '25

RH, Kohnen, Shindler '25

see also Edwards et al. '26

Moments of parton densities



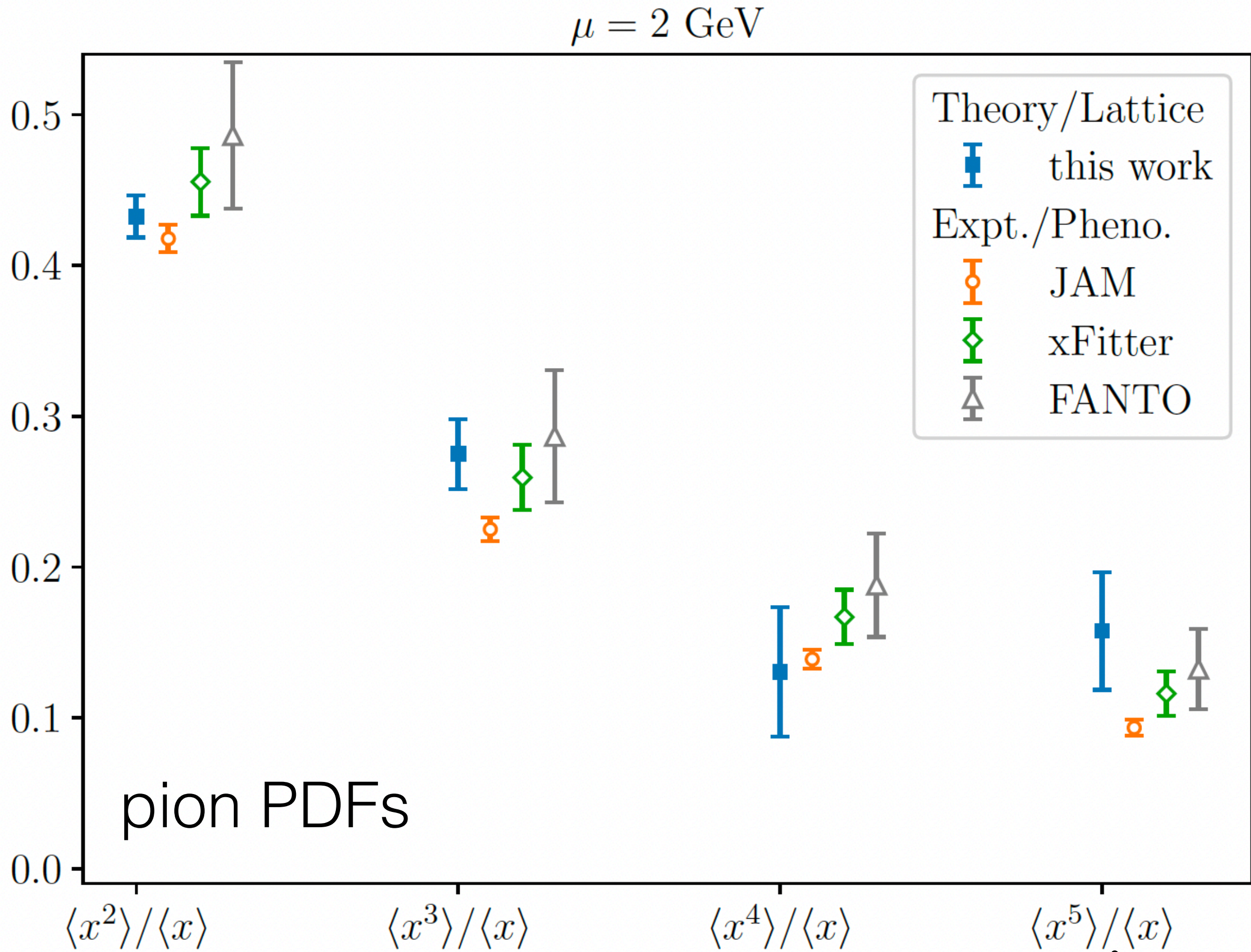
$$\langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}} | h(p) \rangle = \sum_m \zeta_{mn}^{-1}(t) \langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}}(t) | h(p) \rangle$$

Francis, ..., Shindler, ... '25
 Francis, ..., RH, Kohnen, Shindler, ... '25
 RH, Kohnen, Shindler '25

unreachable w/o gradient flow

see also Edwards et al. '26

Moments of parton densities



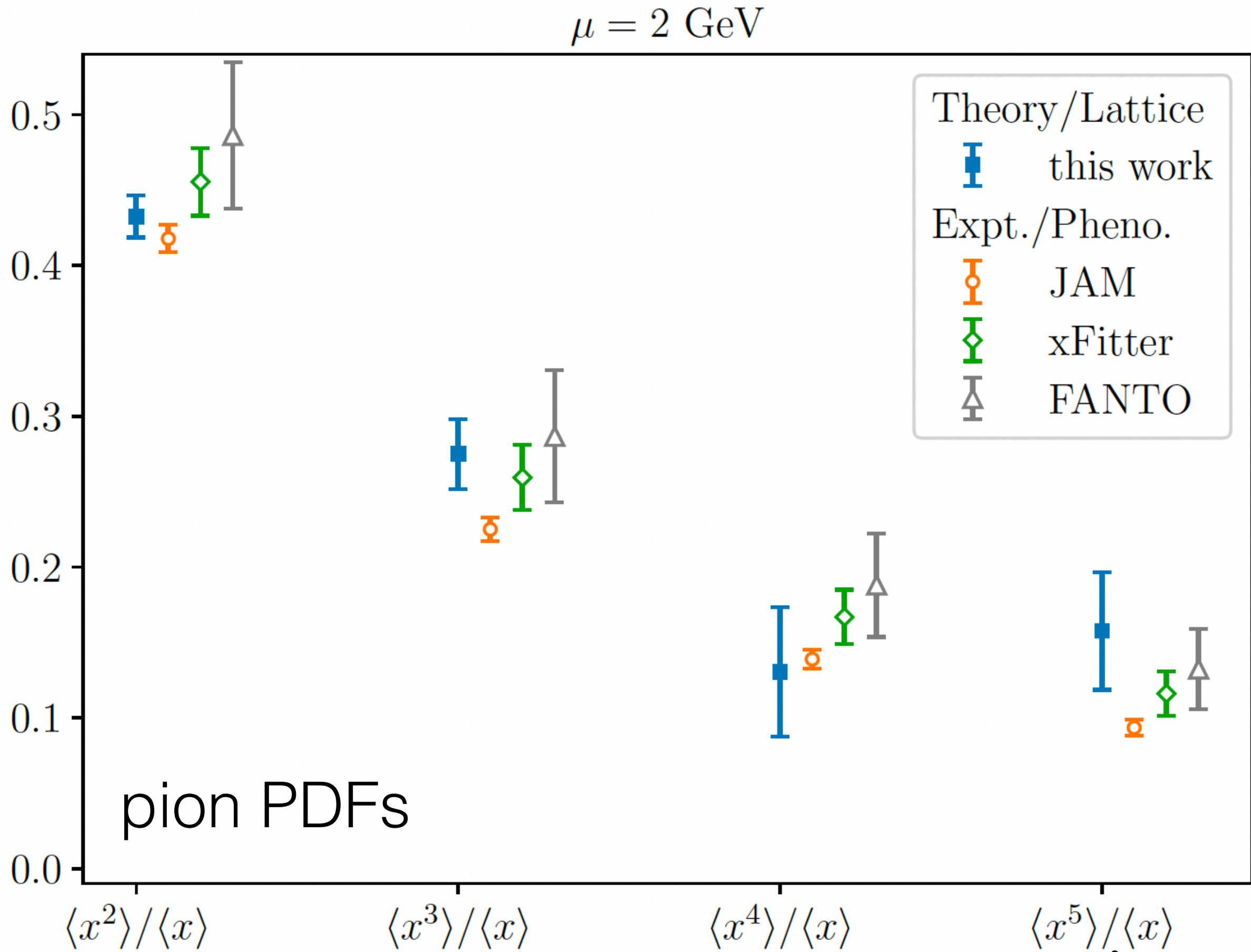
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Moments of parton densities



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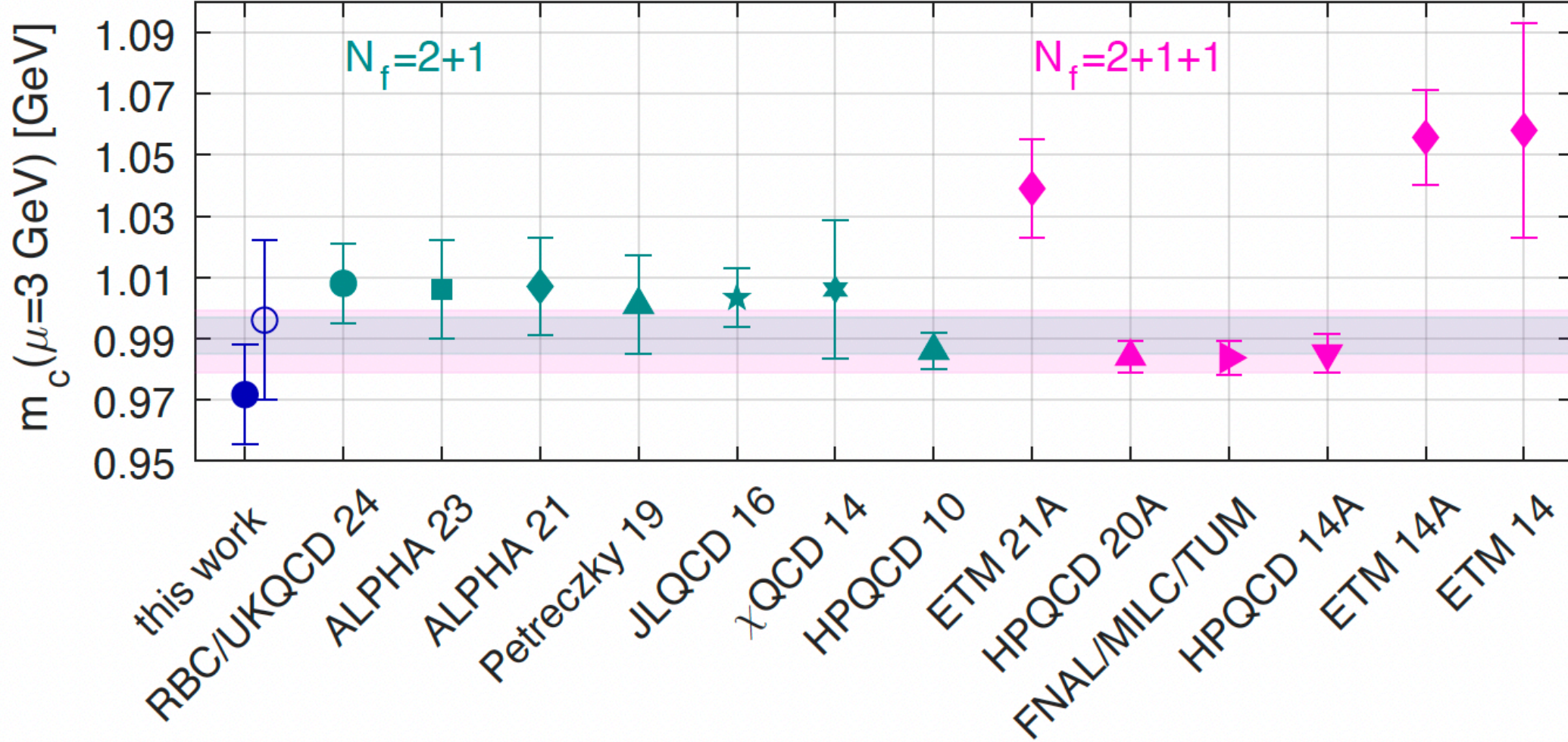
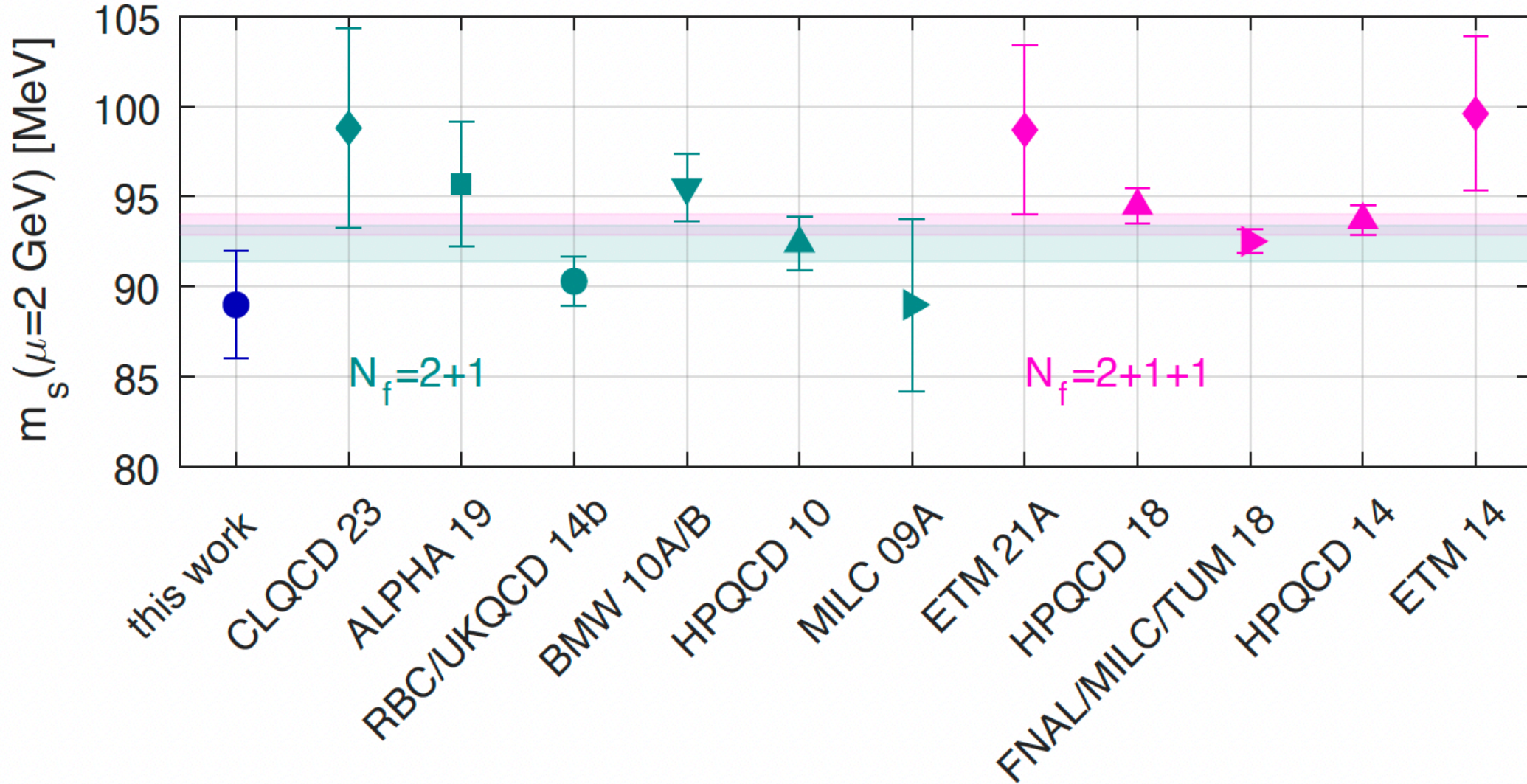
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Andrea Shindler (Fri 09:30)
Chris Monahan (Fri 09:00)

Francis, ..., Shindler, ... '25
Francis, ..., RH, Kohlen, Shindler, ... '25
RH, Kohlen, Shindler '25

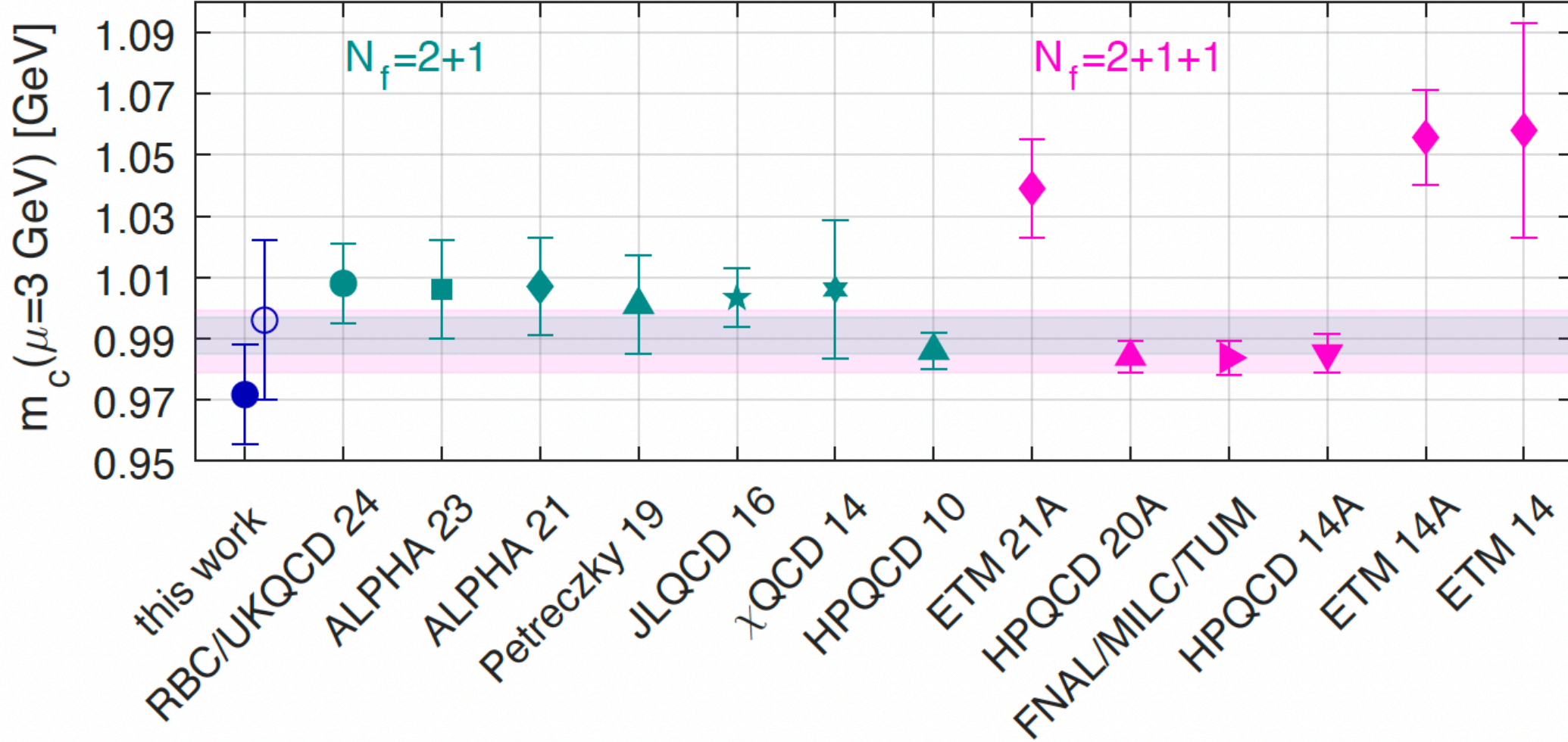
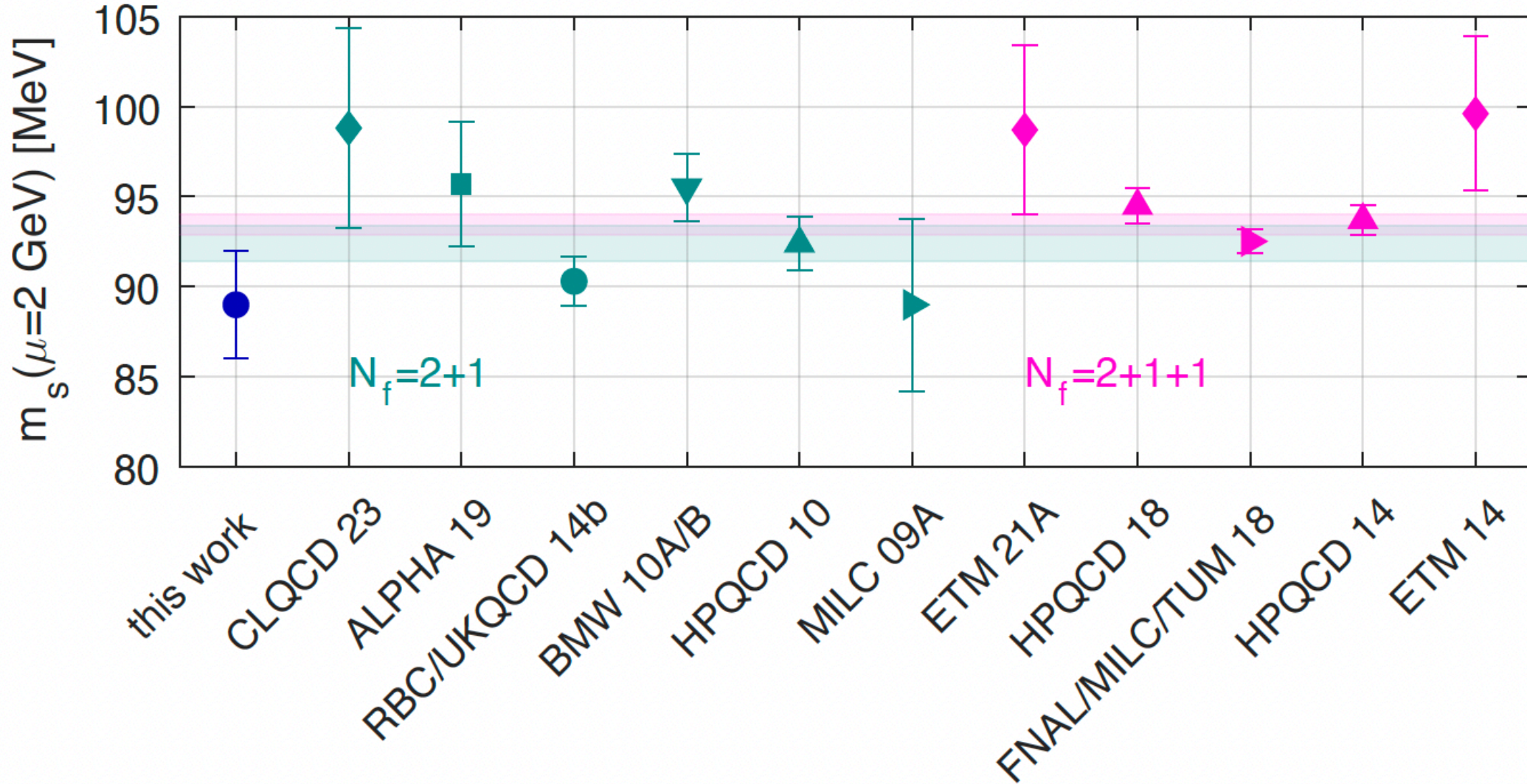
see also Edwards et al. '26

Quark masses



Quark masses

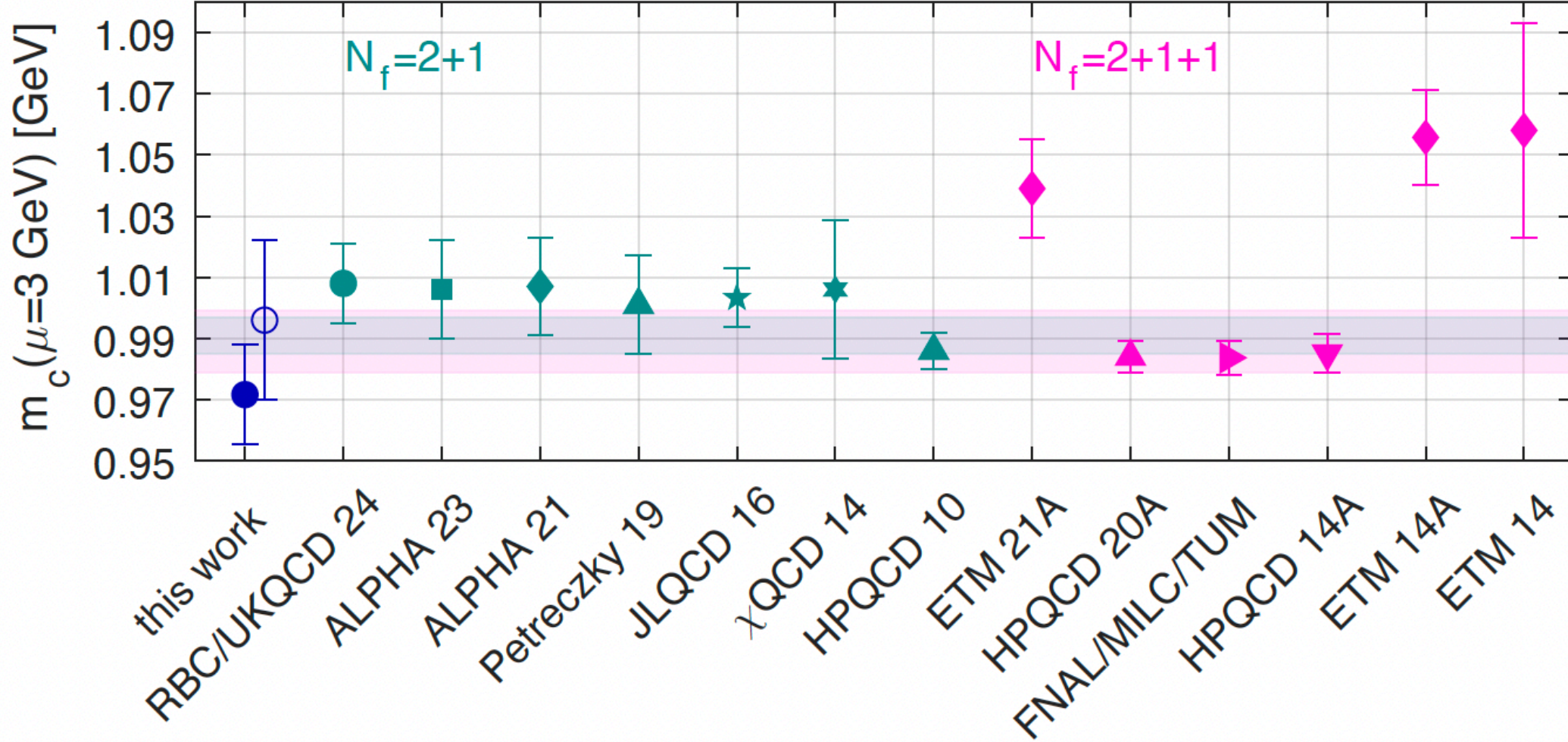
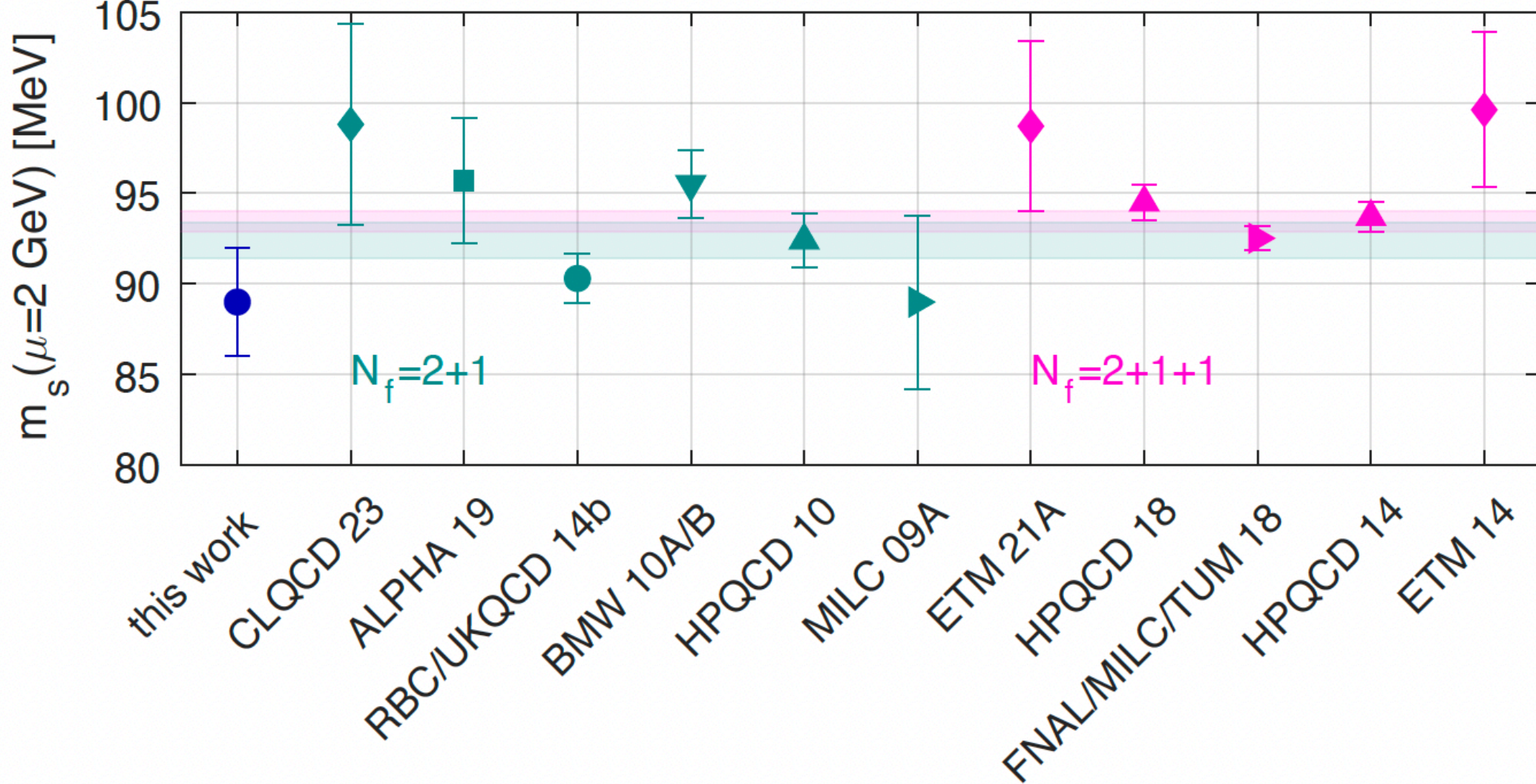
Black, RH, Hasenfratz, Rago, Witzel '25



Quark masses

PCAC:

$$R(t, \tau) = - \frac{\langle j_A^0(t, \tau) j_P(t=0) \rangle}{\langle j_P(t, \tau) j_P(t=0) \rangle}$$



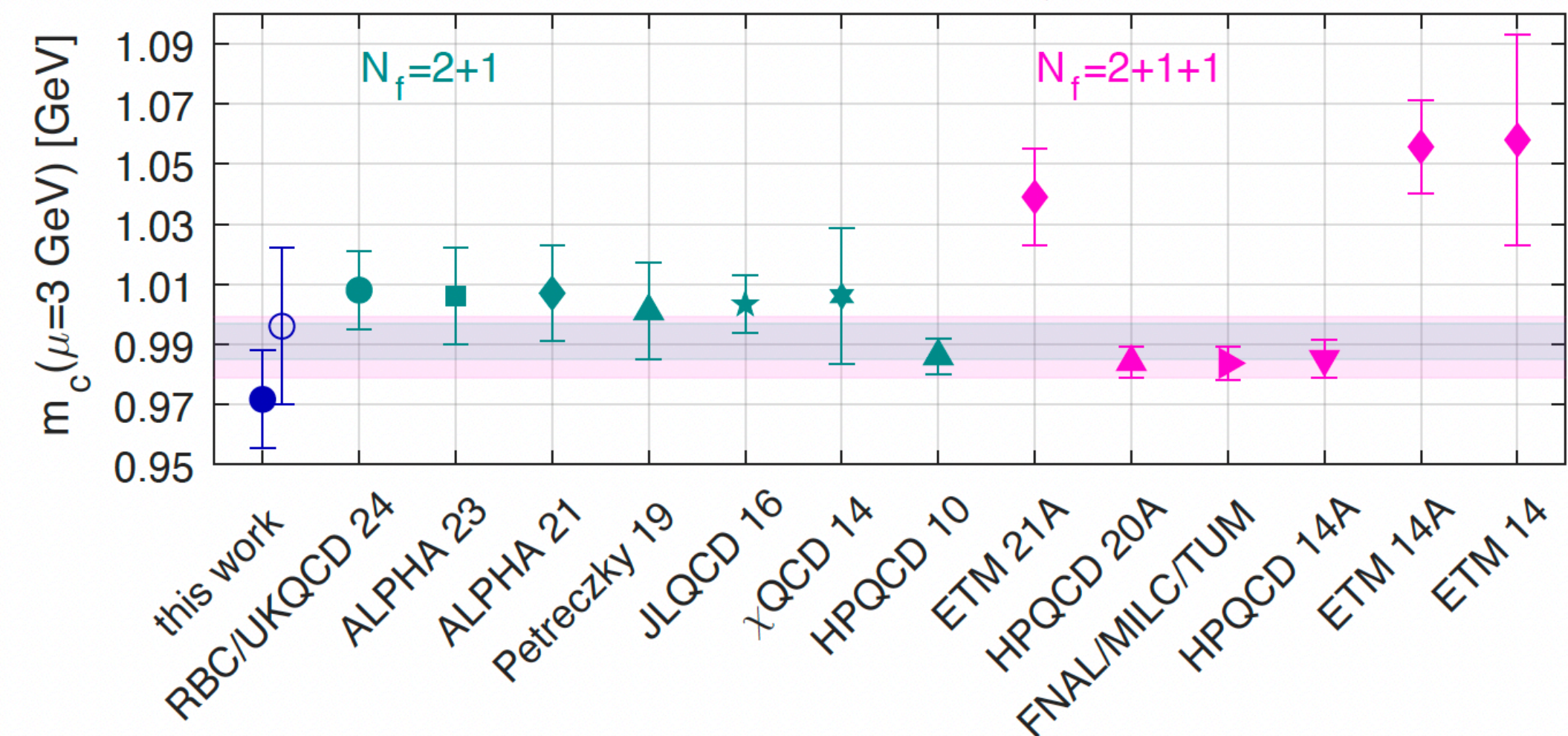
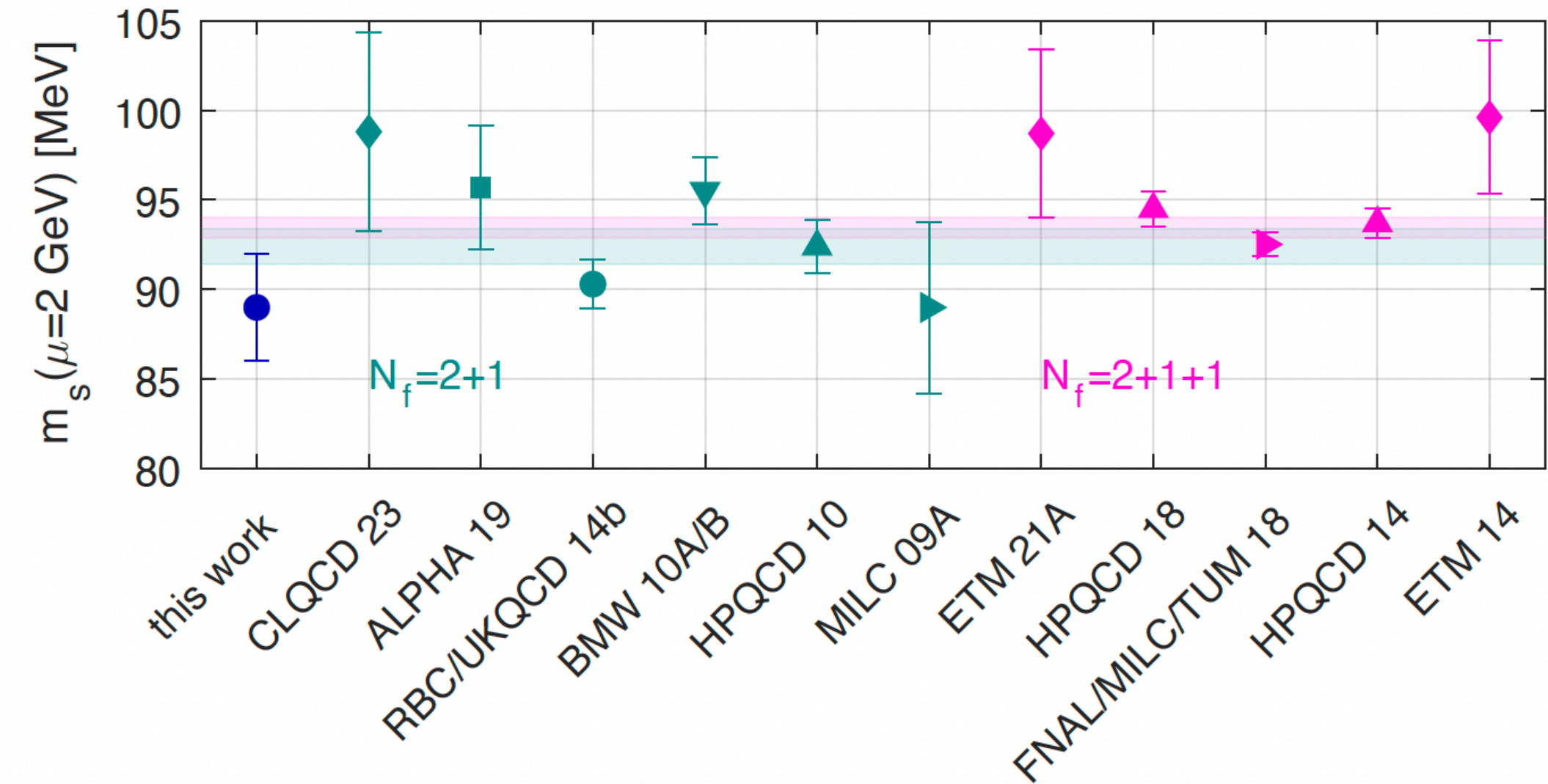
Quark masses

Black, RH, Hasenfratz, Rago, Witzel '25

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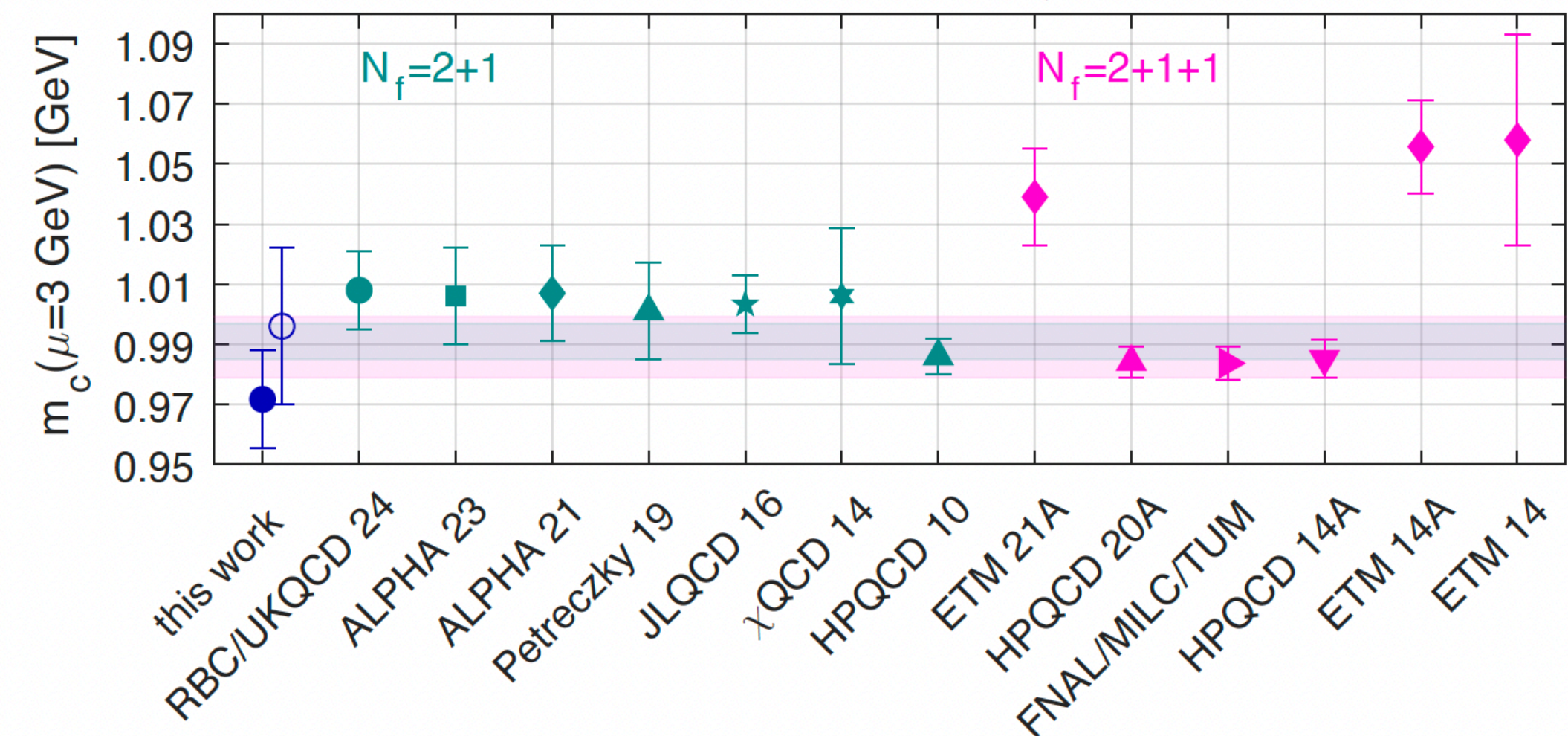
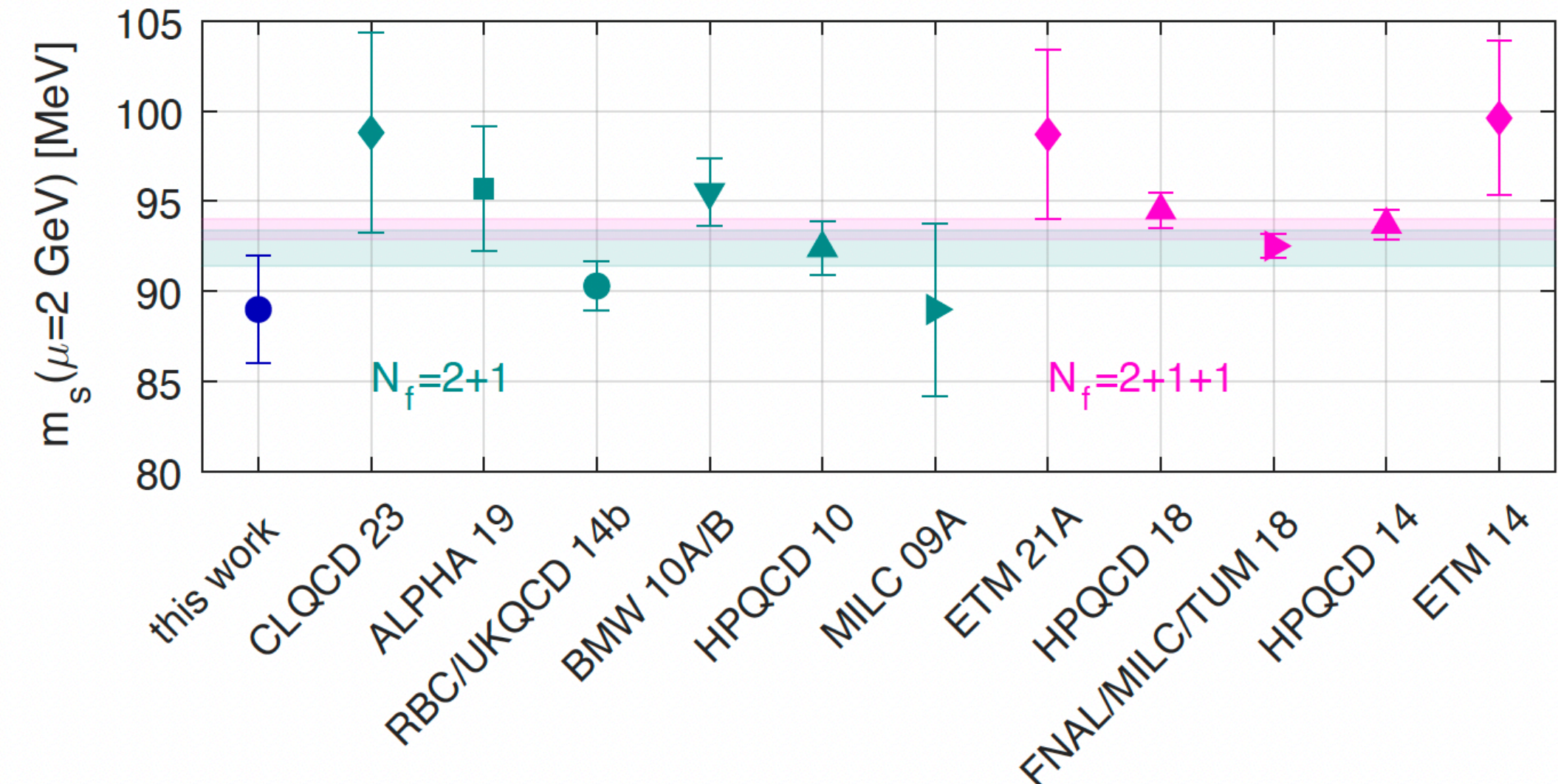
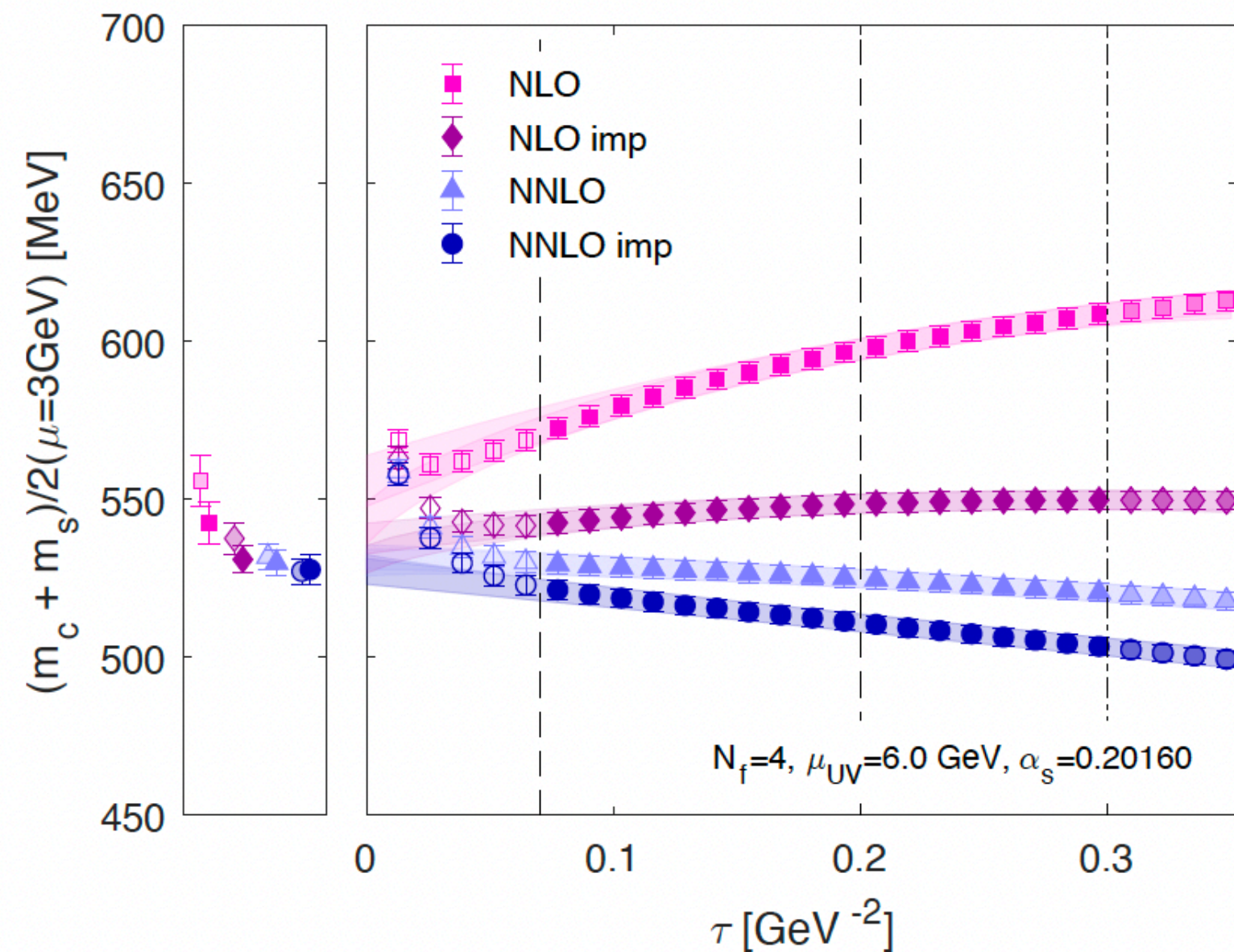
Quark masses

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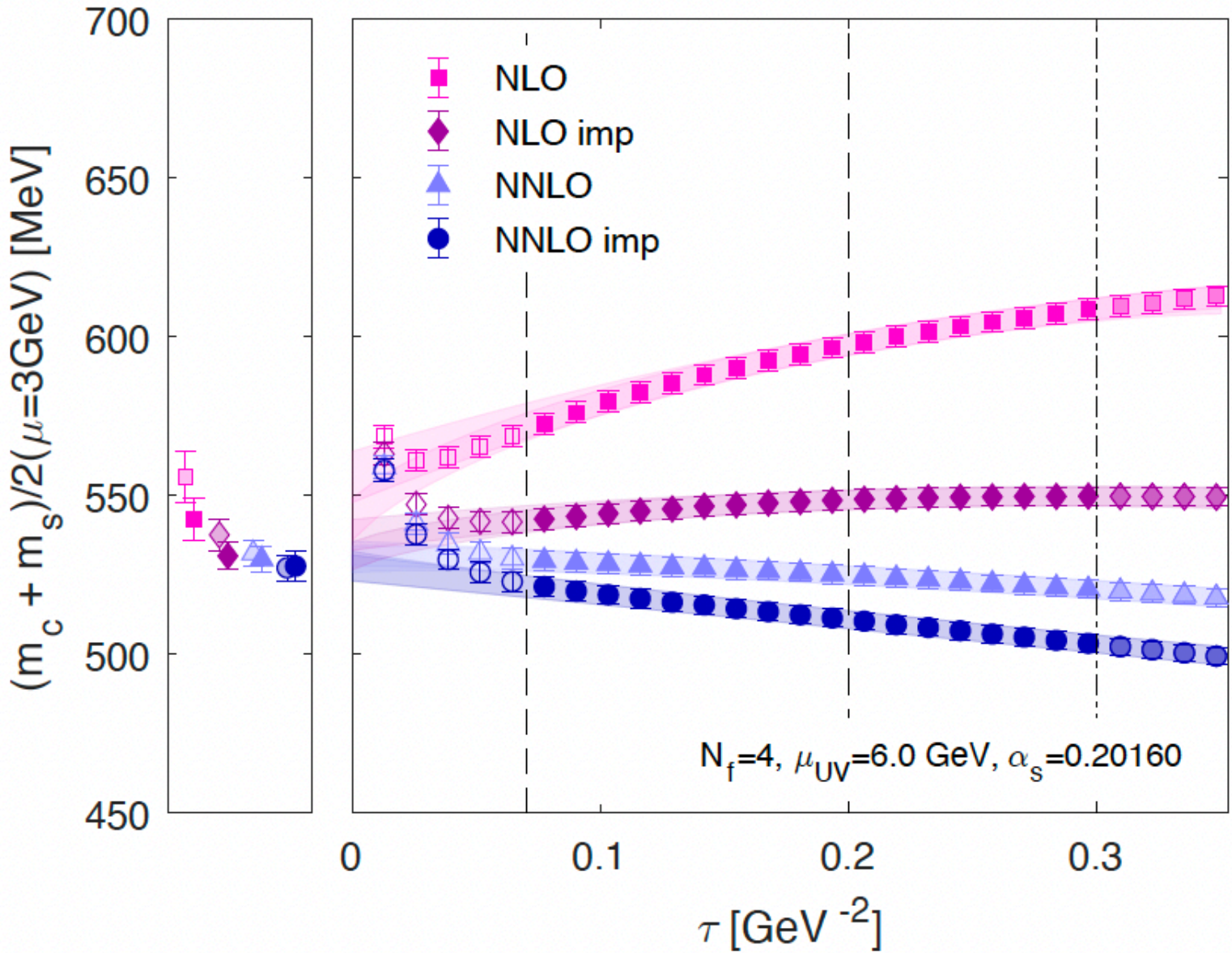


Quark masses

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Oliver Witzel (Tue 10:00)

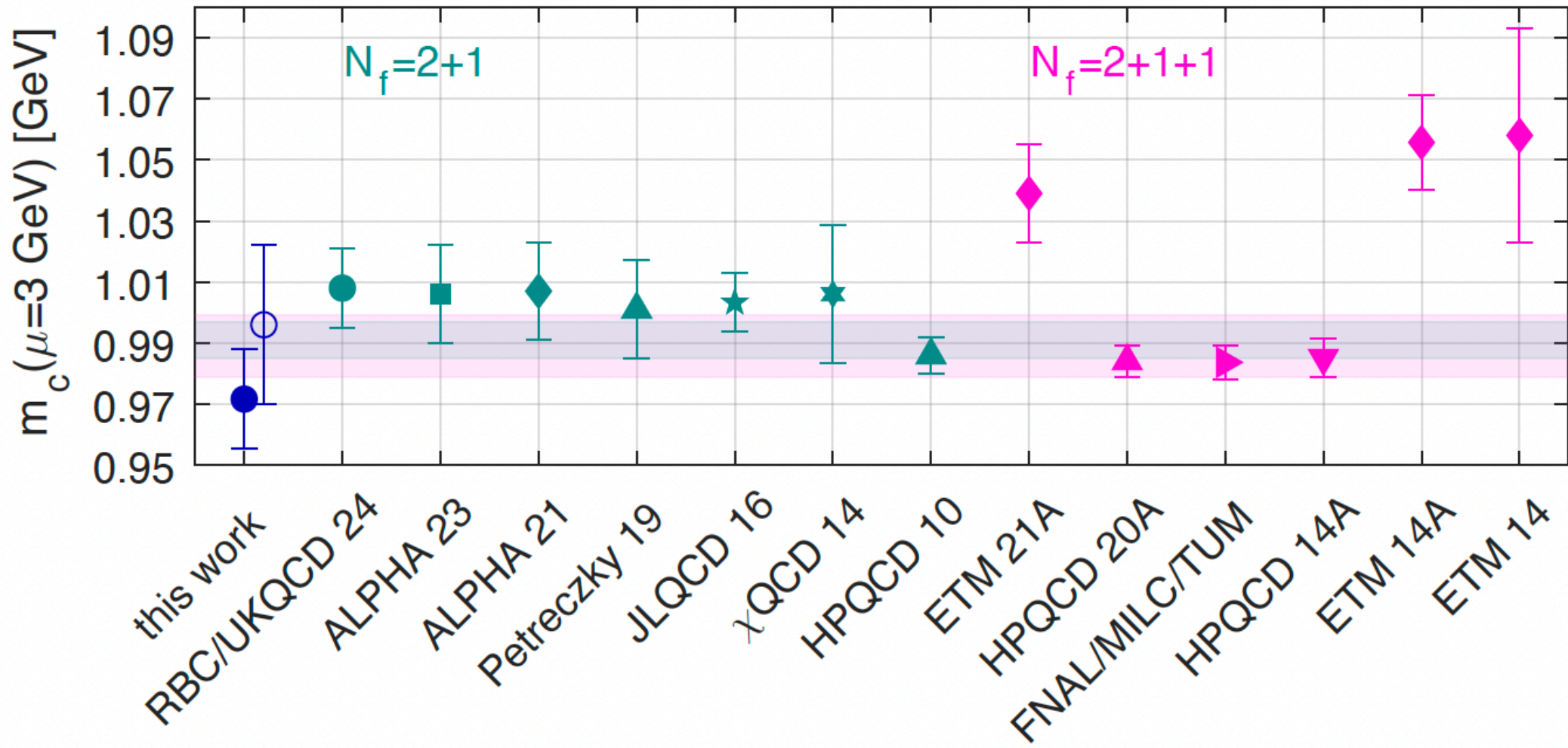
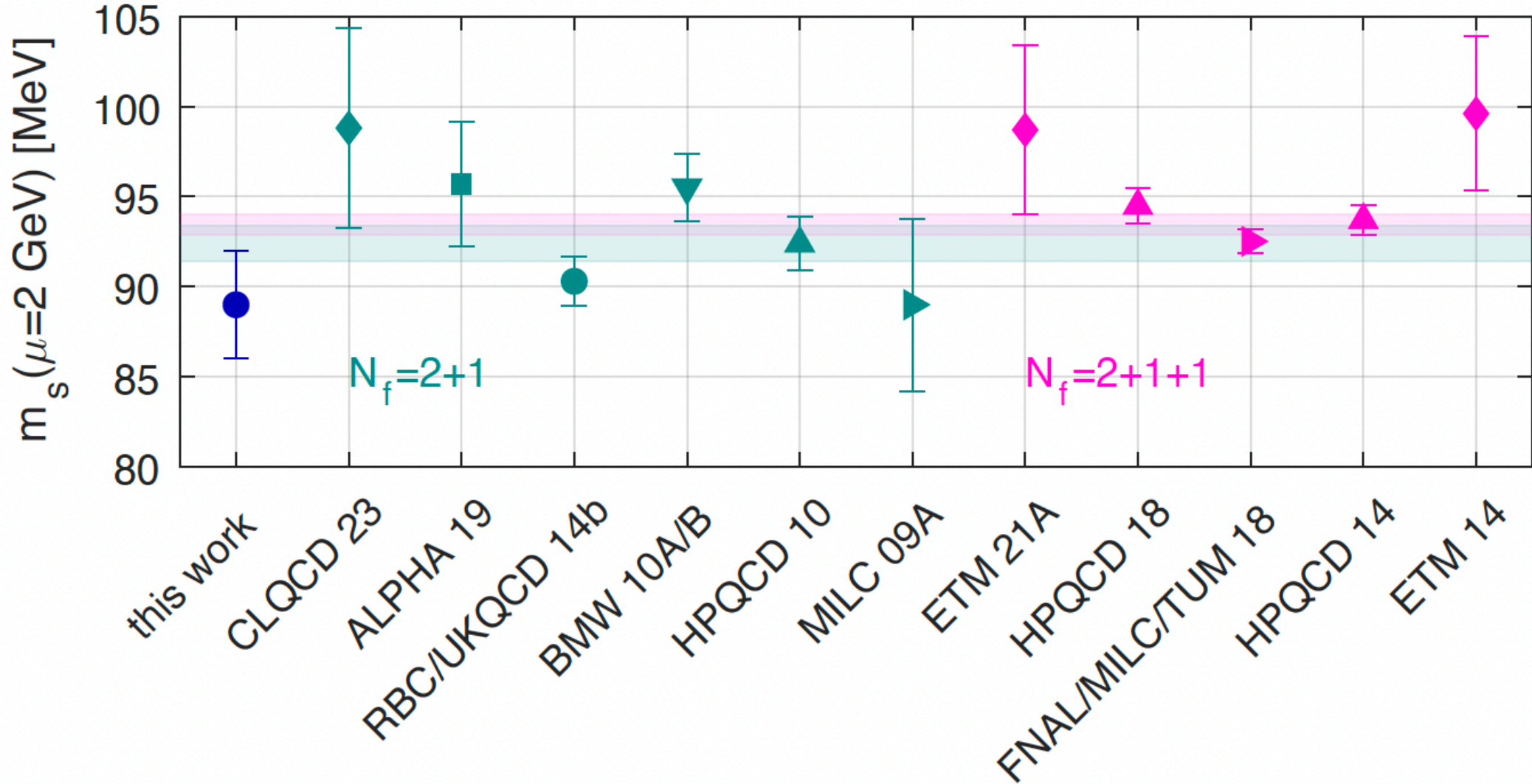


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1. Motivation

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1. Motivation (LHC \rightarrow PDFs \rightarrow SFTX \rightarrow Flavor physics \rightarrow PDFs)

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1. Motivation (LHC \rightarrow PDFs \rightarrow SFTX \rightarrow Flavor physics \rightarrow PDFs)

2. Introduction

Table of contents

1. Motivation (LHC → PDFs → SFTX → Flavor physics → PDFs)

2. Introduction

~~**3. Main part**~~

Table of contents

1. Motivation (LHC → PDFs → SFTX → Flavor physics → PDFs)

2. Introduction

~~**3. Main part**~~

4. Conclusions

Gradient flow Lagrangian

flowed gauge field:

$$\begin{aligned}\frac{\partial}{\partial t} B_\mu(t, x) &= \mathcal{D}_\nu G_{\nu\mu}(t, x) \\ B_\mu(t=0, x) &= A_\mu(x)\end{aligned}$$

flowed quark field:

$$\begin{aligned}\frac{\partial}{\partial t} \chi(t, x) &= \mathcal{D}^2 \chi(t, x) \\ \chi(t=0, x) &= \psi(x)\end{aligned}$$

Gradient flow Lagrangian

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$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

$$\mathcal{L}_B \sim \int_0^\infty dt L_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

$$\mathcal{L}_\chi \sim \int_0^\infty dt \bar{\lambda} \left(\partial_t - \mathcal{D}^2 \right) \chi + \text{h.c.}$$

Lüscher, Weisz 2011

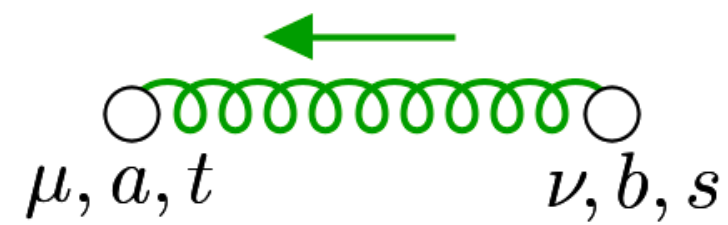
Lüscher 2013

Perturbative approach

$$\mathcal{L}_B \sim \int_0^\infty dt \mathbf{L}_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

Perturbative approach

$$\mathcal{L}_B \sim \int_0^\infty dt L_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$



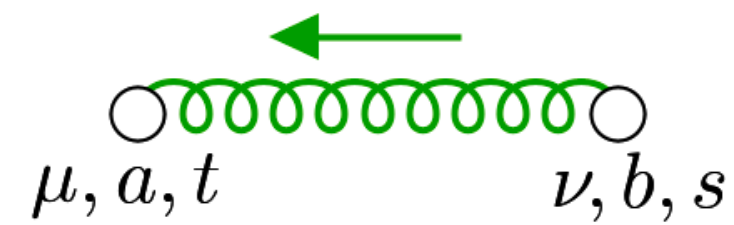
$$\delta_{ab} \delta_{\mu\nu} \theta(t-s) e^{-(t-s)p^2}$$

“gluon flow line”

$$\sim \langle 0 | T L_\mu^a(t, x) B_\nu^b(s, 0) | 0 \rangle$$

Perturbative approach

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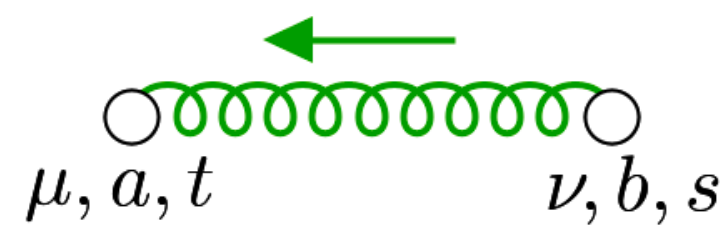


$$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T B_\mu^a(t, x) B_\nu^b(s, 0) | 0 \rangle$$

Perturbative approach

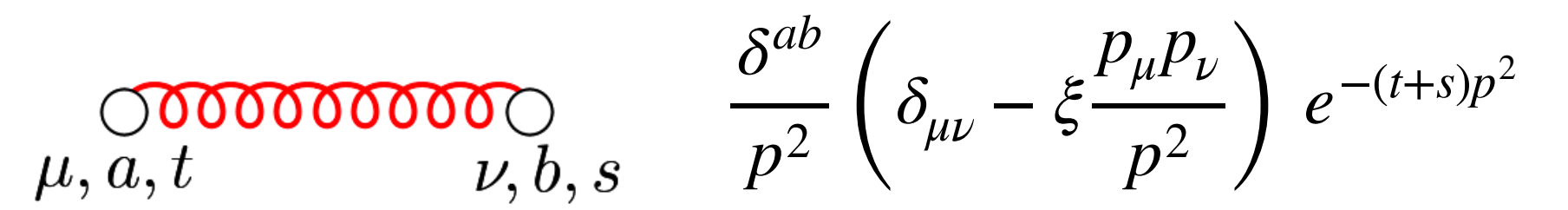
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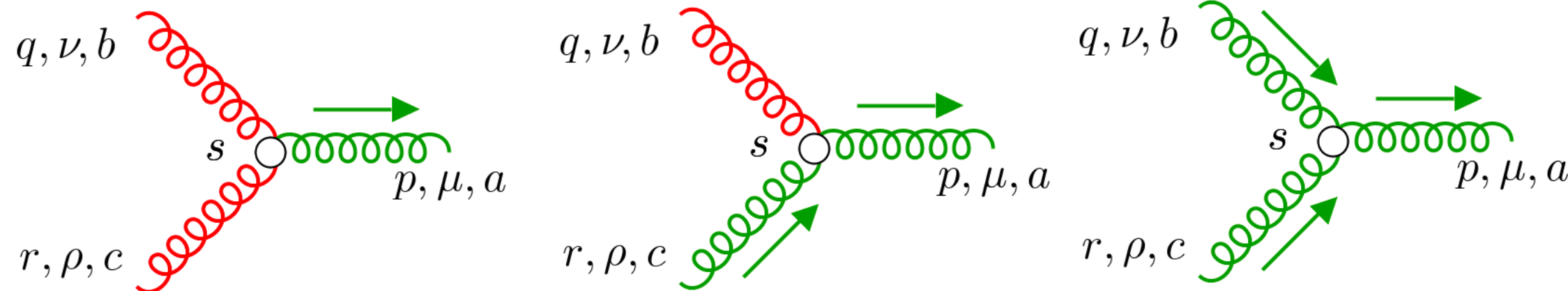
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$$-ig f^{abc} \int_0^\infty ds \left(\delta_{\nu\rho} (r-q)_\mu + 2\delta_{\mu\nu} q_\rho - 2\delta_{\mu\rho} r_\nu + (\kappa - 1)(\delta_{\mu\rho} q_\nu - \delta_{\mu\nu} r_\rho) \right)$$

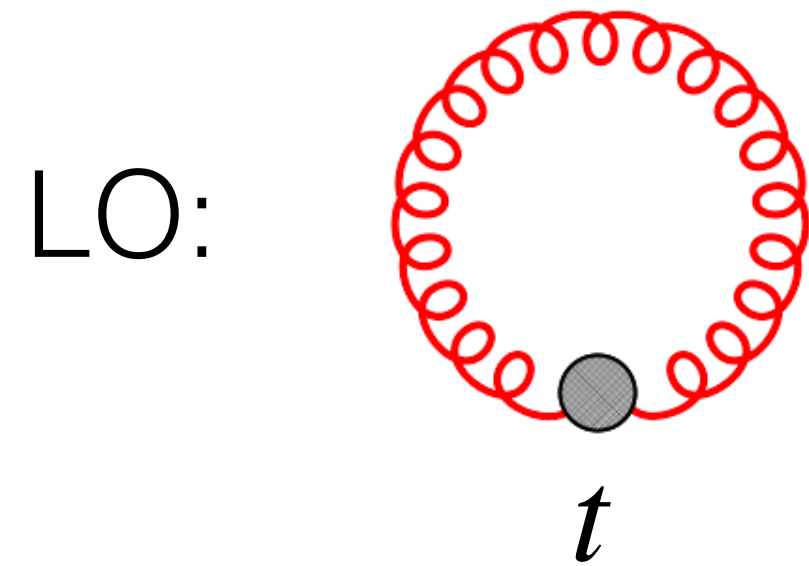
+ 4-gluon vertex

Let's calculate

$$E(t) \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$

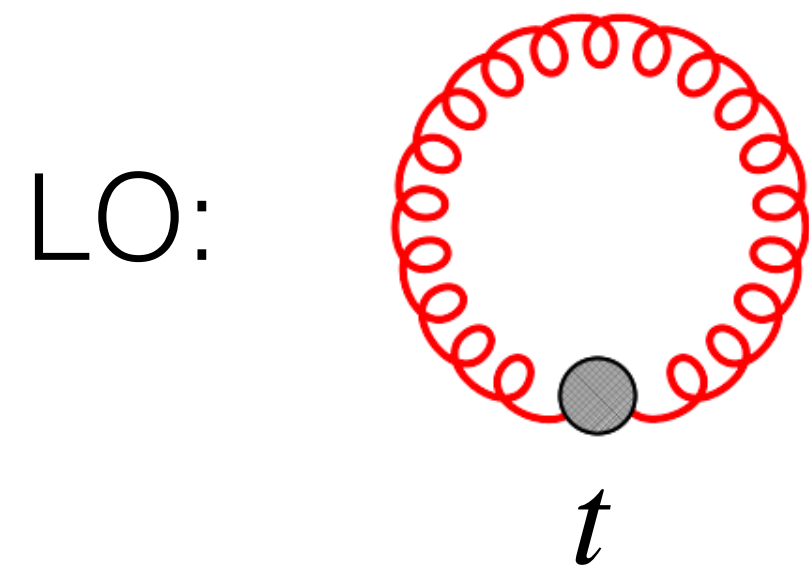
Let's calculate


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Let's calculate

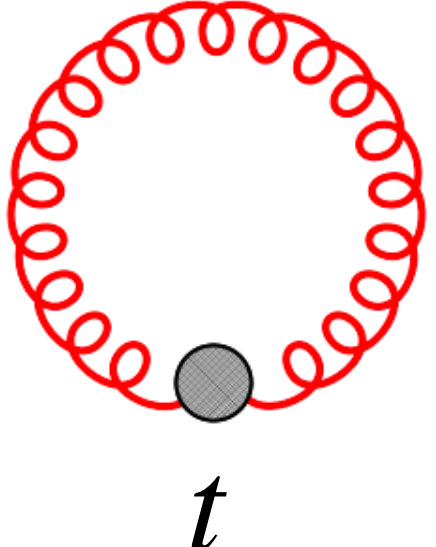
$$E(t) \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$





$$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

Let's calculate


$$E(t) \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$

LO:  $\sim \int d^D p e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

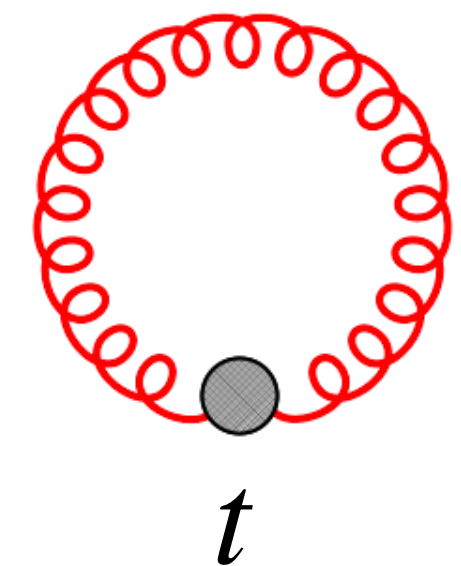
 $\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$

Let's calculate

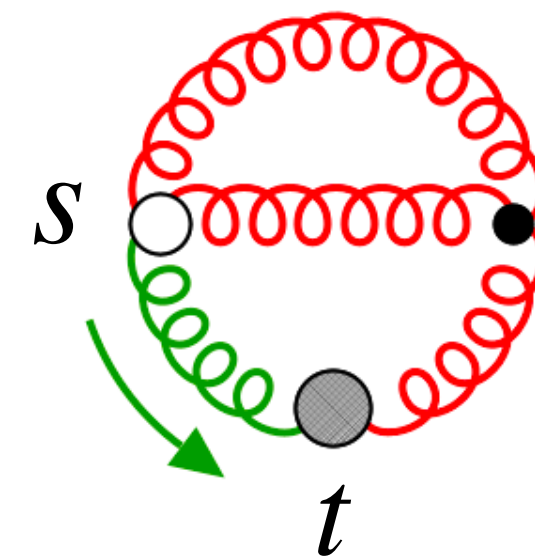
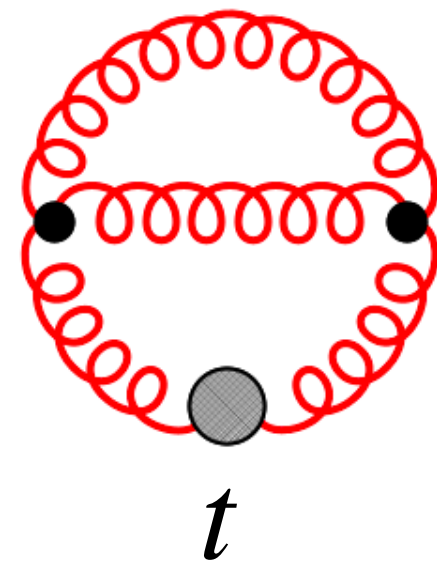
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NLO:




$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4 k^2 (p-k)^2}$$

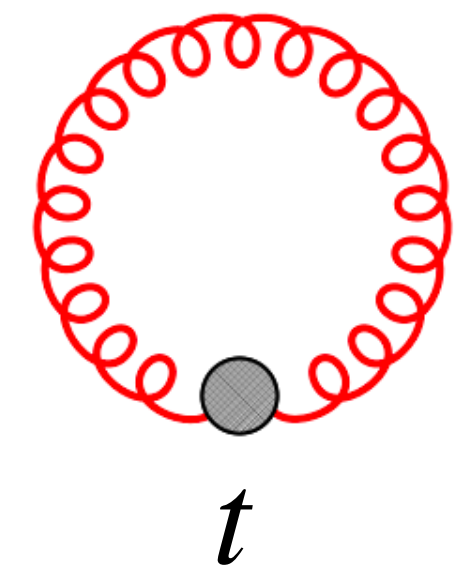
$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

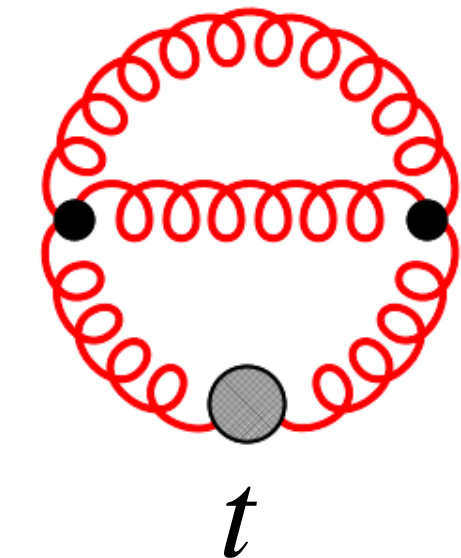
Let's calculate

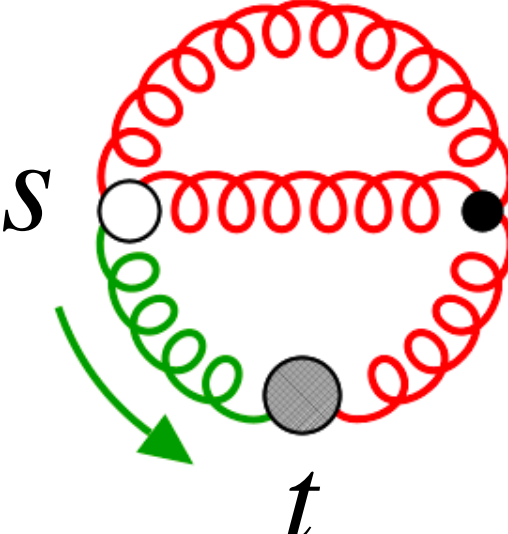
$$E(t) \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$



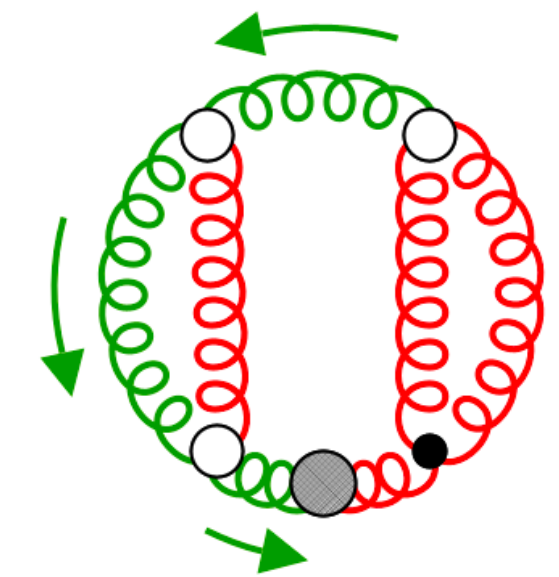
$$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

LO:  $\sim \int d^D p e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

NLO: 



NNLO:



$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4 k^2 (p-k)^2}$$

$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

$$\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_p \int_k \int_l \dots$$

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation:

qgraf Nogueira 1993

diagram analyzation:

q2e/exp RH, Seidensticker, Steinhauser 1997

→ tapir/exp Gerlach, Herren, Lang 2022

algebraic manipulations:

FORM Vermaseren 2000, ...

reduction to masters:

Kira ⊗ FireFly Usovitsch, Uwer, Maierhöfer 2017

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sector decomposition:

Binoth, Heinrich 2002

$$\int d^D k \int d^D p \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p)^2} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \dots$$

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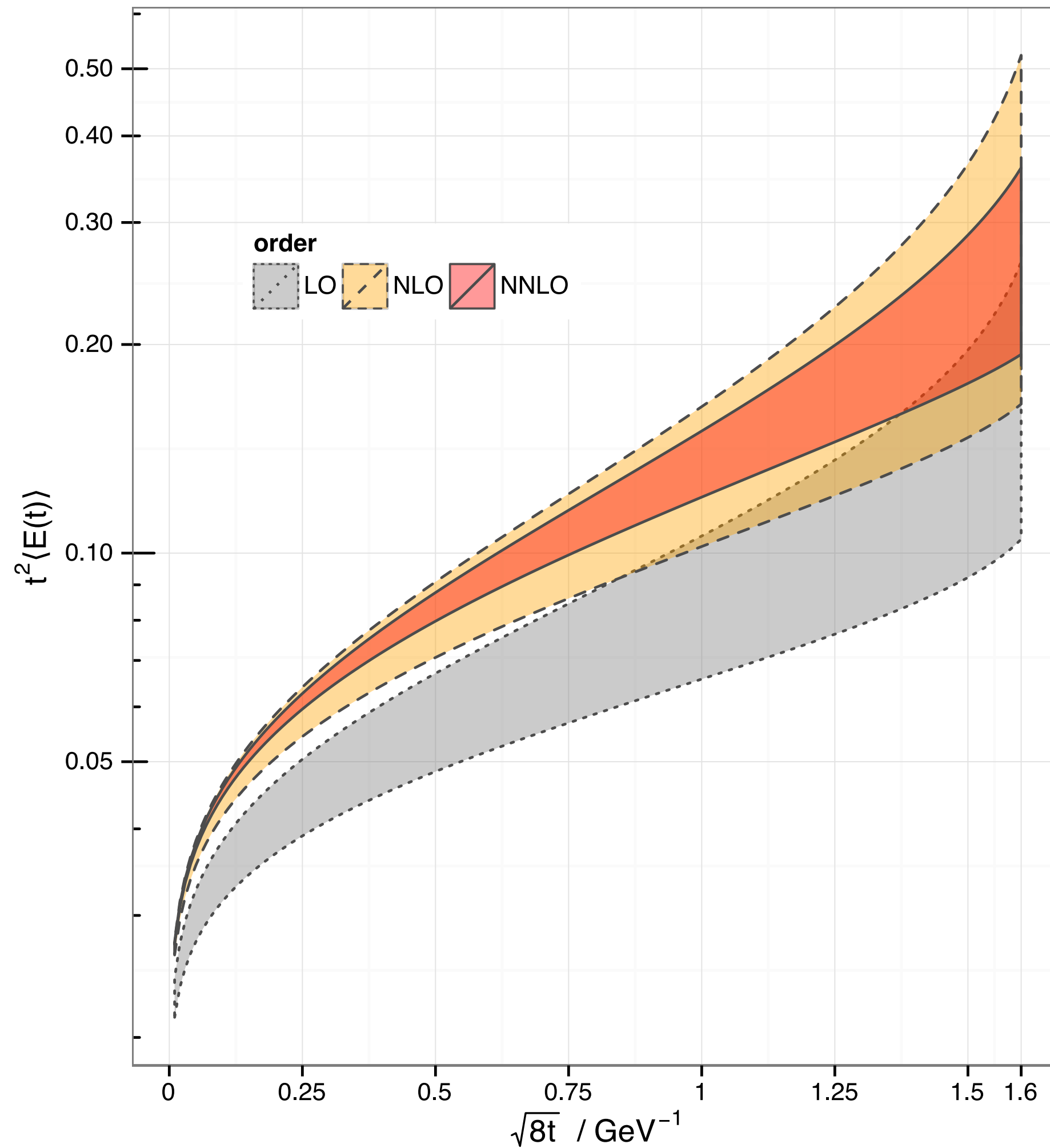
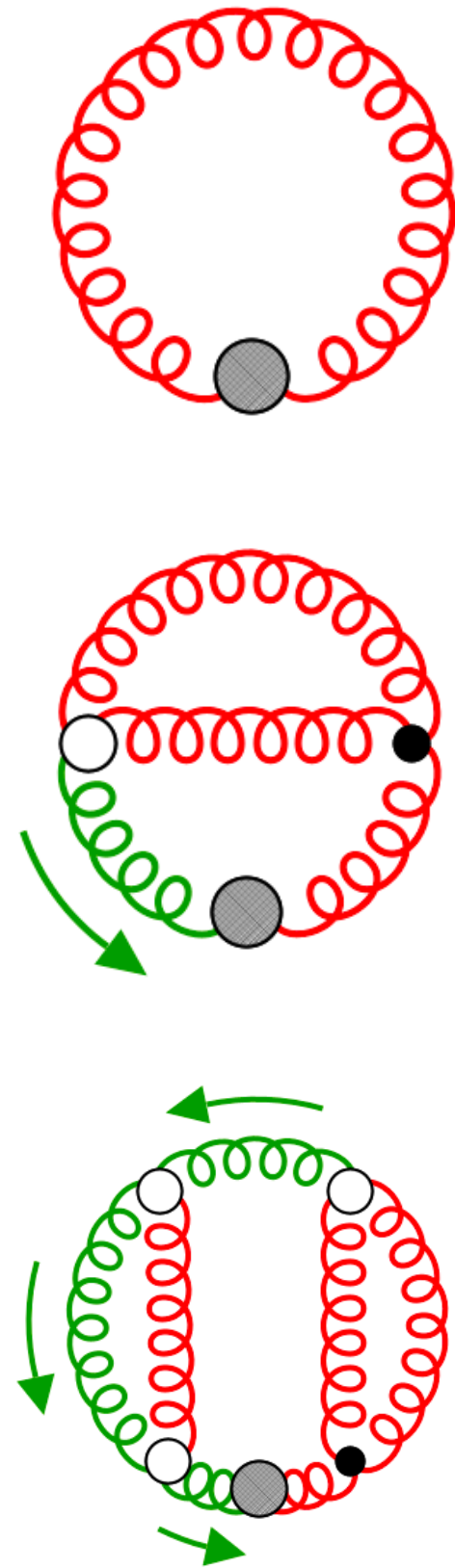
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→ ftint RH, Nellopoulos, Olsson, Wesle '25

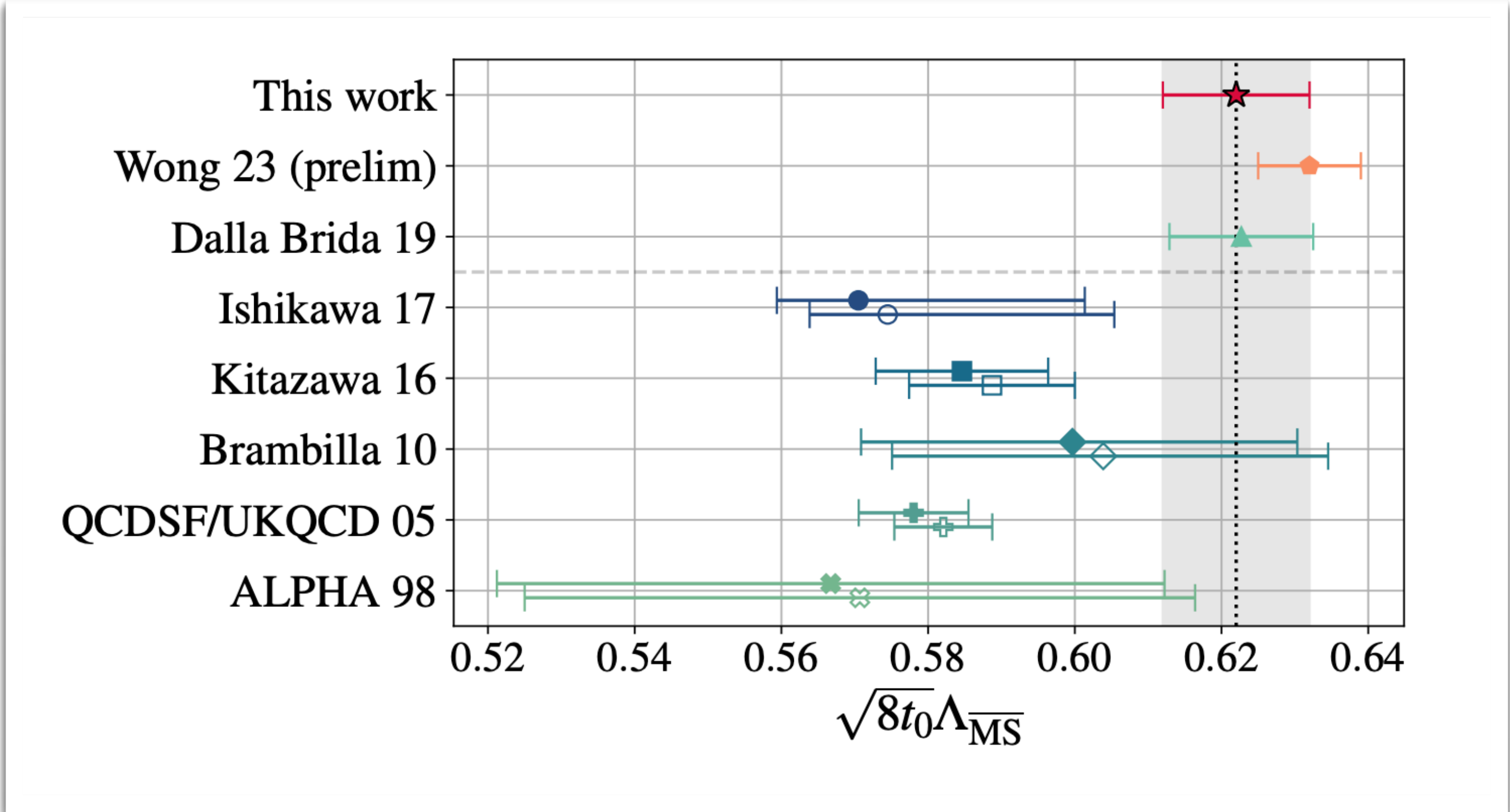
$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right] \equiv \frac{3}{4\pi} \alpha_s^{\text{GF}}(t)$$

RH, Neumann 2016



Renormalization scheme for α_s
compatible with lattice and PT!

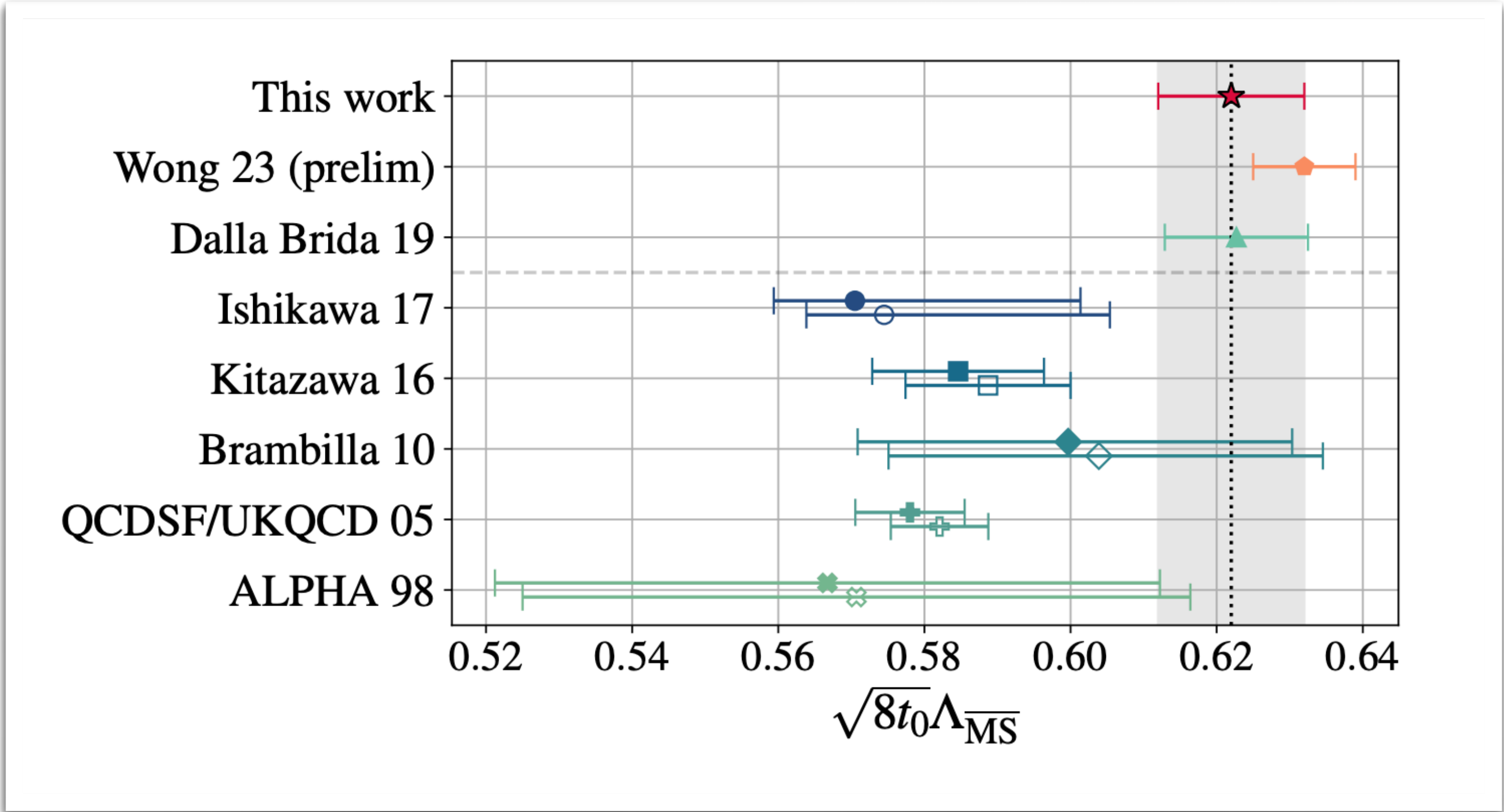
Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel (2023)

see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

Determine Λ_{QCD}



Hasenfratz (Tue 9:00)

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Three thoughts on the GF scheme

1

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$$+ t^2 \tilde{C}_F(t) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \dots \quad \text{non-perturbative cont's}$$

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perturbative contribution: renormalons!

Renormalon subtraction

Beneke, Takaura '25

$$\Pi(q^2) \sim \int d^4x e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$

Adler function:
$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \dots$$

Baikov, Chetyrkin, Kühn '08 \swarrow perturbative \nwarrow RH '98

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RH '98

perturbative

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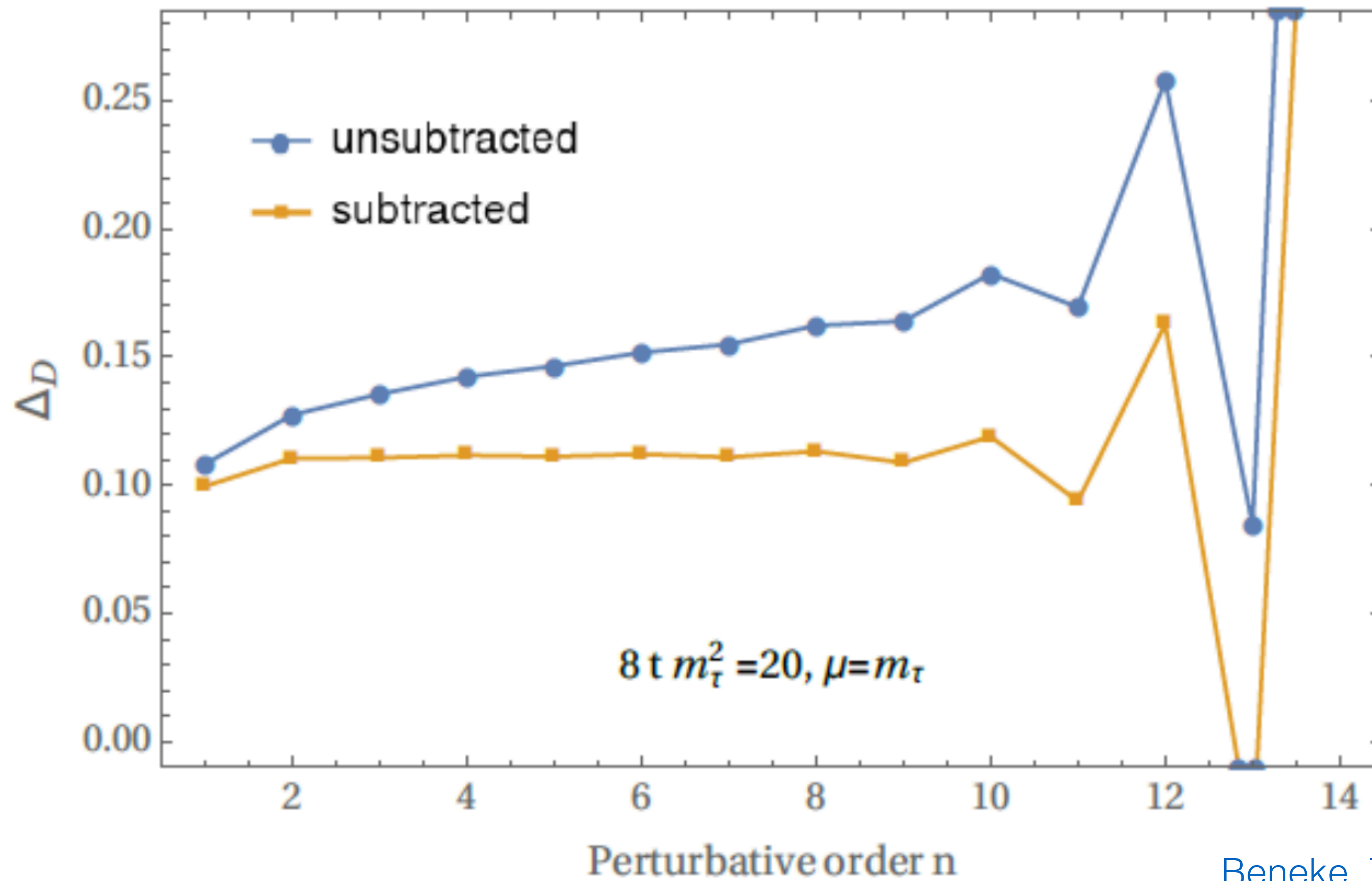
RH, Neumann '16

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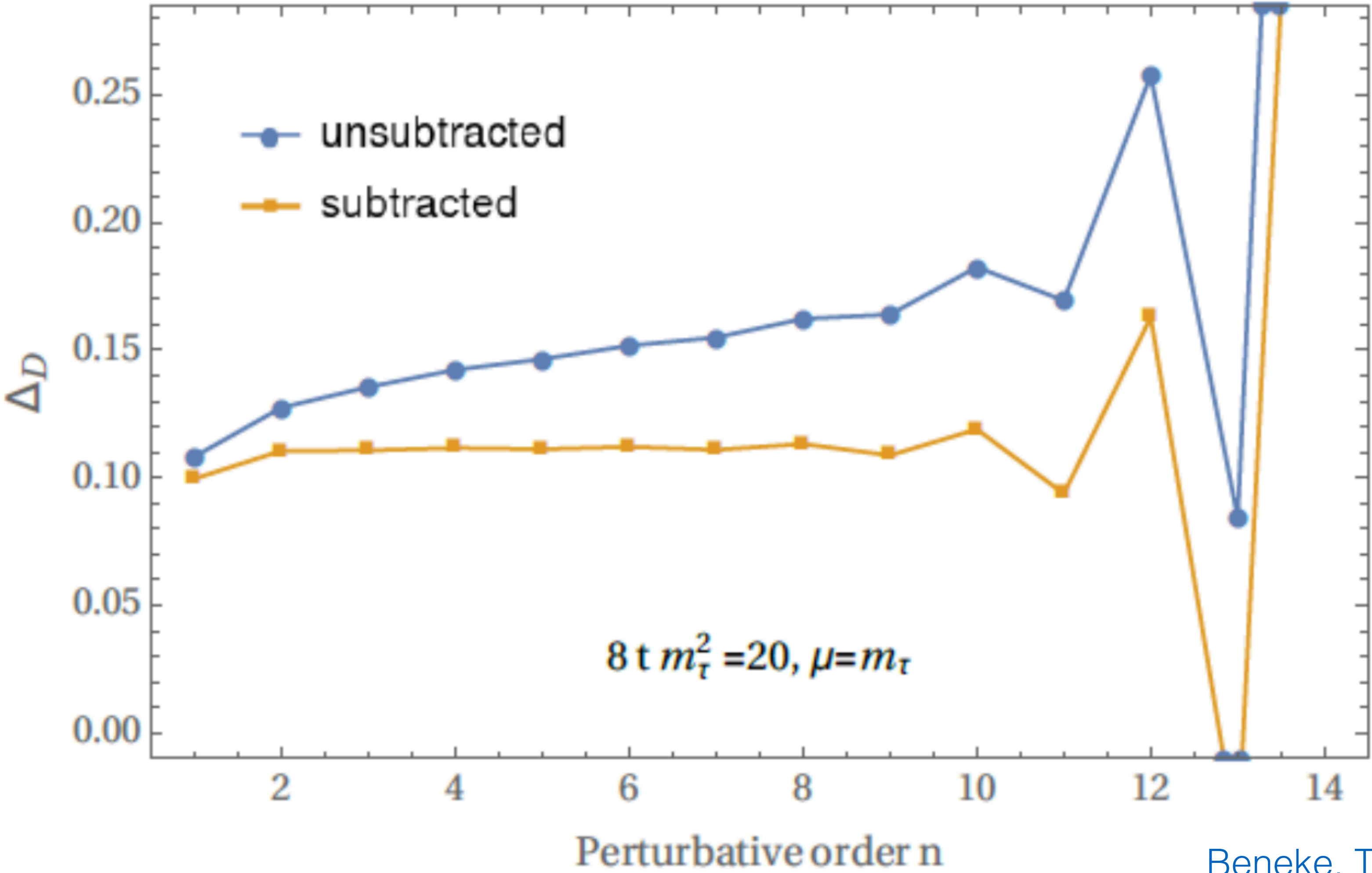
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renormalon cancels



Beneke, Takaura '25



Beneke, Takaura '25

Three thoughts on the GF scheme

2

Three thoughts on the GF scheme

2

Problems with \overline{MS} : mass independent

Three thoughts on the GF scheme

2

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decoupling must be implemented “by hand”

Three thoughts on the GF scheme

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e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

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for $\mu > m_c$, switch to $\alpha_s^{(4)}(\mu)$

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$$\beta^{\overline{\text{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi} \left(11 - \frac{2}{3}n_f \right) + \dots$$

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$\alpha_s(\mu)$ has a kink/step at some “decoupling scale” μ_{thr}

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$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right] \equiv \frac{3}{4\pi} \alpha_s^{\text{GF}}(t)$$

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$$k_1 = k_1 \Big|_{n_f=0} - \sum_q \frac{1}{6\pi} \left(1 + \Omega_{1q}(m_q, t) \right)$$

Lüscher 2010
RH, Neumann 2016

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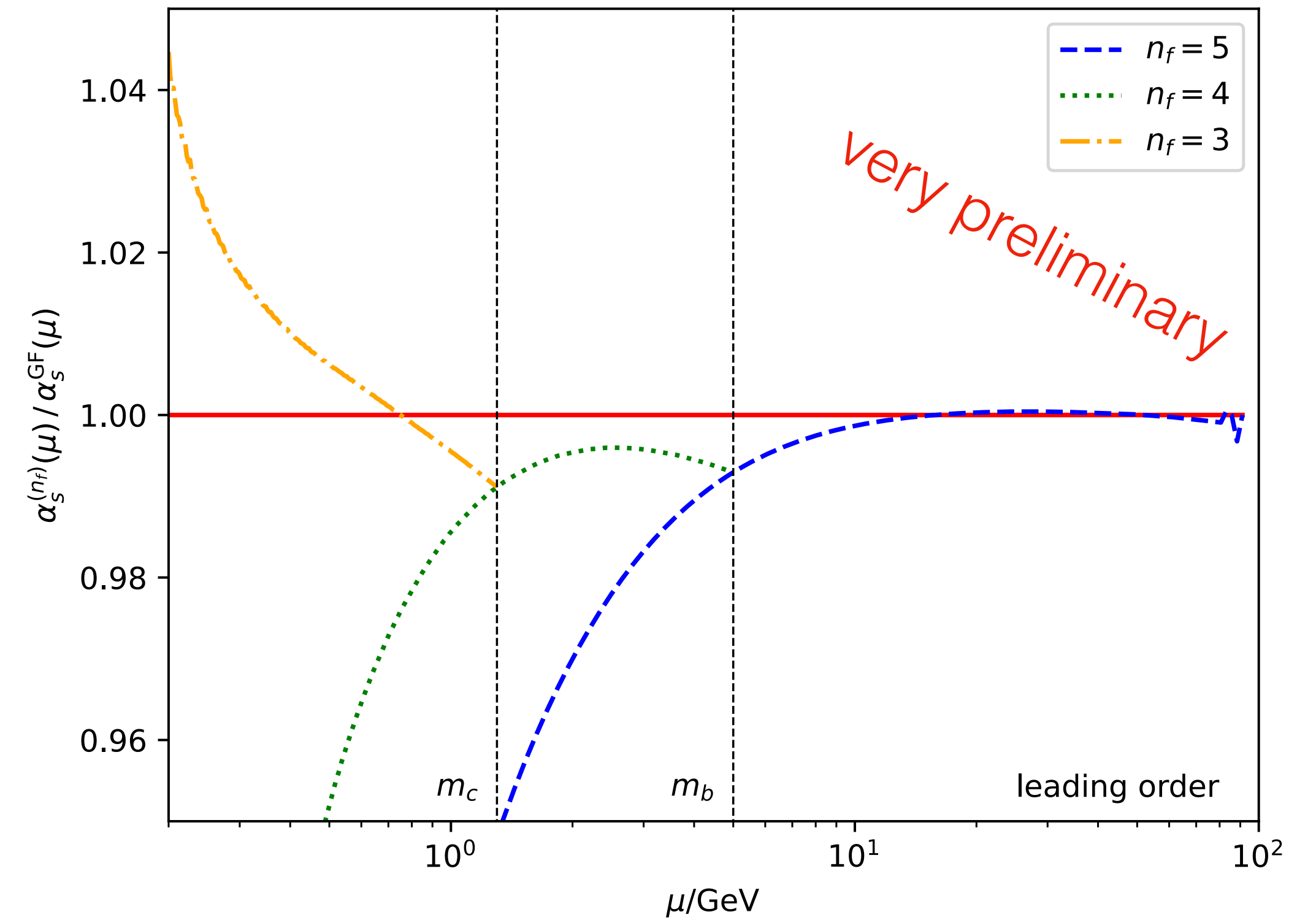
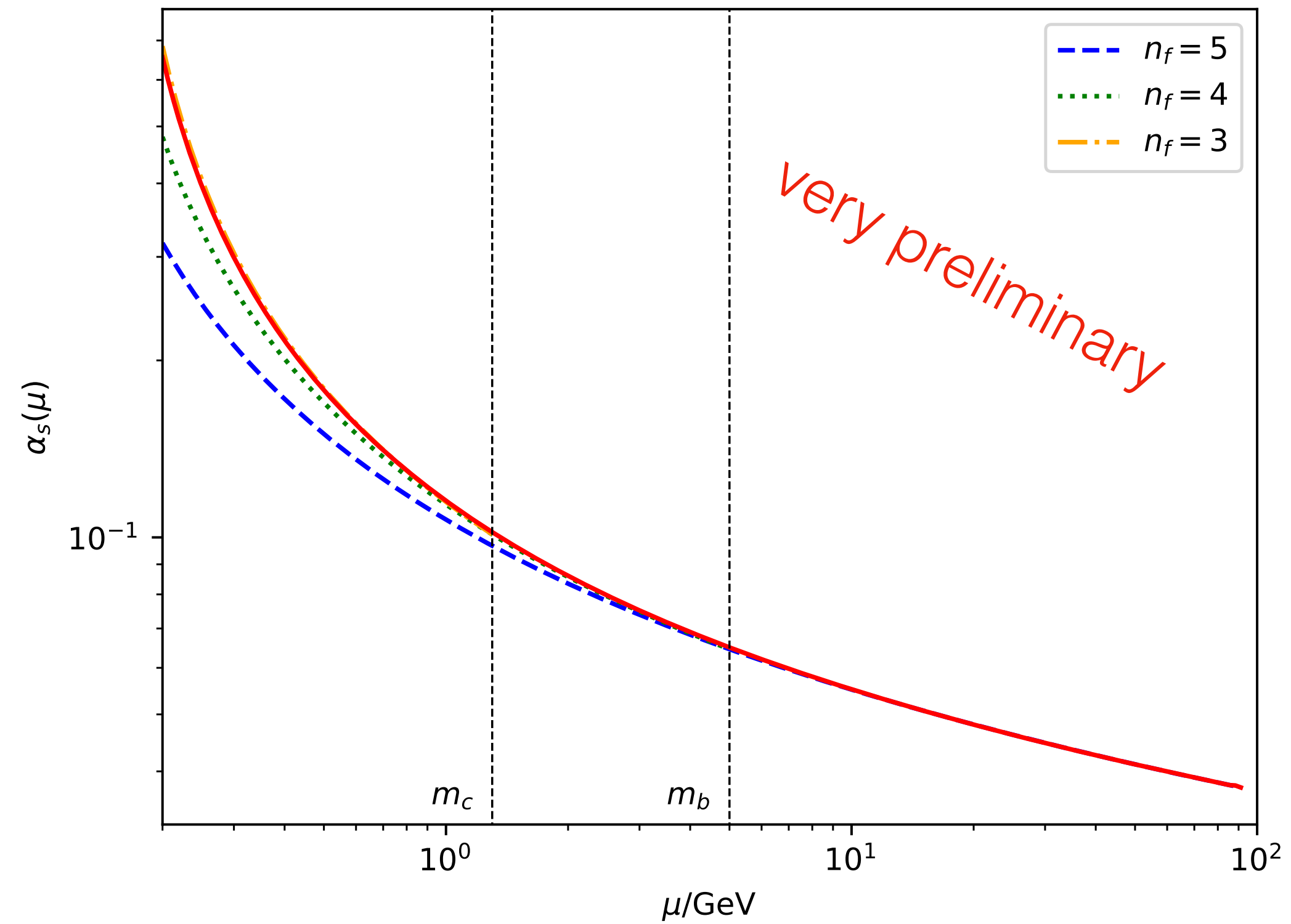
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GF vs \overline{MS}



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NNLO: RH, Mason 2025

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Robert Mason (Wed 11:00)

Three thoughts on the GF scheme

Kluth '24, '26

3

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with \overline{MS} : insensitive to power divergences

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2} \Gamma(1 - D/2)$$

Three thoughts on the GF scheme

Kluth '24, '26

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Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2} \Gamma(1 - D/2)$$
$$D = 4 - 2\epsilon \quad = m^{2-2\epsilon} \Gamma(-1 + \epsilon) = \frac{m^{2-2\epsilon}}{\epsilon - 1} \Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

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Kluth '24, '26

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i.e., only logarithmic divergence contributes!

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→ Fixed point in gravity is hidden in $\overline{\text{MS}}$!

Three thoughts on the GF scheme

Kluth '24, '26

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Need mass dependent renormalization scheme

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$

$$B_\mu(t = 0, x) = A_\mu(x)$$

Application to gravity

RH, Kluth, Kohlen, Werthenbach '26

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gravity
→

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

Application to gravity

RH, Kluth, Kohlen, Werthenbach '26

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$$\begin{aligned}\frac{\partial}{\partial t} B_\mu(t, x) &= \mathcal{D}_\nu G_{\nu\mu}(t, x) \\ B_\mu(t=0, x) &= A_\mu(x)\end{aligned}$$

gravity
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Ricci flow

flowed metric requires field renormalization (analogous to Z_χ in QCD)

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\begin{aligned}\frac{\partial}{\partial t} B_\mu(t, x) &= \mathcal{D}_\nu G_{\nu\mu}(t, x) \\ B_\mu(t=0, x) &= A_\mu(x)\end{aligned}$$

gravity \longrightarrow

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

flowed metric requires field renormalization (analogous to Z_χ in QCD)

define “fixed-volume scheme”:

$$\langle \sqrt{g(t)} \rangle \equiv -\frac{3G_N}{4\pi t}$$

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RH, Kluth, Kohlen, Werthenbach '26

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flowed metric requires field renormalization (analogous to Z_χ in QCD)

define “fixed-volume scheme”:

$$\langle \sqrt{g(t)} \rangle \equiv -\frac{3G_N}{4\pi t}$$

cf.: $\langle \dot{\chi}(t) \mathbb{D} \dot{\chi}(t) \rangle \equiv -\frac{2n_c}{(4\pi t)^2}$

Application to gravity

RH, Kluth, Kohlen, Werthenbach '26

define Newton coupling in Ricci flow scheme:

$$g_{\text{RF}}(\mu) \equiv -\frac{t}{3\nu} \langle \sqrt{g(t)} R(t) \rangle \Big|_{t=c/\mu^2}$$

$$\text{cf. } \alpha_s^{\text{GF}}(\mu) \equiv \frac{4\pi t^2}{3} \langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle \Big|_{t=c/\mu^2}$$

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$$= \frac{G_{\text{N}}}{4\pi t\nu} \left[1 - \frac{5G_{\text{N}}}{4\pi t} + \dots \right]$$

Application to gravity

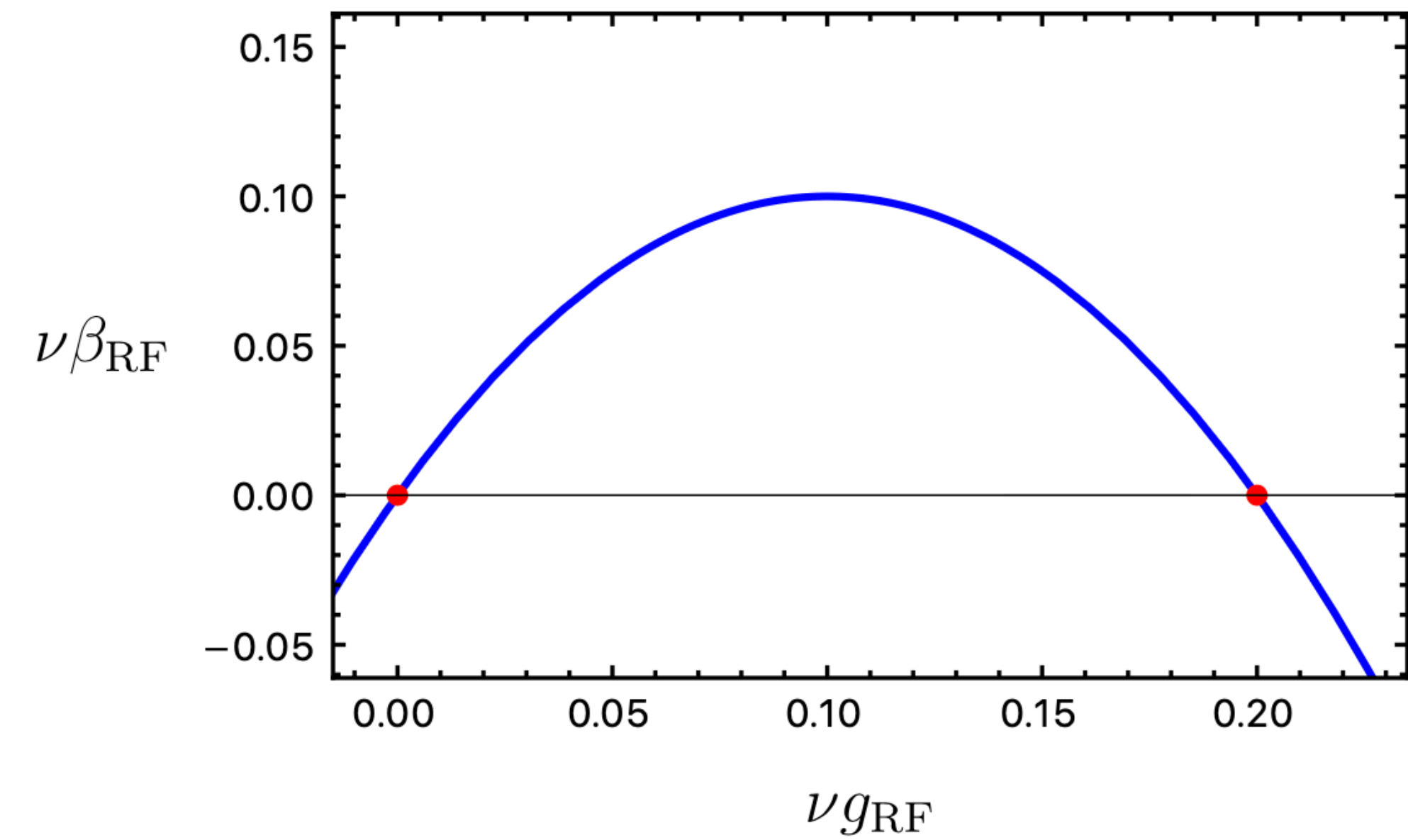
RH, Kluth, Kohlen, Werthenbach '26

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Application to gravity

RH, Kluth, Kohlen, Werthenbach '26

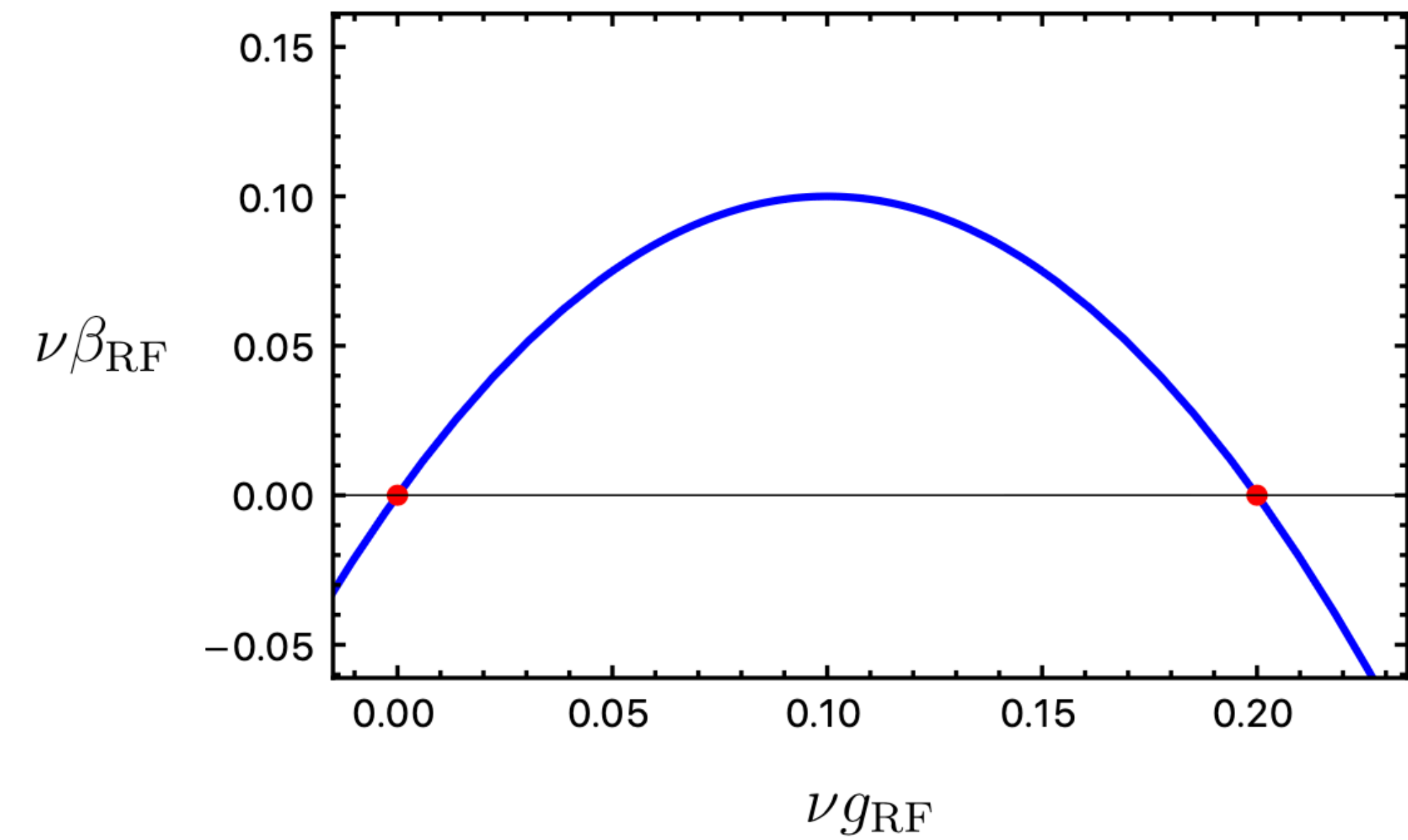
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Henry Werthenbach (Tue 16:00)

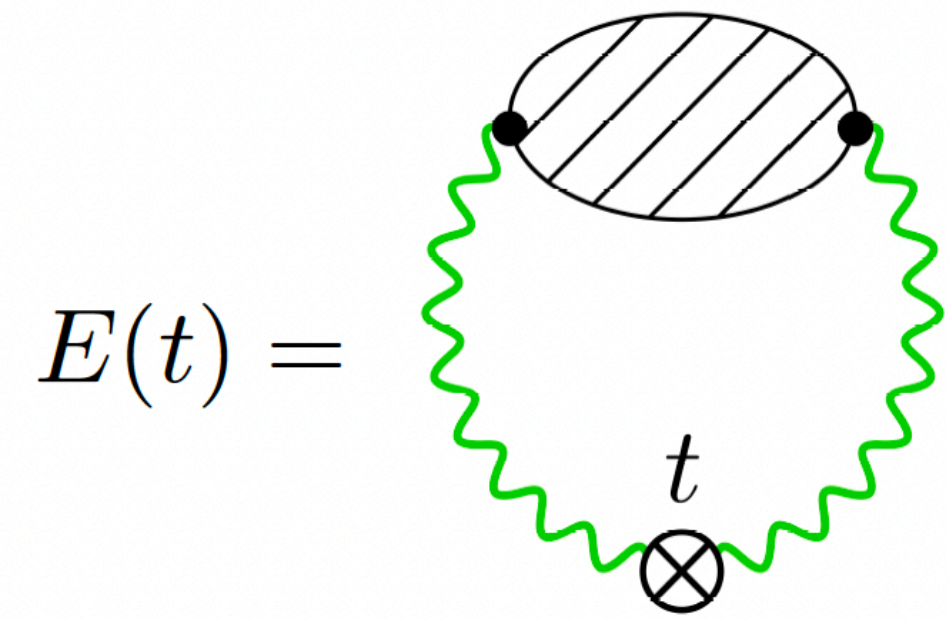


Application to QED₃

$$E(t) = \text{[Diagram]} = \langle F_{\mu\nu}(t)F_{\mu\nu}(t) \rangle$$

The diagram shows a green wavy loop with a shaded oval at the top and a cross symbol at the bottom, labeled with t .

Application to QED₃

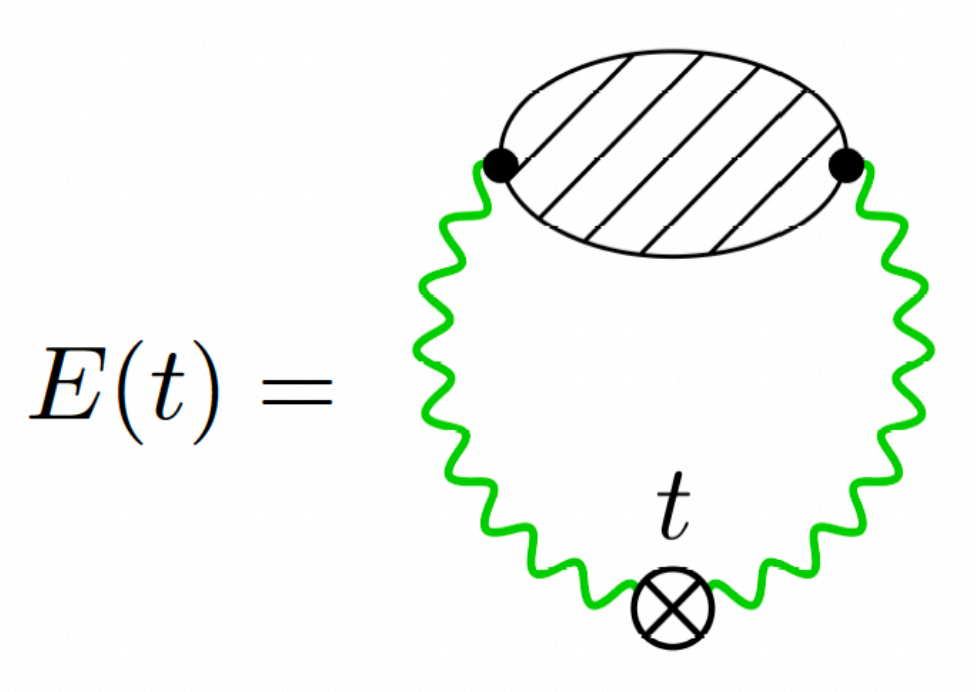


in D dimensions:

$$E(t) = \langle F_{\mu\nu}(t)F_{\mu\nu}(t) \rangle = \frac{e^2}{2} (D - 1) \int \frac{d^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 + \Pi_R(p)}$$

Georg, RH, Mason '26

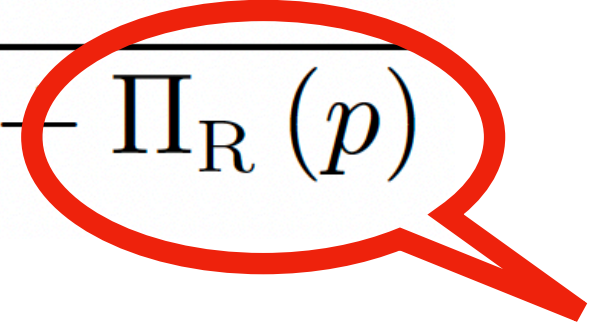
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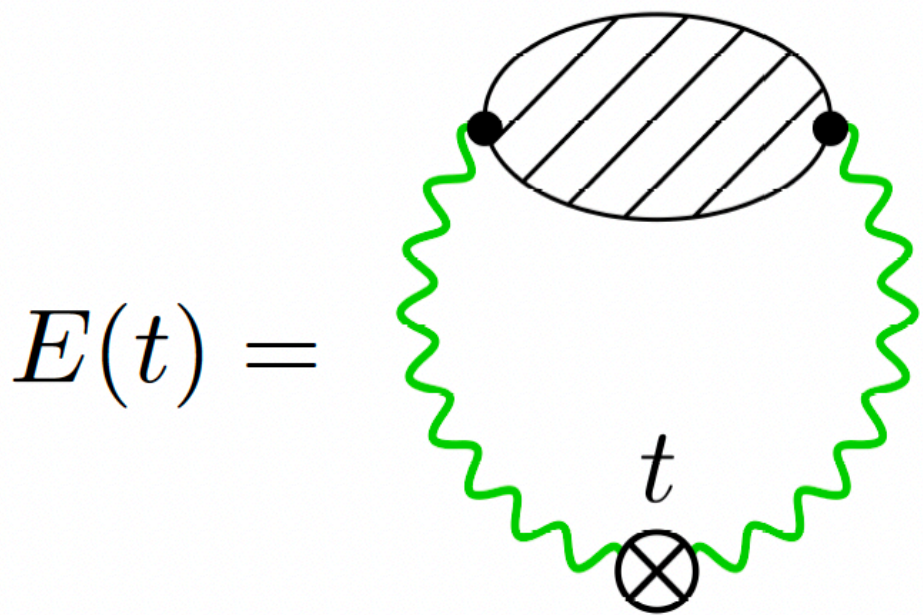
Georg, RH, Mason '26



regular photon polarization function

Application to QED₃

in D dimensions:



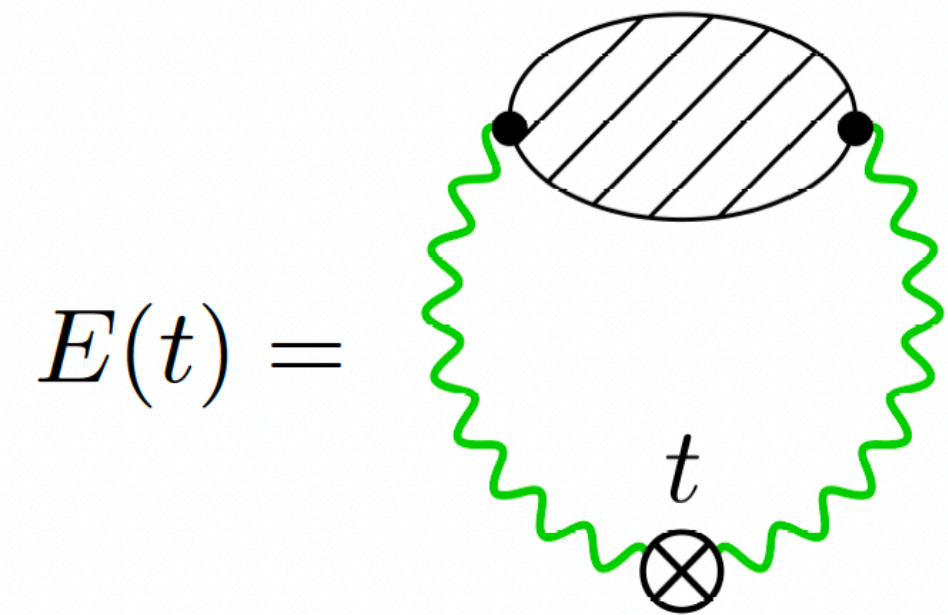
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Georg, RH, Mason '26

regular photon polarization function

D=3:

Application to QED₃



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Georg, RH, Mason '26

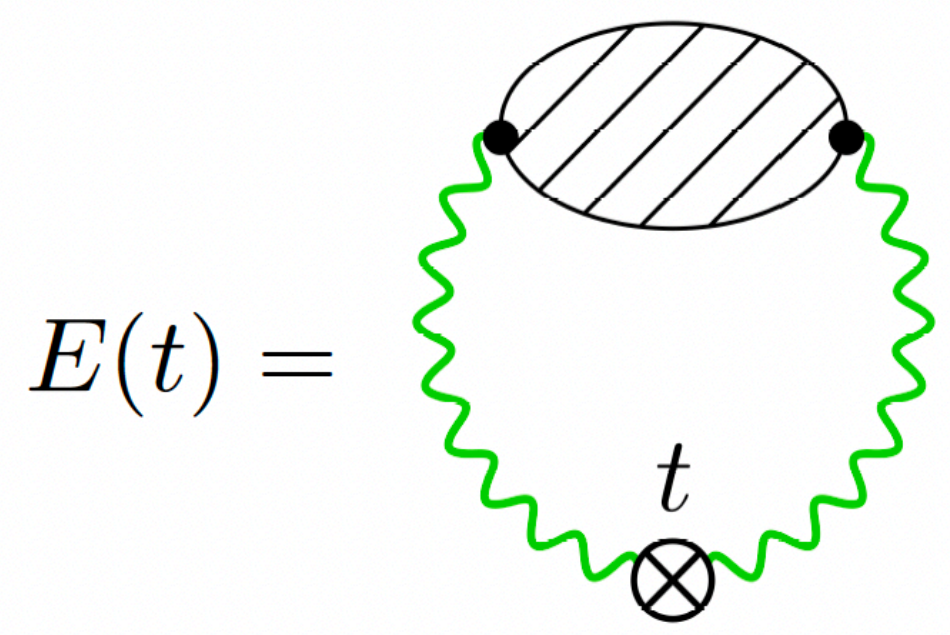
regular photon polarization function

D=3: α mass dimension = 1

$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski '81
Pisarski '84

Application to QED₃



in D dimensions:

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Georg, RH, Mason '26

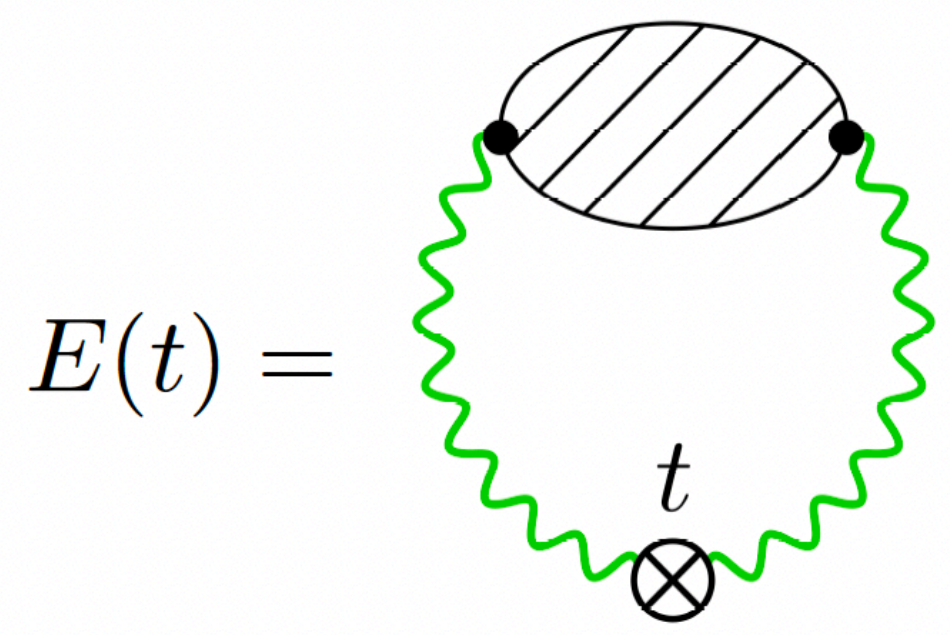
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D=3: α mass dimension = 1
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Georg, RH, Mason '26

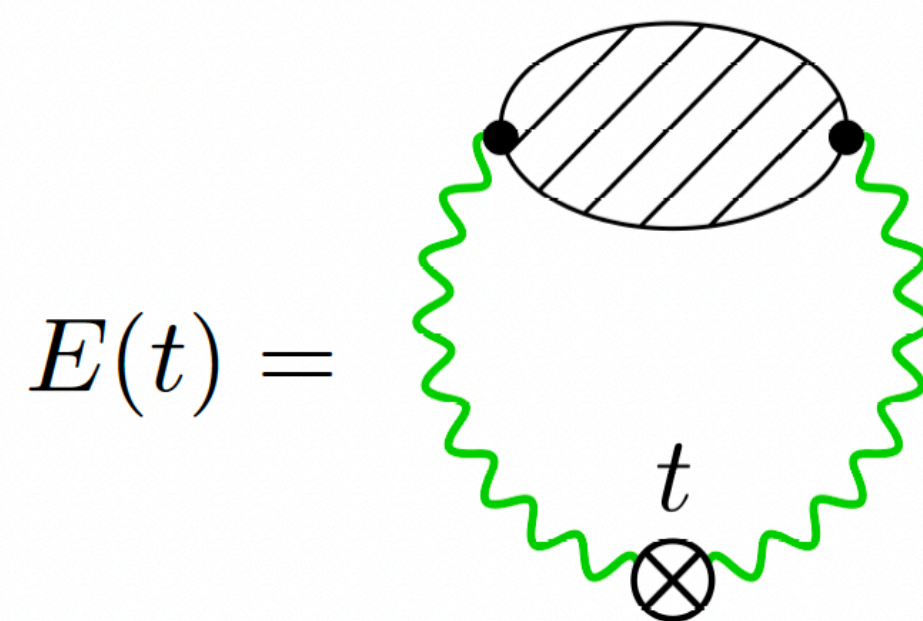
regular photon polarization function

D=3: α mass dimension = 1
 super-renormalizable
 strongly interacting in IR

$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski '81
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Application to QED₃



$$E(t) = \langle F_{\mu\nu}(t)F_{\mu\nu}(t) \rangle$$

in D dimensions:

$$= \frac{e^2}{2} (D - 1) \int \frac{d^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 - \Pi_R(p)}$$

Georg, RH, Mason '26

regular photon polarization function

D=3: α mass dimension = 1
 super-renormalizable
 strongly interacting in IR
 IR fixed point at large n_f

$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski '81
 Pisarski '84

Application to QED₃

$$E(t) = \text{[Diagram: A loop with a shaded top arc and a cross on the bottom arc, labeled with } t \text{]} = \langle F_{\mu\nu}(t)F_{\mu\nu}(t) \rangle = \frac{e^2}{2} (D - 1) \int \frac{d^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 - \Pi_R(p)}$$

in D dimensions:

Georg, RH, Mason '26

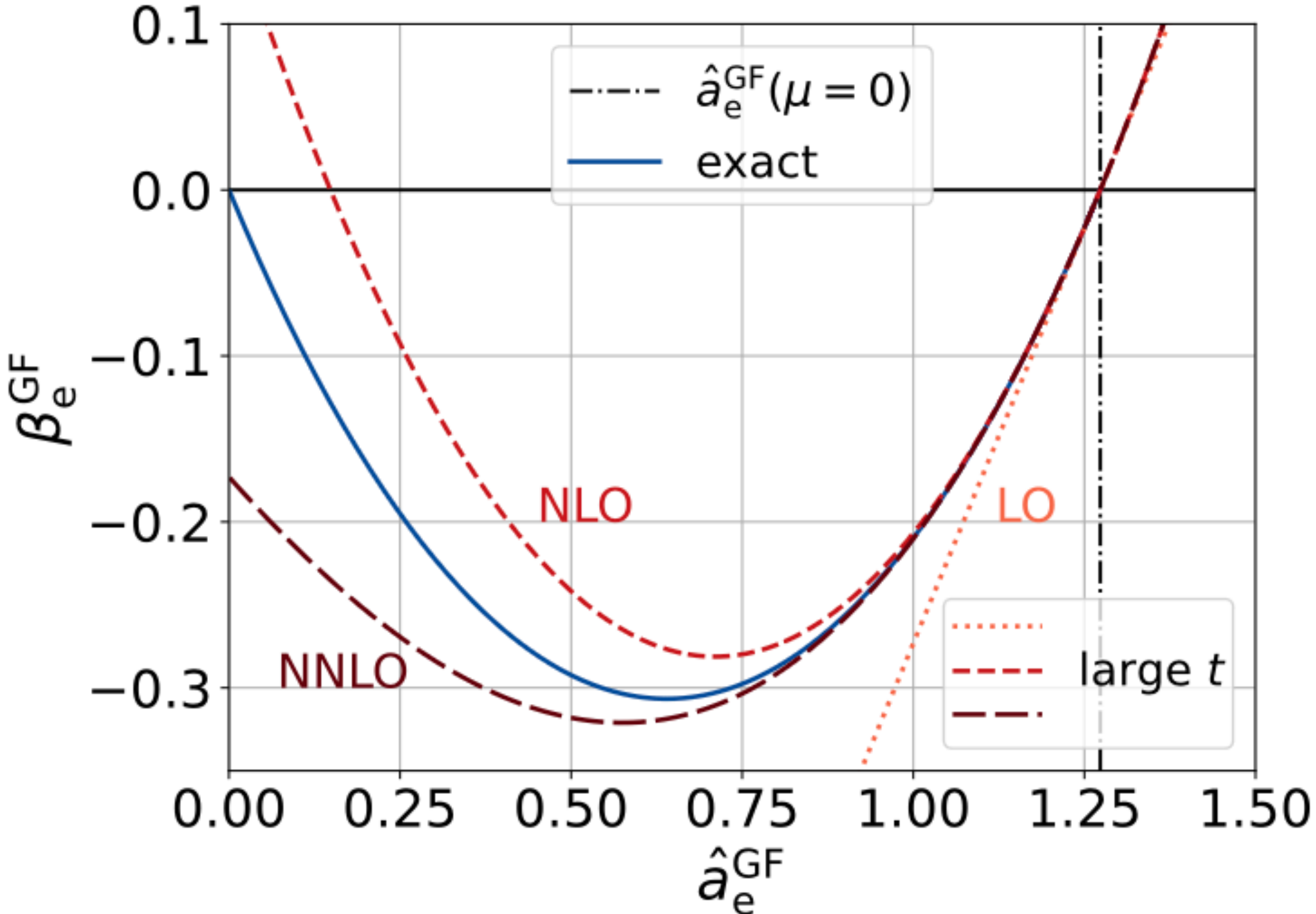
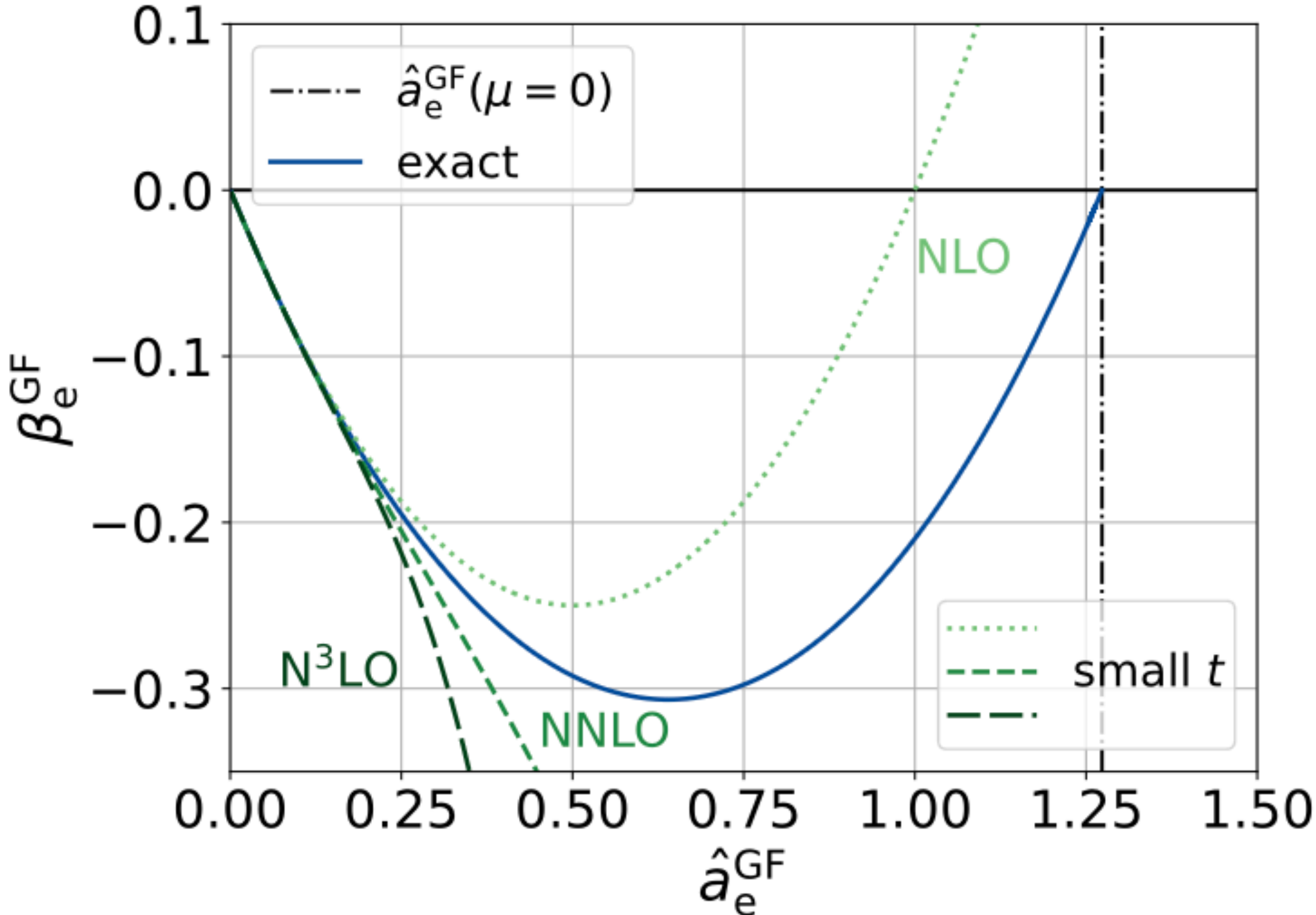
regular photon polarization function

- D=3:** α mass dimension = 1
- super-renormalizable
- strongly interacting in IR
- IR fixed point at large n_f
- dynamical fermion mass generation

$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski '81
Pisarski '84

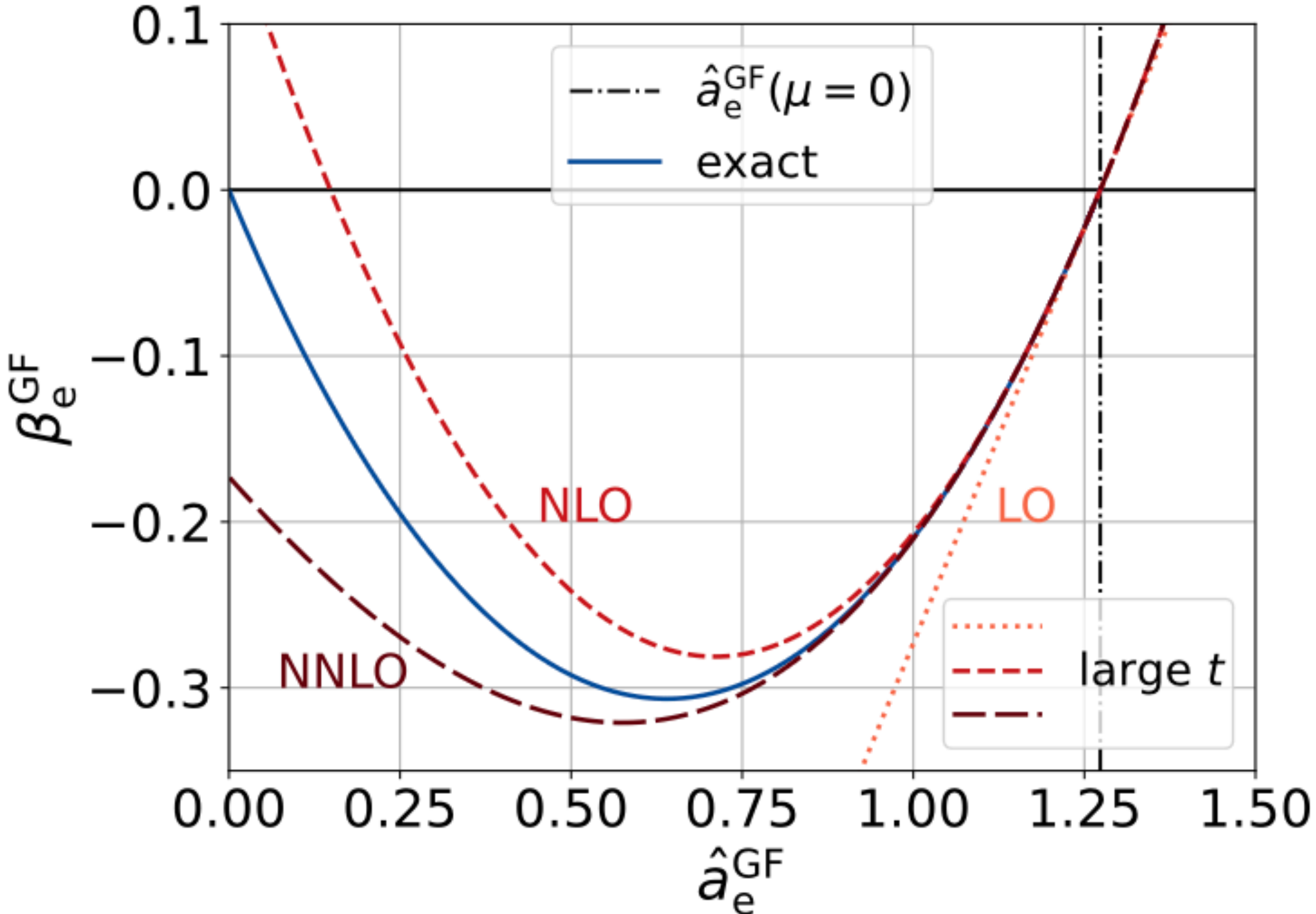
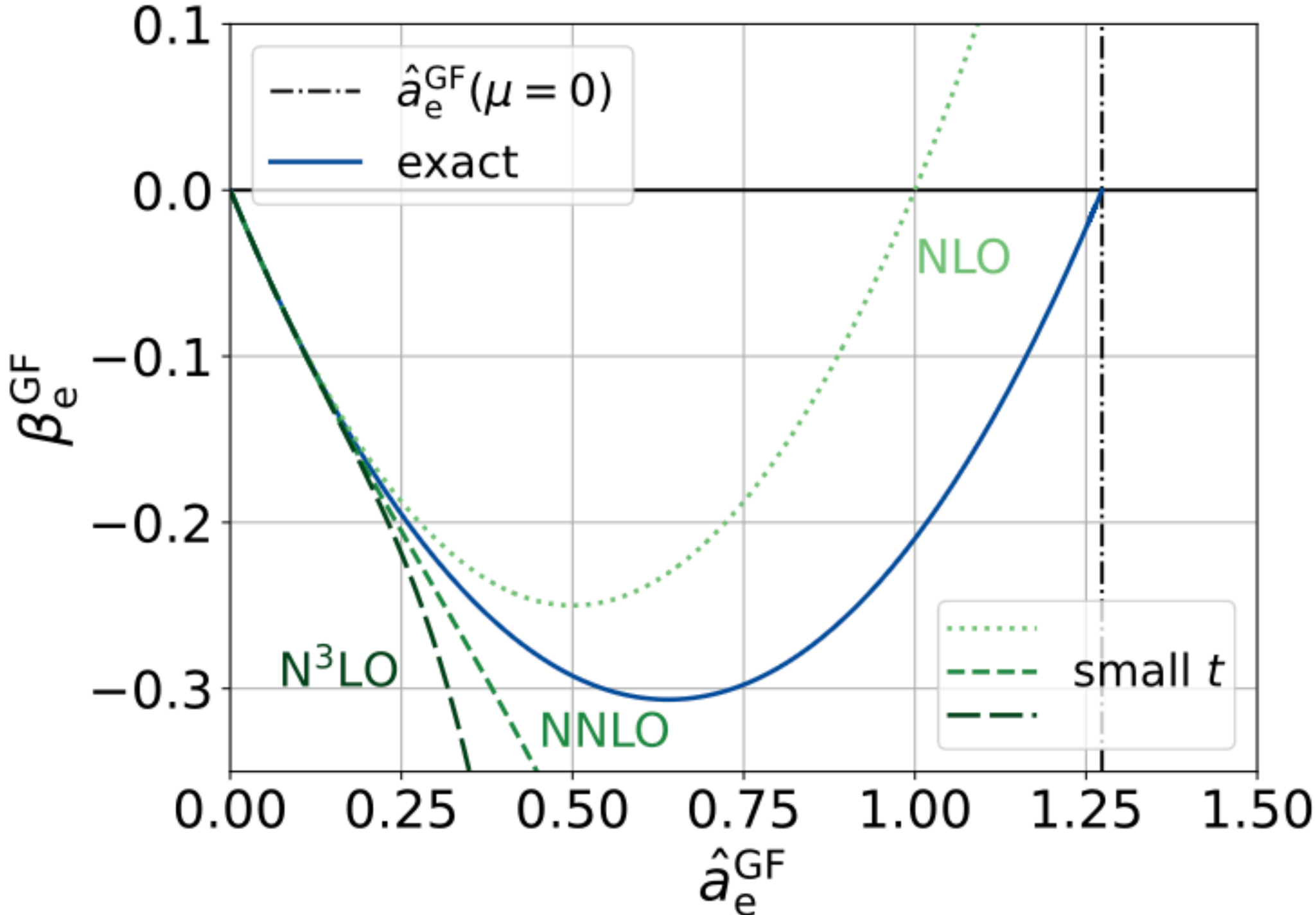
Application to QED₃



Georg, RH, Mason 2025

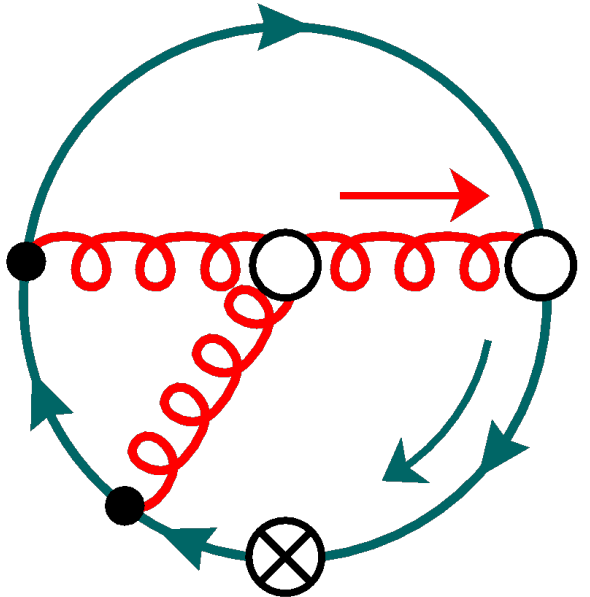
Application to QED₃

Lars Georg (Thu 16:30)



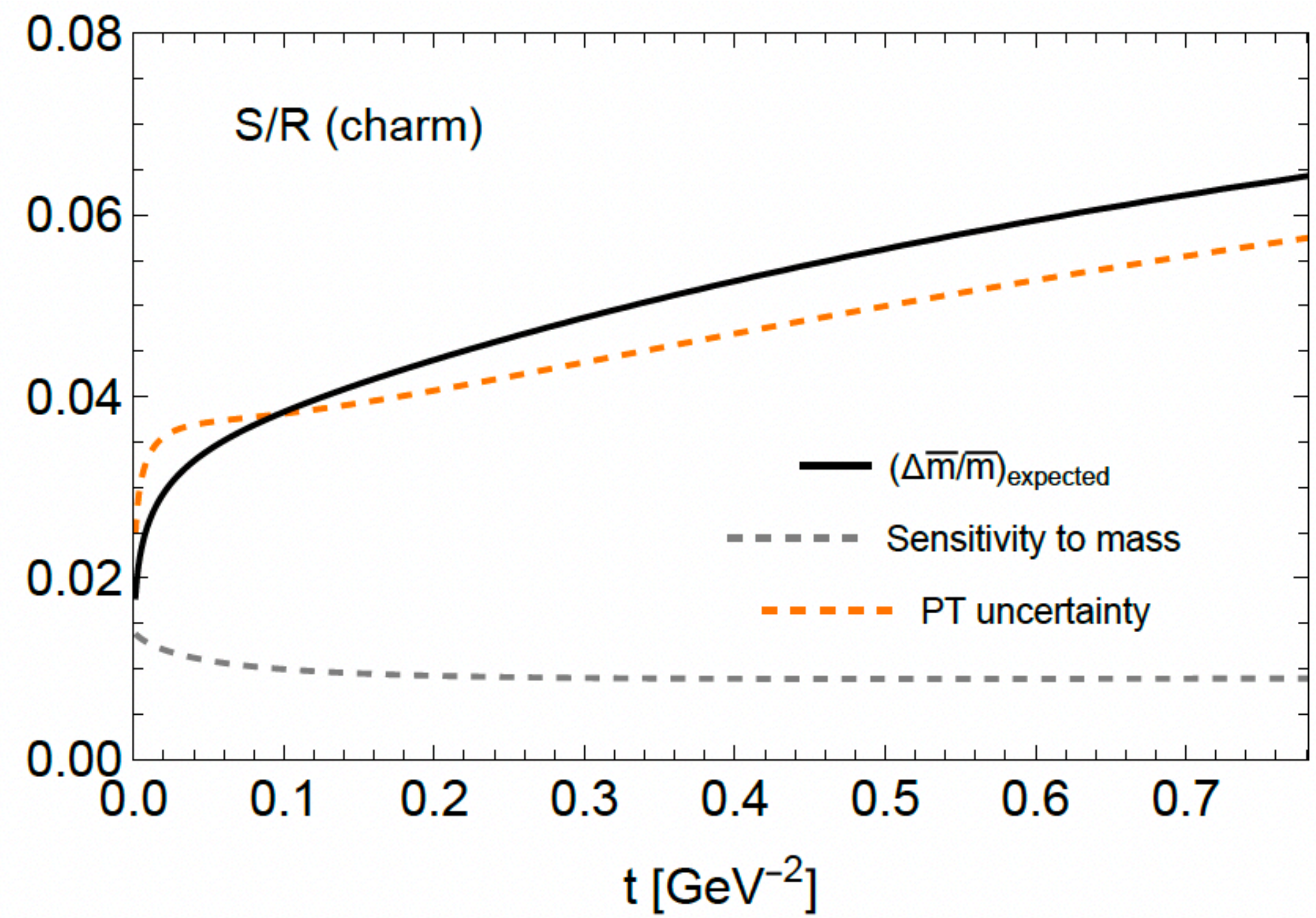
Georg, RH, Mason 2025

Quark masses



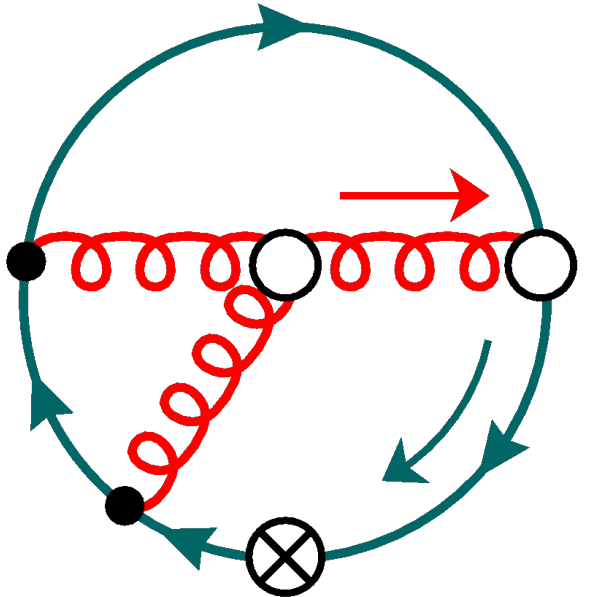
$$\equiv \langle \bar{\chi}(t)\chi(t) \rangle = -\frac{3}{8\pi^2 t} m(\mu) + \text{higher orders}$$

Artz, RH, Lange, Neumann, Prausa '19
Georg, RH, Mason '26



Takaura, RH, Lange '25

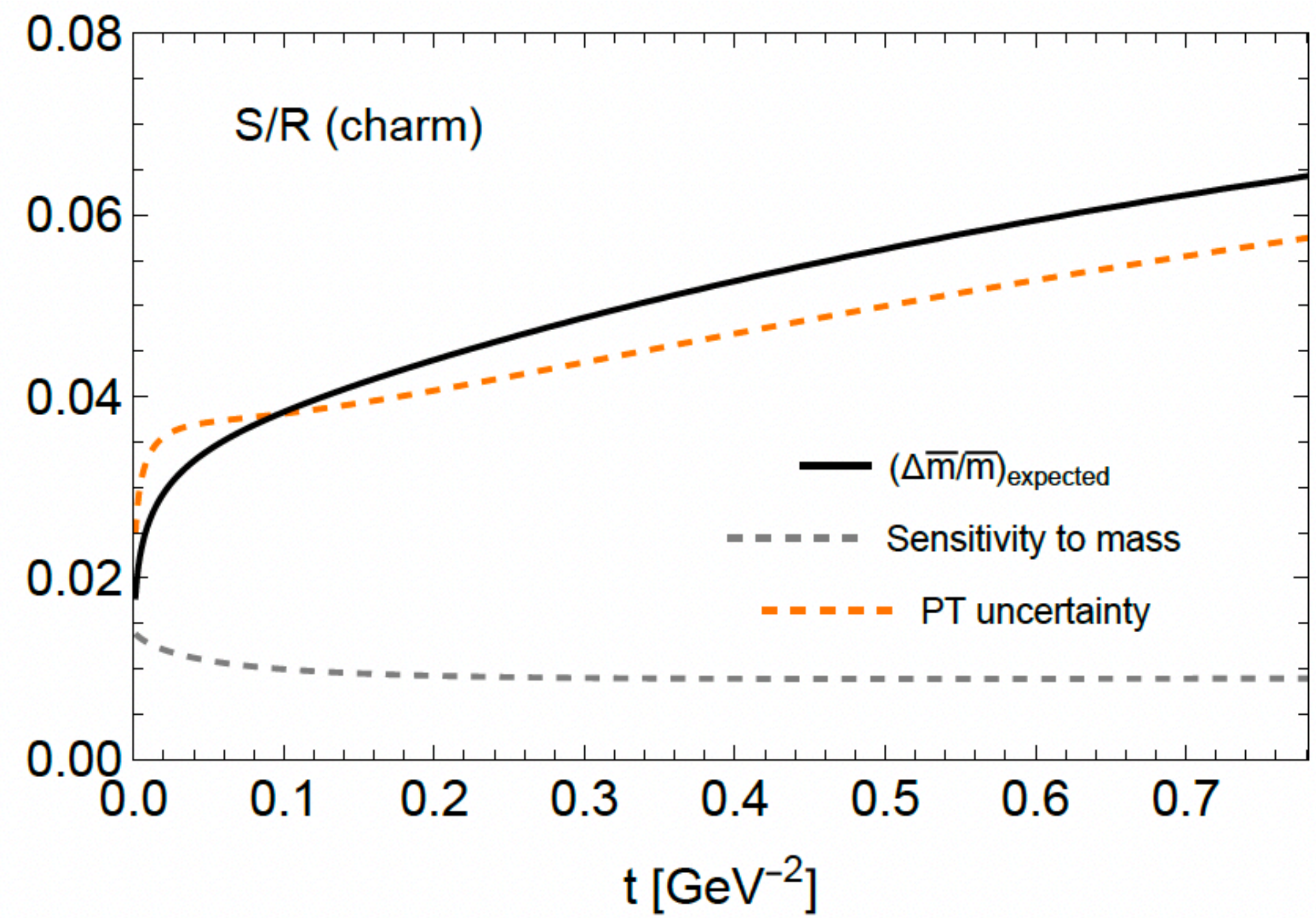
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+ higher orders

Artz, RH, Lange, Neumann, Prausa '19
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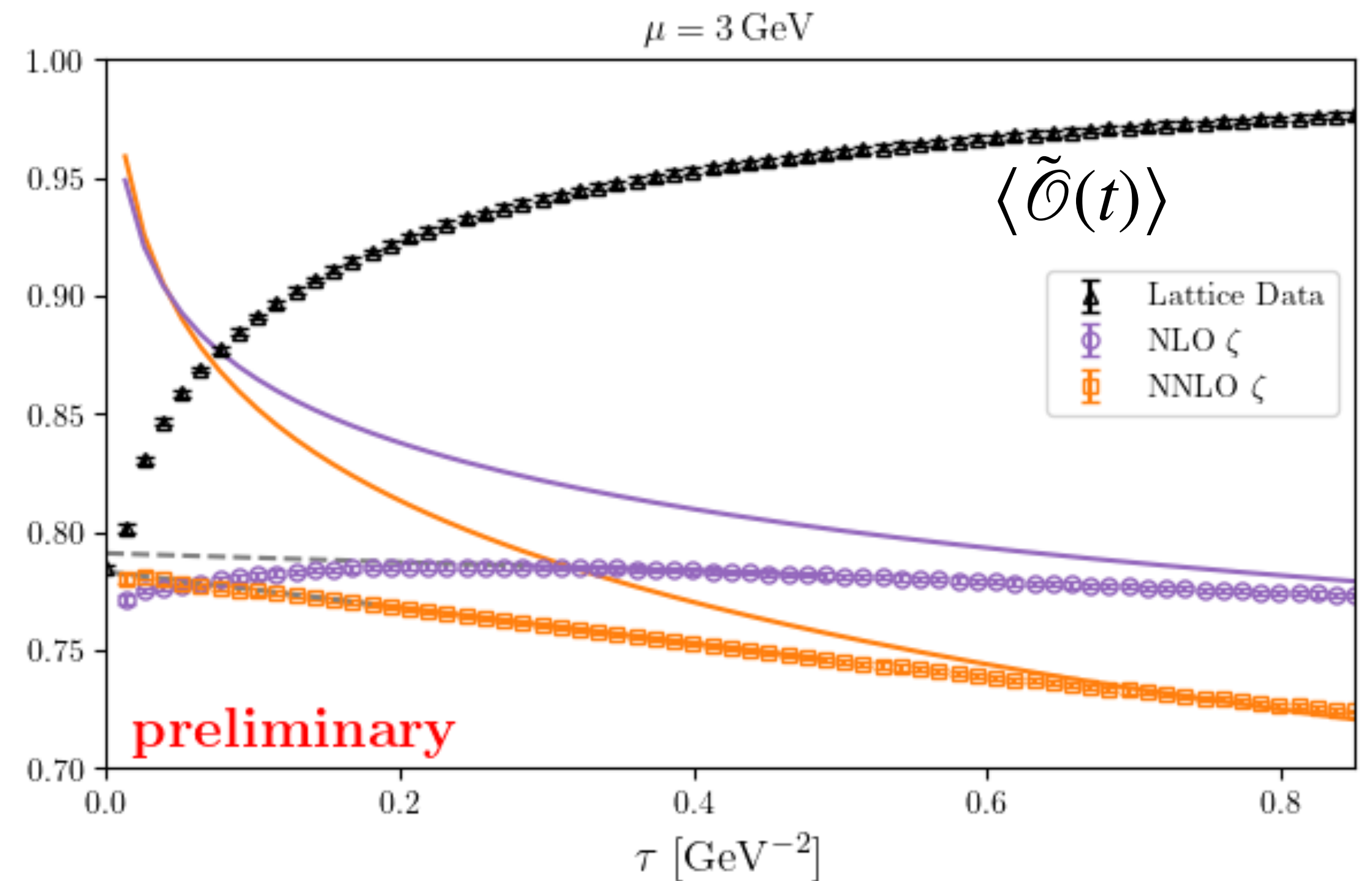
Fabian Lange (Tue 11:00)
Robert Mason (Wed 11:00)

Takaura, RH, Lange '25

The gradient flow scheme

From Zürich 2025 Workshop

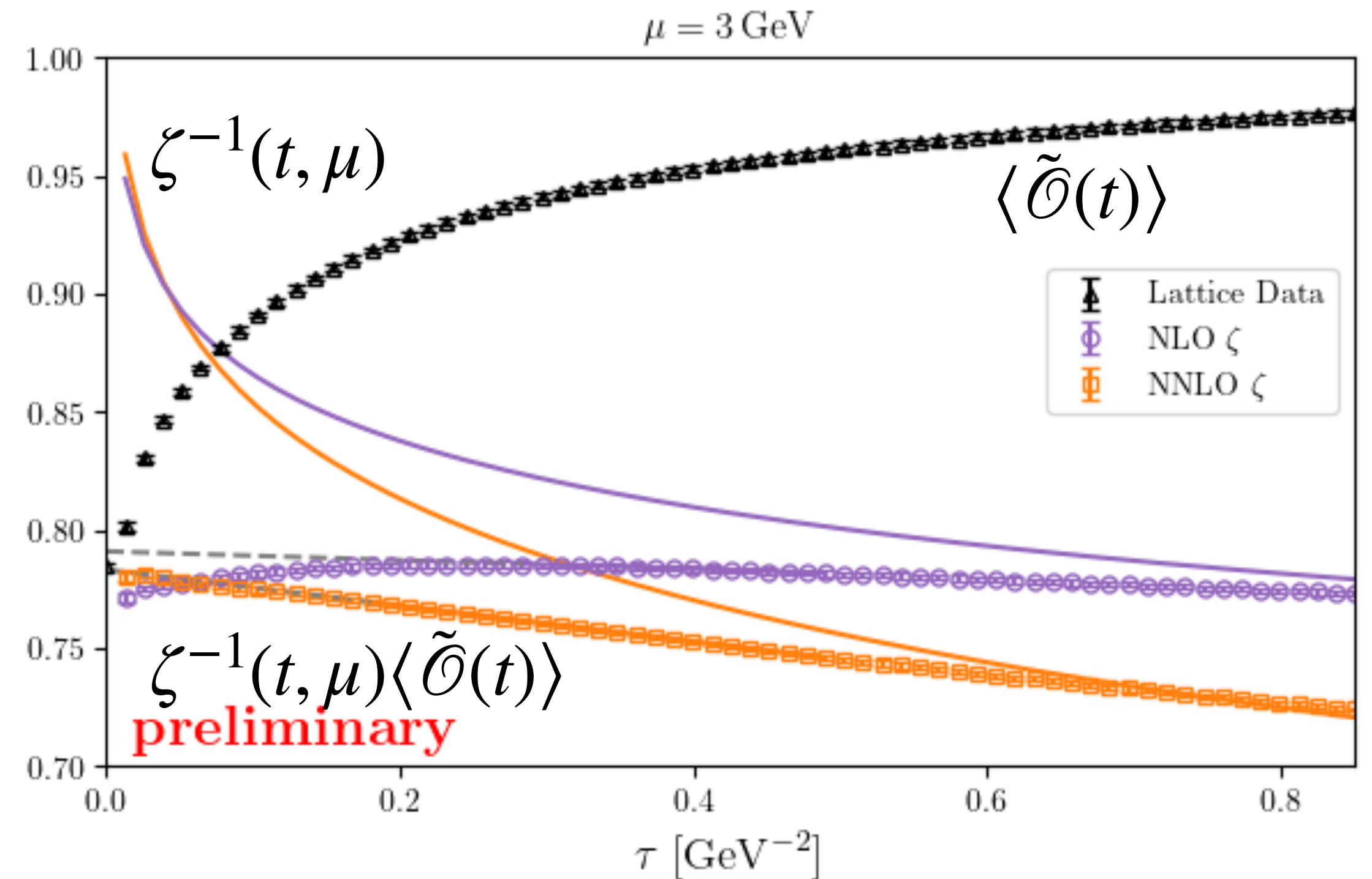
$$\begin{aligned}\Delta\Gamma &\sim \sum_n C_n^{\text{R}}(\mu) \langle \mathcal{O}_n^{\text{R}} \rangle(\mu) \\ &= \sum_n C_n^{\text{R}}(\mu) \left[\zeta^{-1}(t, \mu) \langle \tilde{\mathcal{O}}_n \rangle(t) \right]\end{aligned}$$



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From Zürich 2025 Workshop

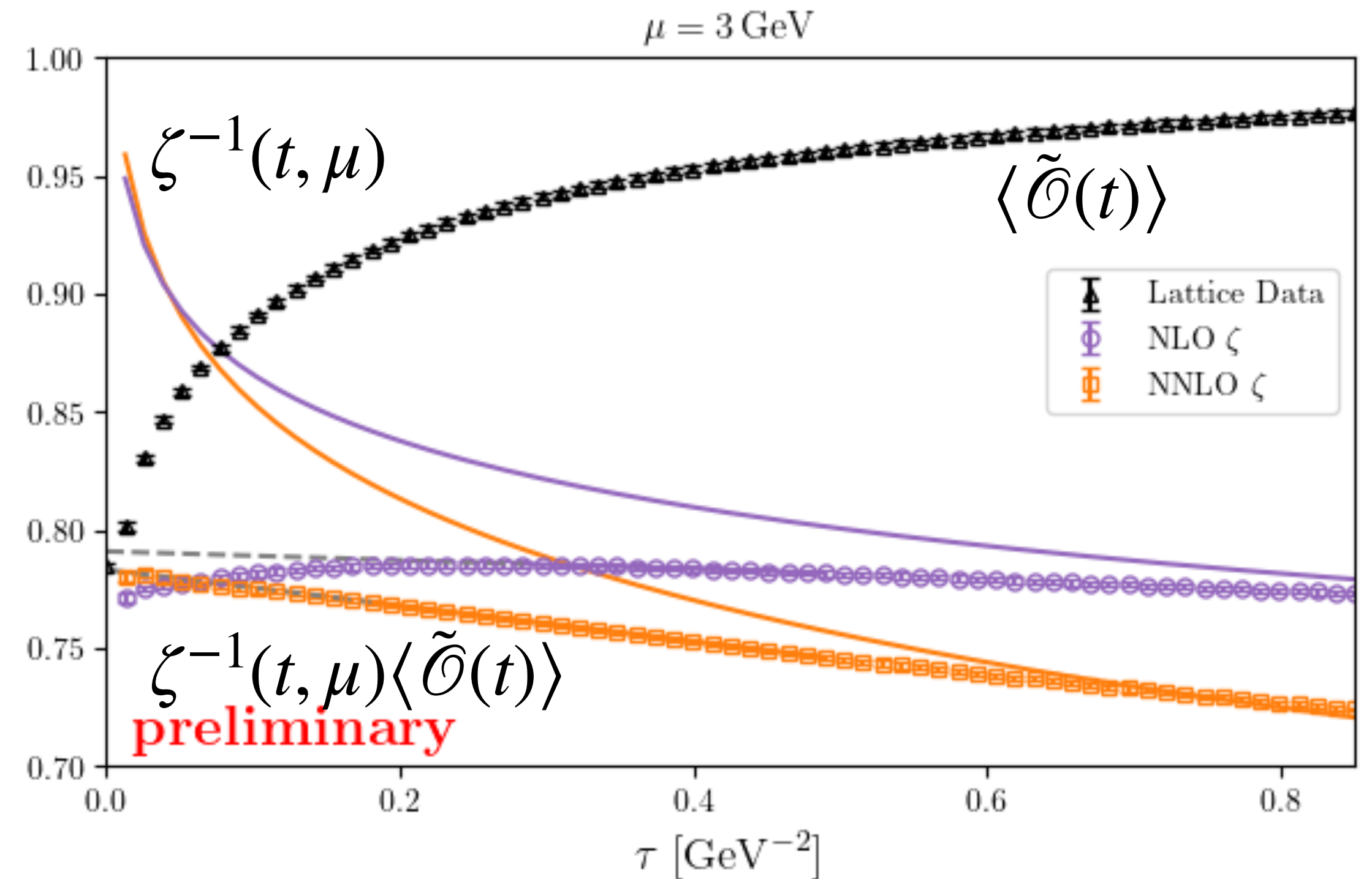
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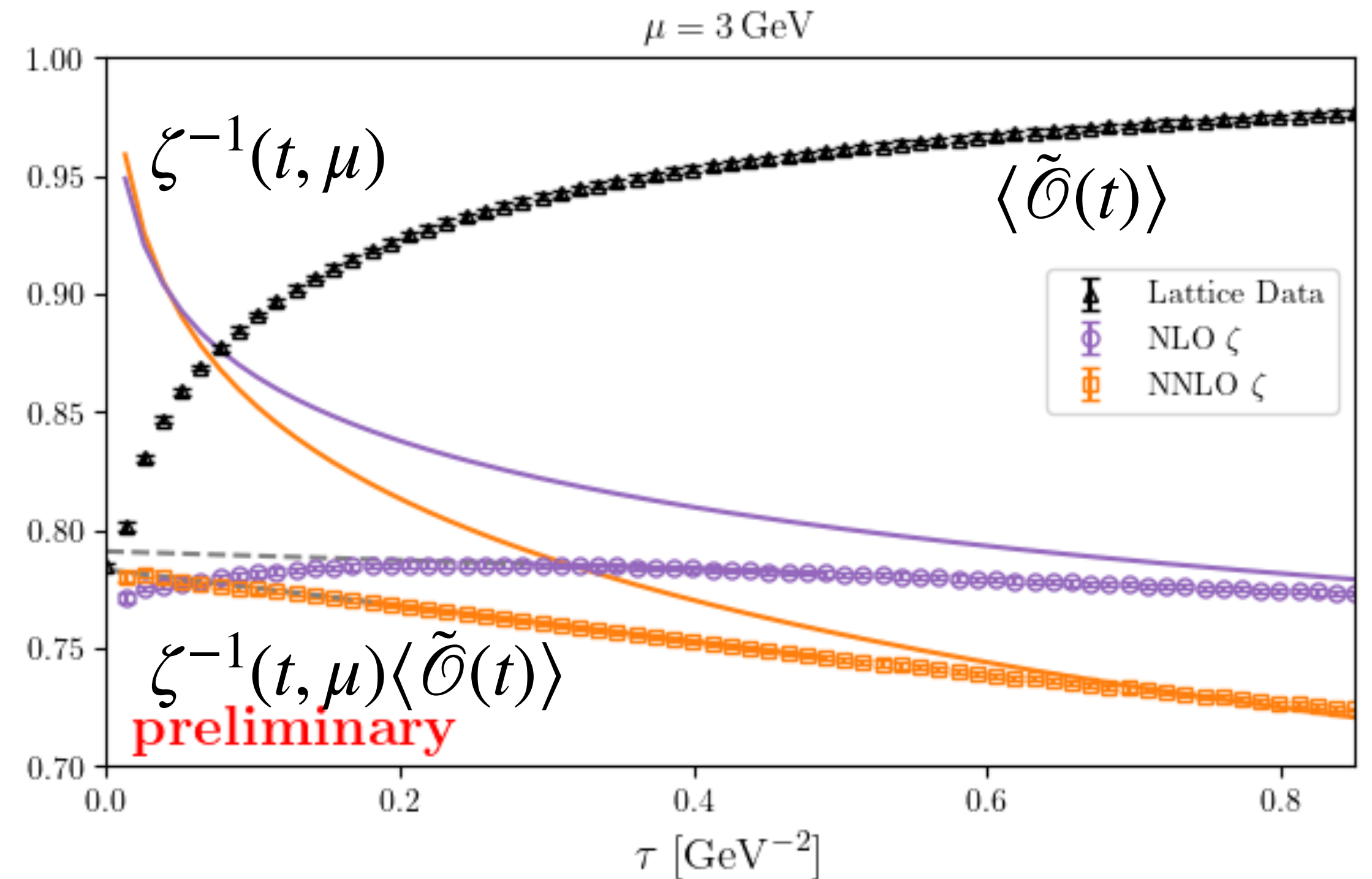
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The gradient flow scheme

From Zürich 2025 Workshop

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 &= \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n \rangle(t)
 \end{aligned}$$



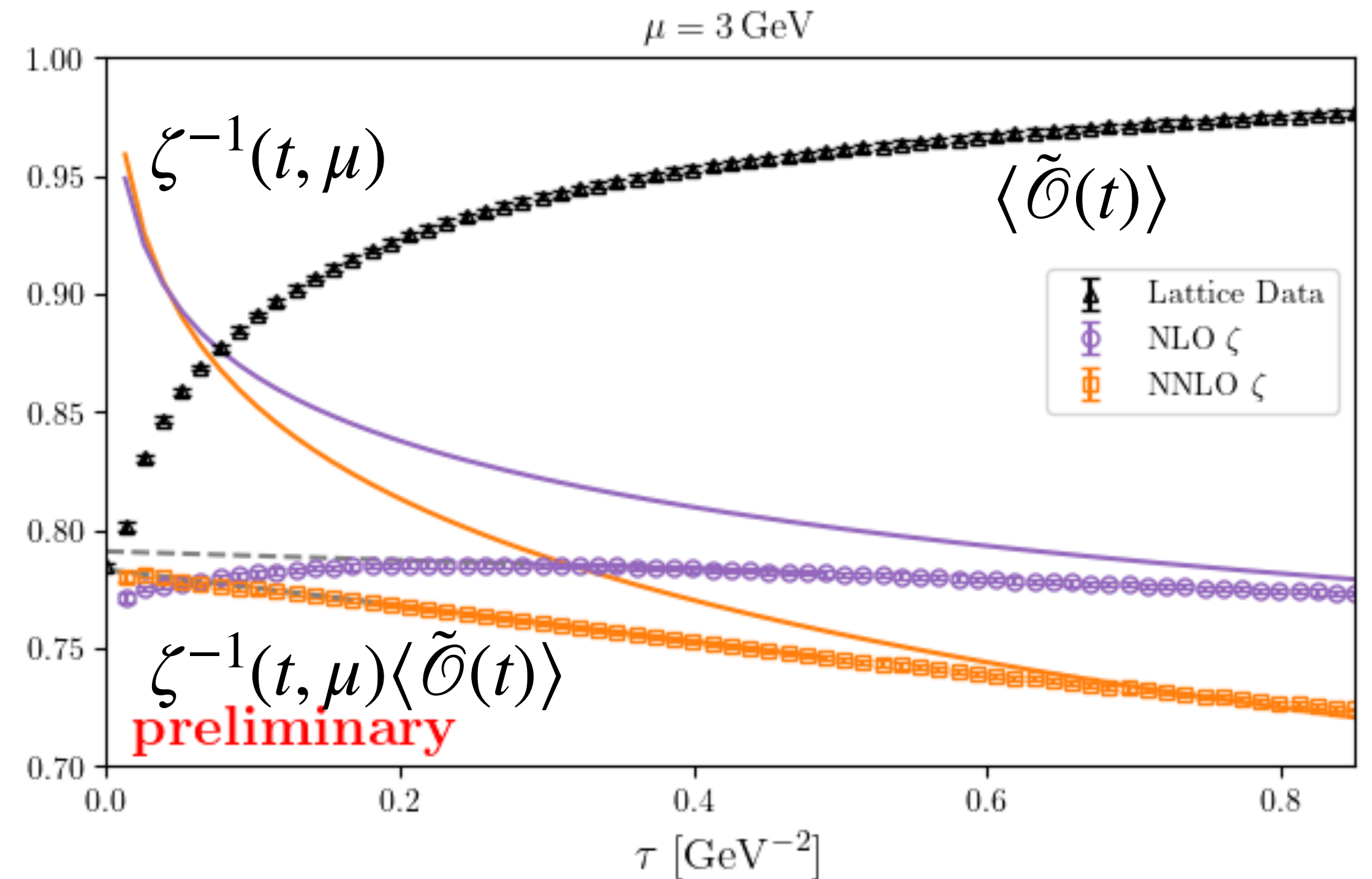
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 &= \sum_n \left[C_n^{\text{R}}(\mu) \zeta^{-1}(t, \mu) \right] \langle \tilde{\mathcal{O}}_n \rangle(t) \\
 &= \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n \rangle(t)
 \end{aligned}$$

[...] Hence, no conversion to the $\overline{\text{MS}}$ scheme is needed any more, the advantage being that the renormalization scheme employed is well-defined beyond perturbation theory.

Ammer, Dürr '24



The GF scheme

$$\mathcal{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF

$\overline{\text{MS}}$

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$$\mathcal{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

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$\overline{\text{MS}}$

$$= \sum_n (C \zeta^{-1}(t))_n \langle \zeta(t) \mathcal{O} \rangle_n$$

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$$= \sum_n C_n^{\text{R}}(\mu) \langle \mathcal{O}_n^{\text{R}} \rangle(\mu)$$

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$$t \frac{d}{dt} \tilde{C}(t) = \tilde{C}(t) \tilde{\gamma}$$

$$\mu \frac{d}{d\mu} C^{\text{R}}(\mu) = C^{\text{R}}(\mu) \gamma$$

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$$\tilde{\gamma} = -t \frac{d}{dt} \ln \zeta(t)$$

$$\gamma_{nm} = -\mu \frac{d}{d\mu} \ln Z$$

The GF scheme

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$\overline{\text{MS}}$

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RH, Lange, Neumann '20

Borgulat, Felten, RH, Kohlen '25

Left out

SFTX applied to

LEFT (QCD x QED)

[Crosas, Stoffer 2026](#)

scalar QCD

[Borgulat, Felten, RH, Kohlen 2025](#)

Standard Model

[Borgulat \(Diss\), Borgulat et al. \(in prep\)](#)

...

Left out

SFTX applied to

LEFT (QCD x QED)

Crosas, Stoffer 2026

Oscar Crosas (Wed 11:30)

scalar QCD

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Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Conclusions

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Lattice/PT interplay picks up momentum:

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Lattice/PT interplay picks up momentum:
first lattice **PDF moments** beyond $\langle x^3 \rangle$

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Full potential not yet fully explored (I think...)

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quark masses, strong coupling, ...

Full potential not yet fully explored (I think...)

Many things not even discussed:

Static QCD potential, EDMs, ...