The perturbative gradient flow and its applications

Robert Harlander

RWTH Aachen University

04 June 2024

Hadronic physics and heavy quarks on the lattice

Trinity College Dublin (4-7 June 2024)

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TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Motivation

NEWS**FOCUS**

Physicists' Nightmare Scenario: The Higgs and Nothing Else

Many fear the LHC will cough up only the one particle they've sought for decades. Some would rather see nothing new at all

Suppose you are a particle physicist. A score of nations has given you several billion Swiss francs to build a machine that will probe the origins of mass, that ineffable something that keeps an object in steady motion unless shoved by a force. Your proposed explanation of mass requires a new particle, cryptically dubbed the Higgs boson, that your machine aims to espy.

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If, on the other hand, the LHC sees no new particles at all, then the very rules of quantum mechanics and even Einstein's special theory of relativity must be wrong. "It

The particular challenge is to give mass to particles called the W and Z bosons, which convey the weak nuclear force. According to the standard model, the weak force that causes a type of radioactive decay and the electromagnetic force that powers lightning and laptop computers are two facets of the same single thing. The two forces aren't precisely interchangeable: Electromagnetic forces can stretch between the stars, whereas the weak force doesn't even reach across the atomic nucleus. That range difference arises because photons, the quantum particles that make up an electromagnetic field, have no mass. In contrast, the particles that make up the weak force field, the W and Z bosons, are about 86 and 97 times as

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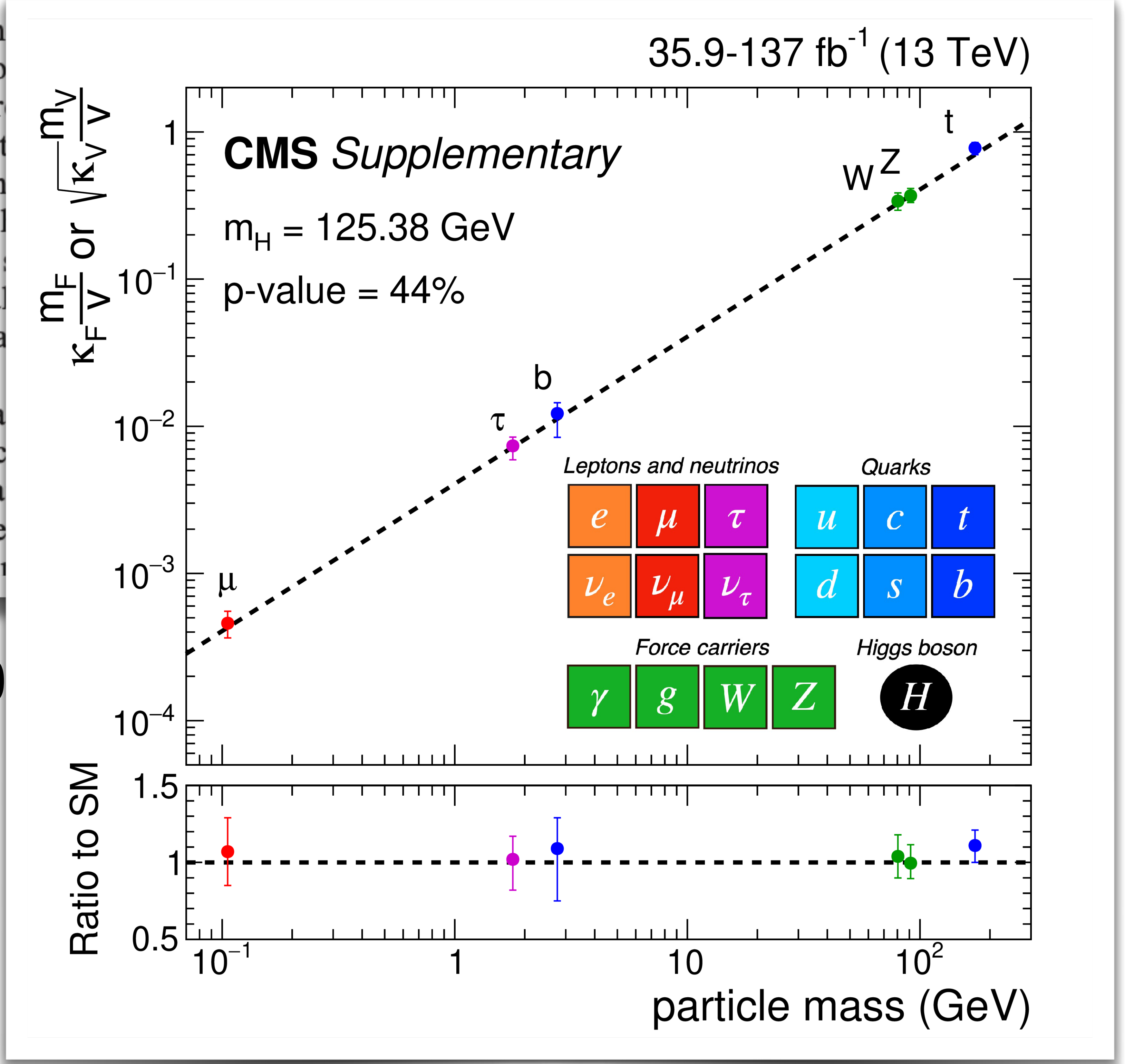

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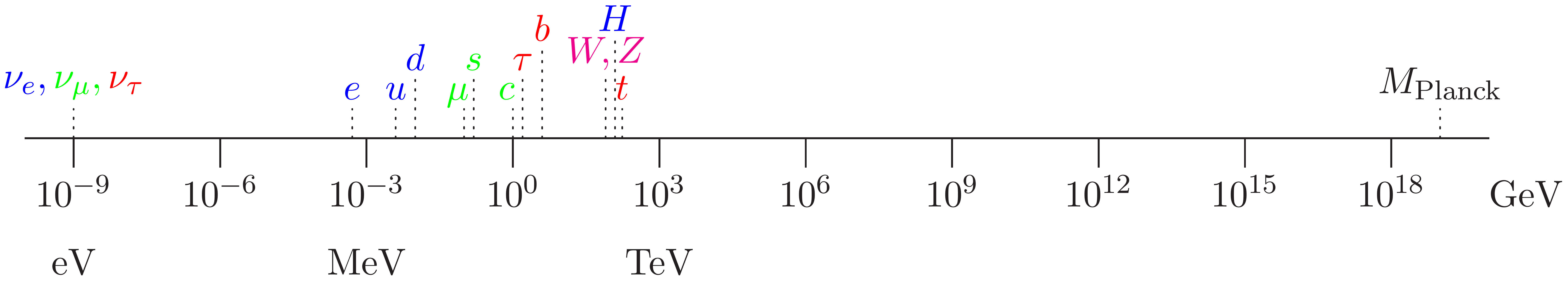

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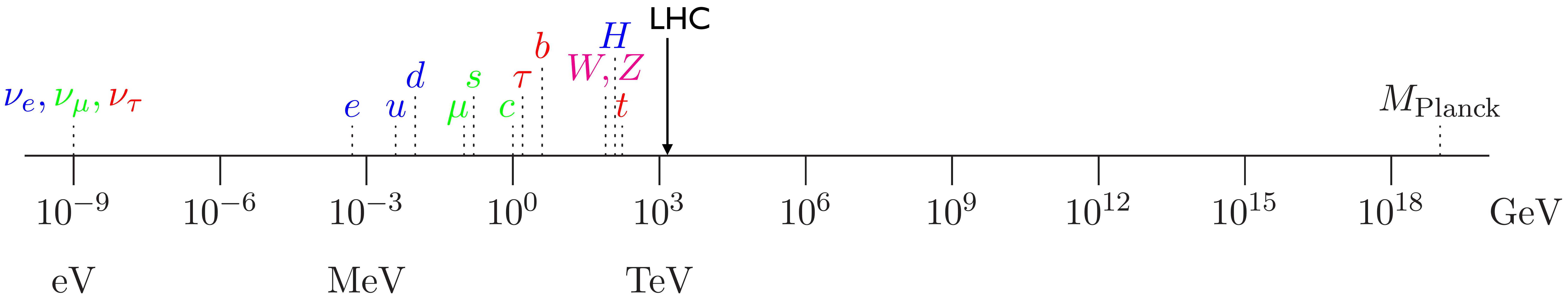
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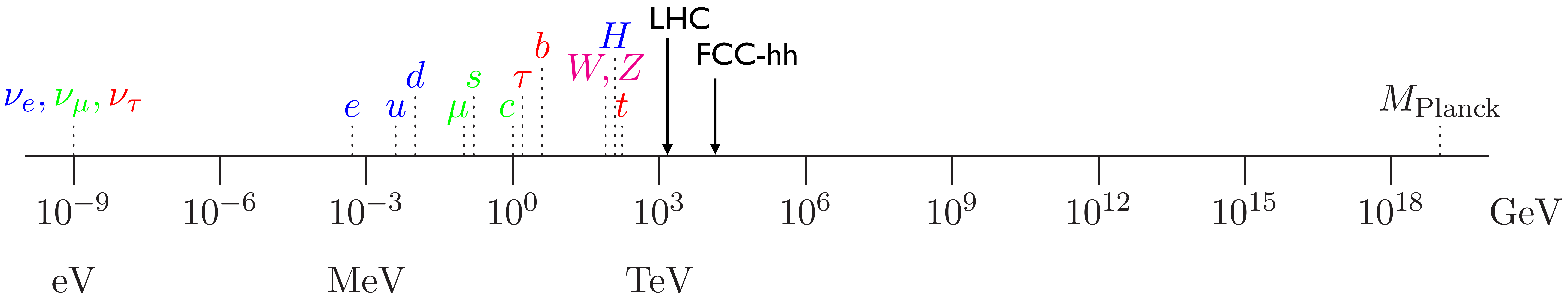

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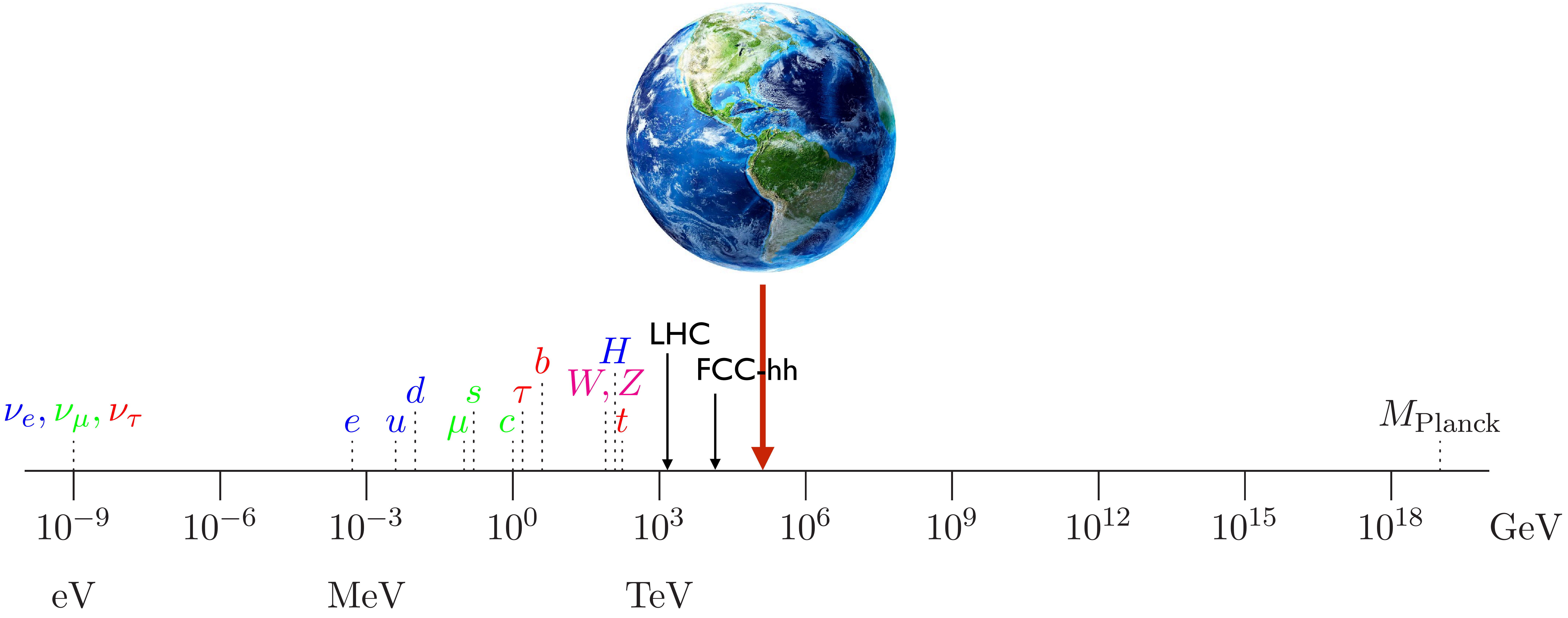

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Eur. Phys. J. H (2023)48:6
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THE EUROPEAN PHYSICAL JOURNAL H

Regular Article

The end of the particle era?

Robert Harlander[1], Jean-Philippe Martinez[1,a], and Gregor Schiemann[2]

[1] Institute for Theoretical Particle Physics and Cosmology, RWTH Aachen University, Aachen, Germany
[2] Faculty of Humanities and Cultural Studies, University of Wuppertal, Wuppertal, Germany

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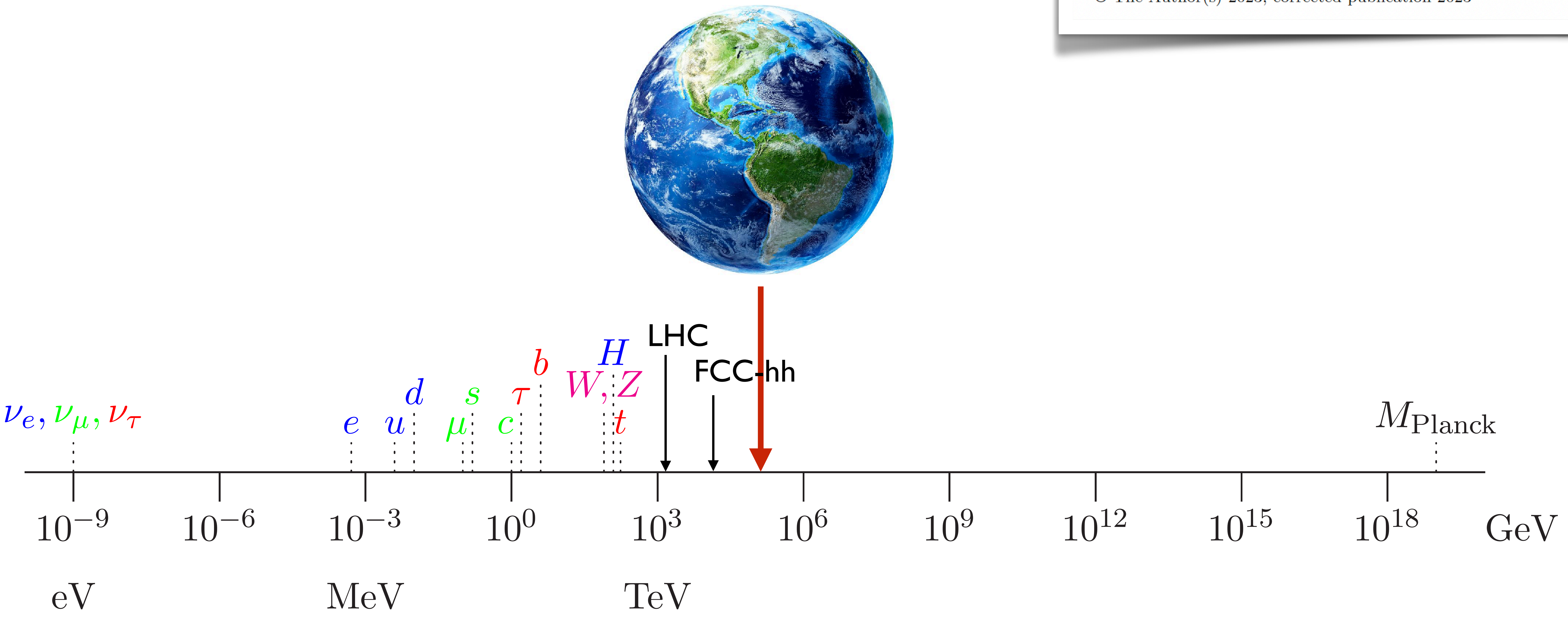

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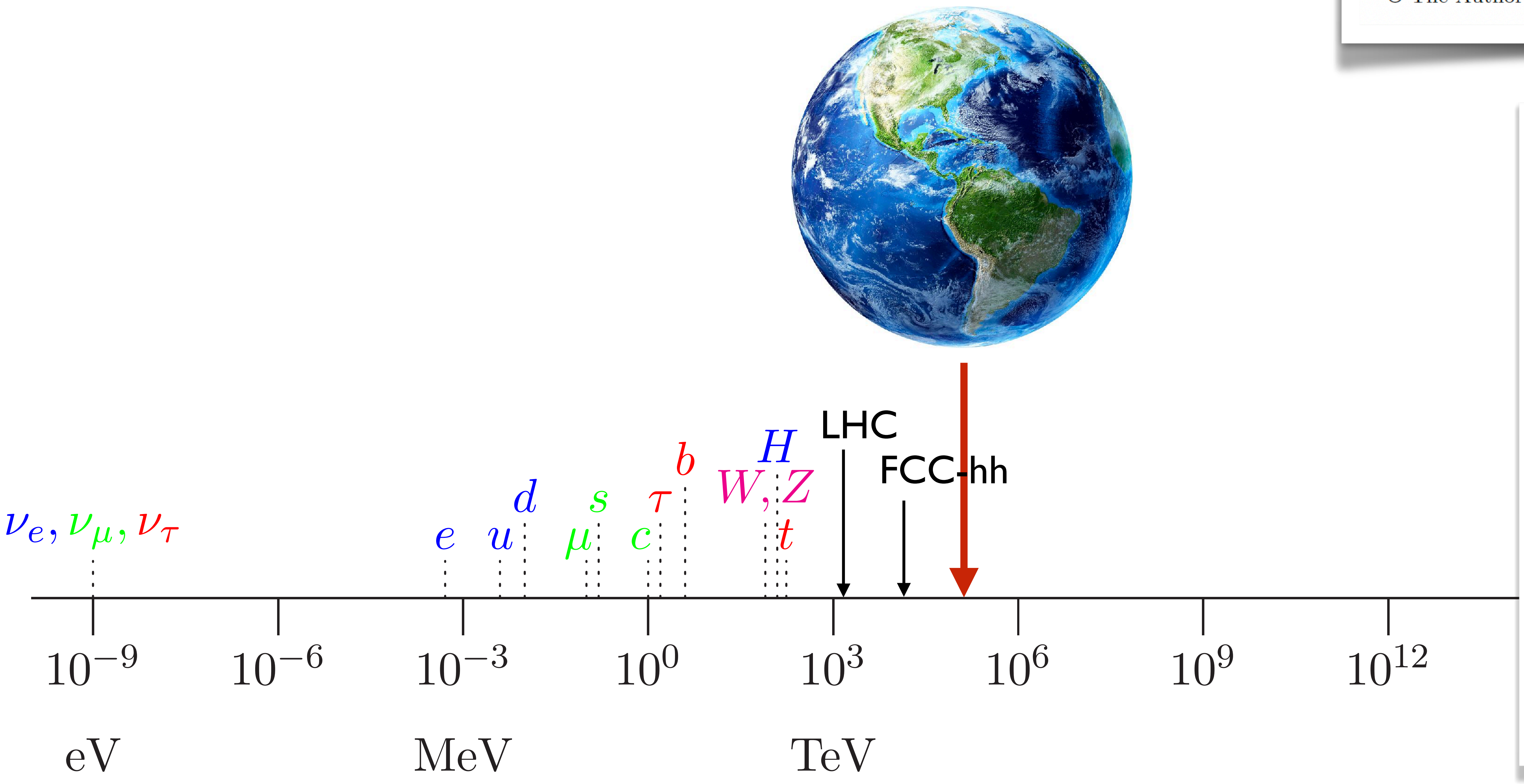

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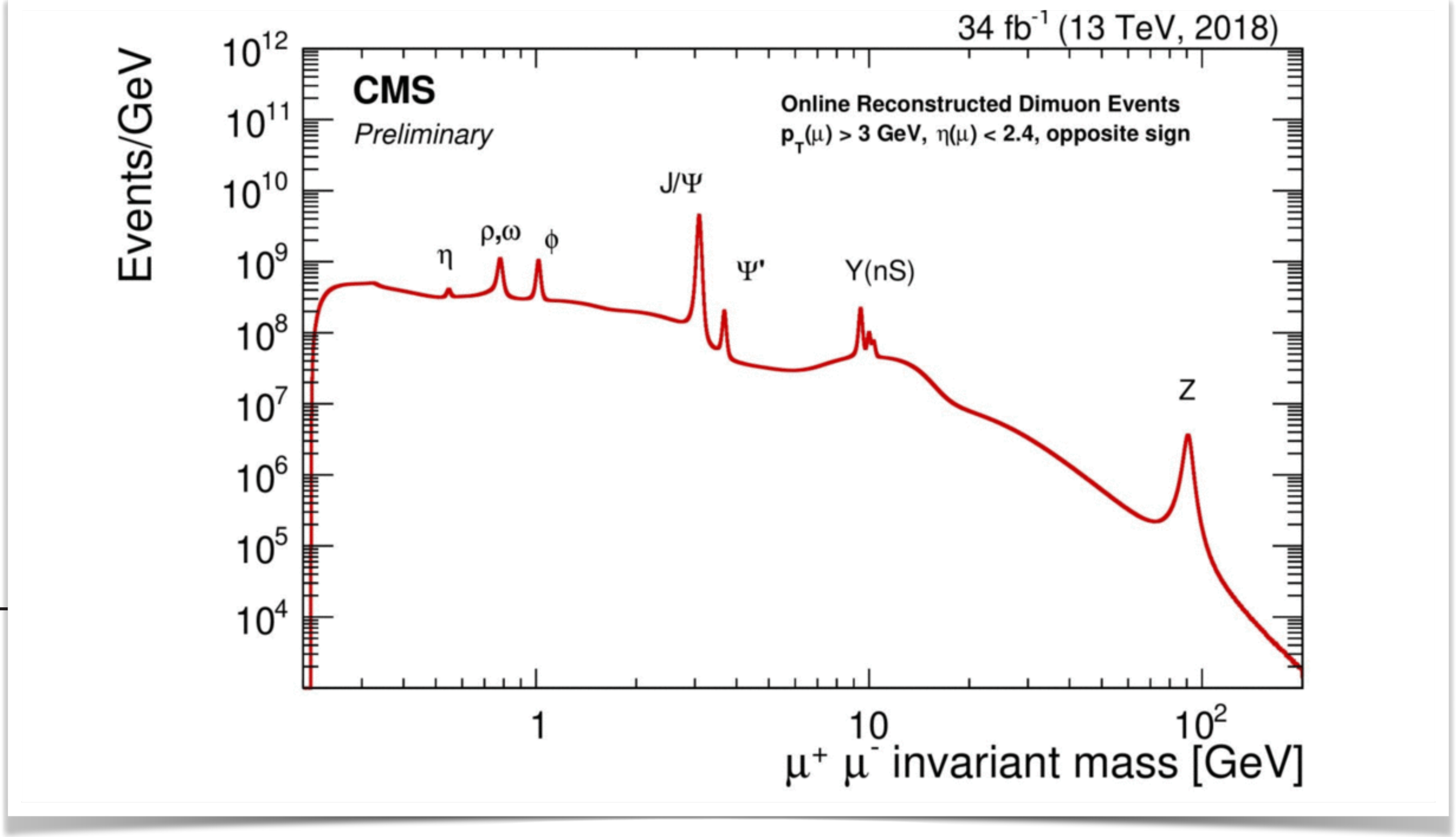

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- Other means of "discovery"
- Requires input from various directions

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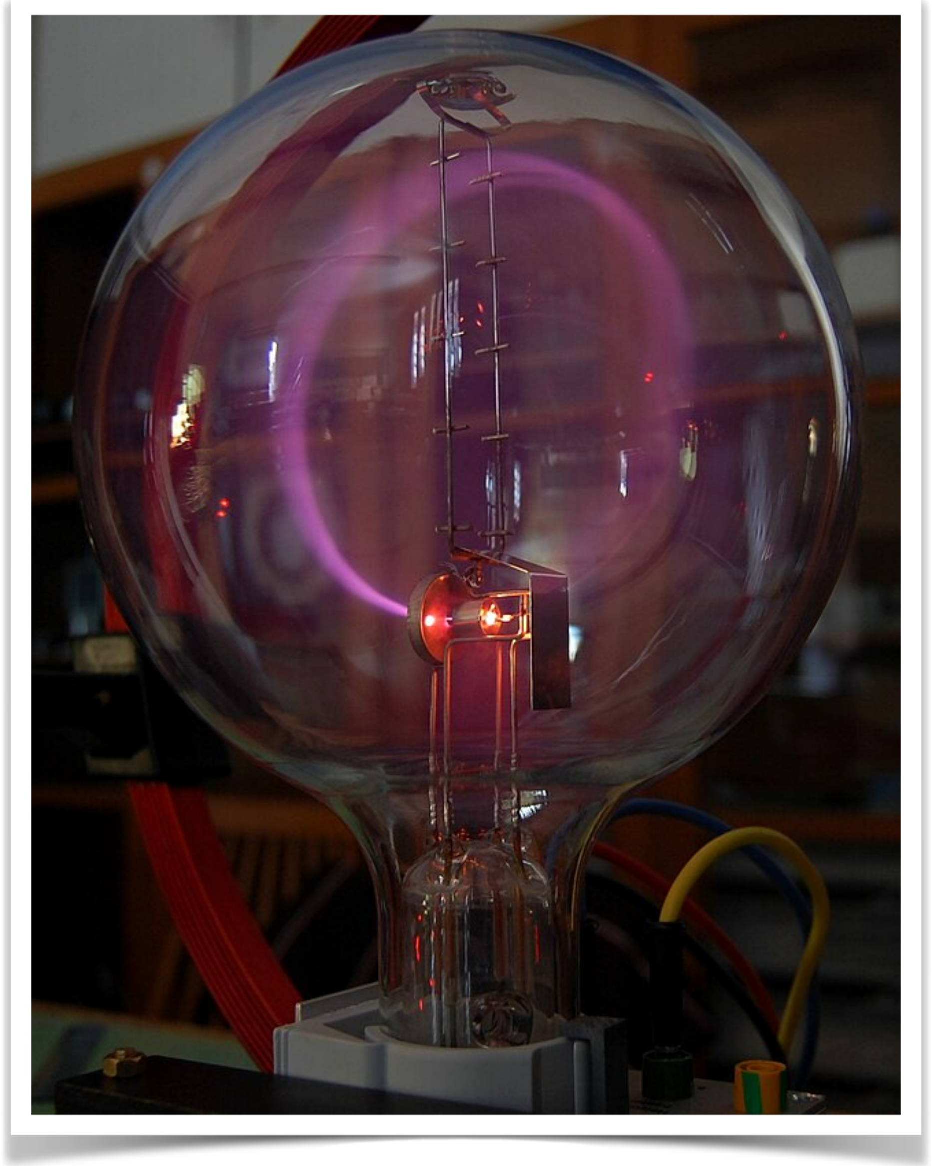
cf. J.J. Thompson, 1897:
I can see no escape from the conclusion that [cathode rays] are charges of negative electricity carried by particles of matter



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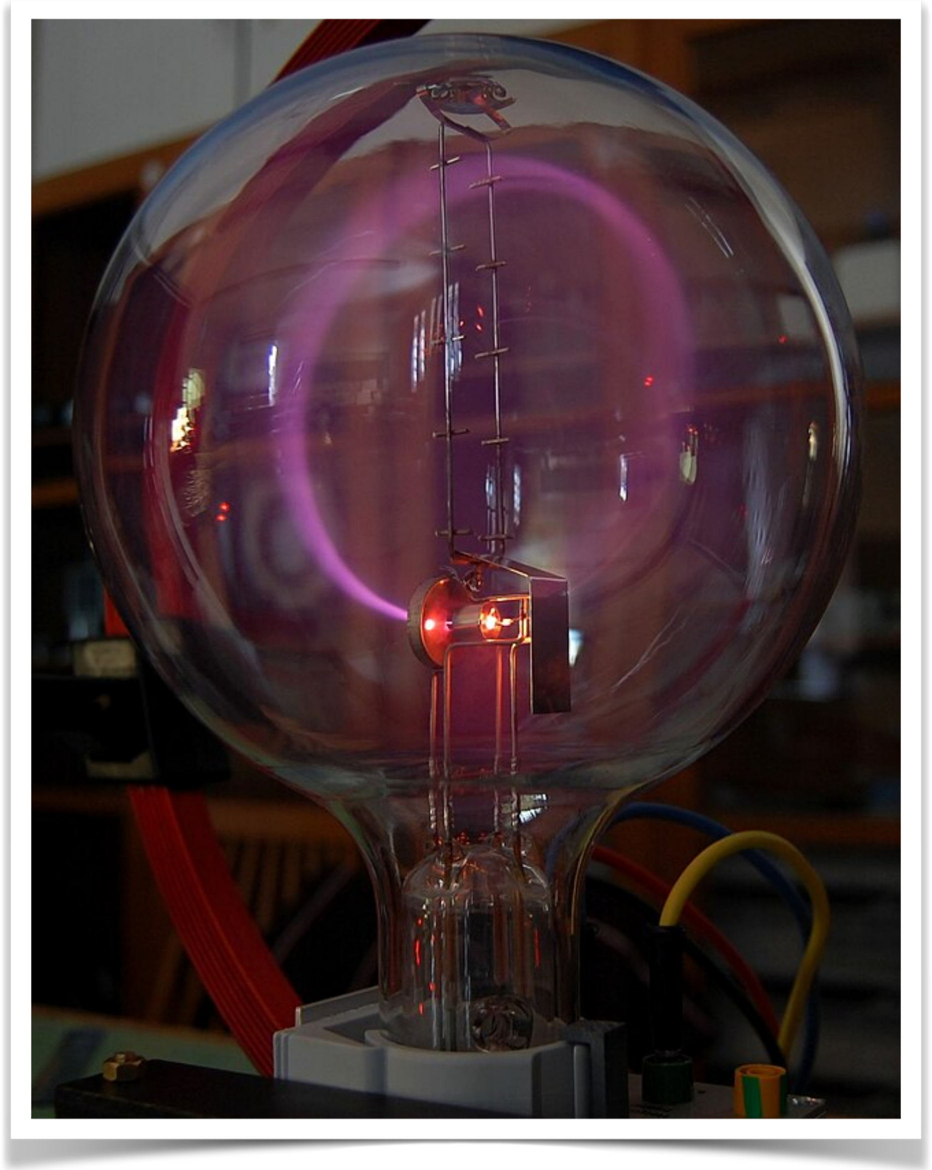

- Low- and high-energy experiments
- perturbative and non-perturbative physics

The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t}B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t}\chi(t,x) = \mathscr{D}^2\chi(t,x)$$
$$\chi(t=0,x) = \psi(x)$$

Lüscher 2010
Lüscher, Weisz 2011
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see also: Narayanan, Neuberger 2006, Lüscher 2010

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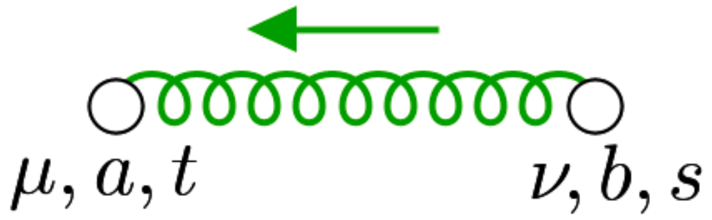
$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$
$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \mathscr{D}^2\right) \chi + \mathrm{h.c.}$$

Perturbative approach

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T B^a_\mu(t,x)\, B^b_\nu(s,0) | 0 \rangle$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

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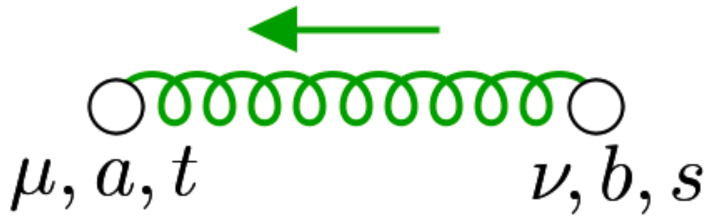

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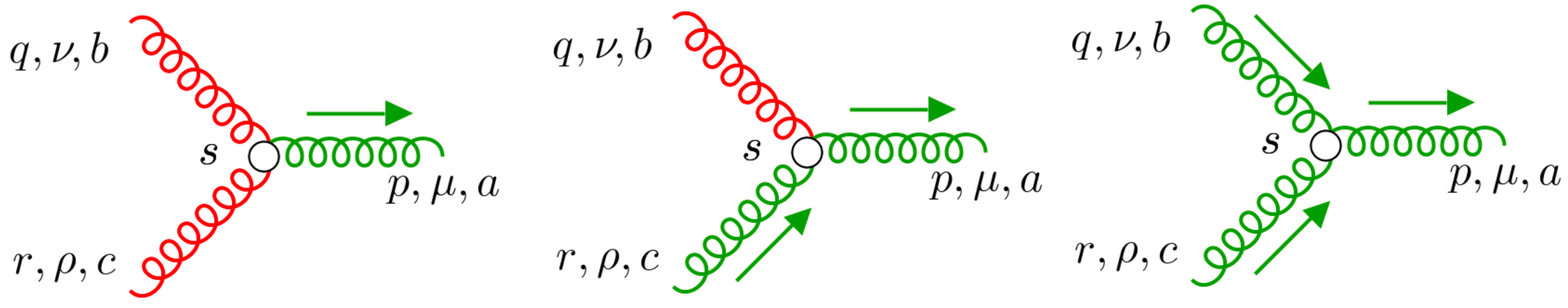
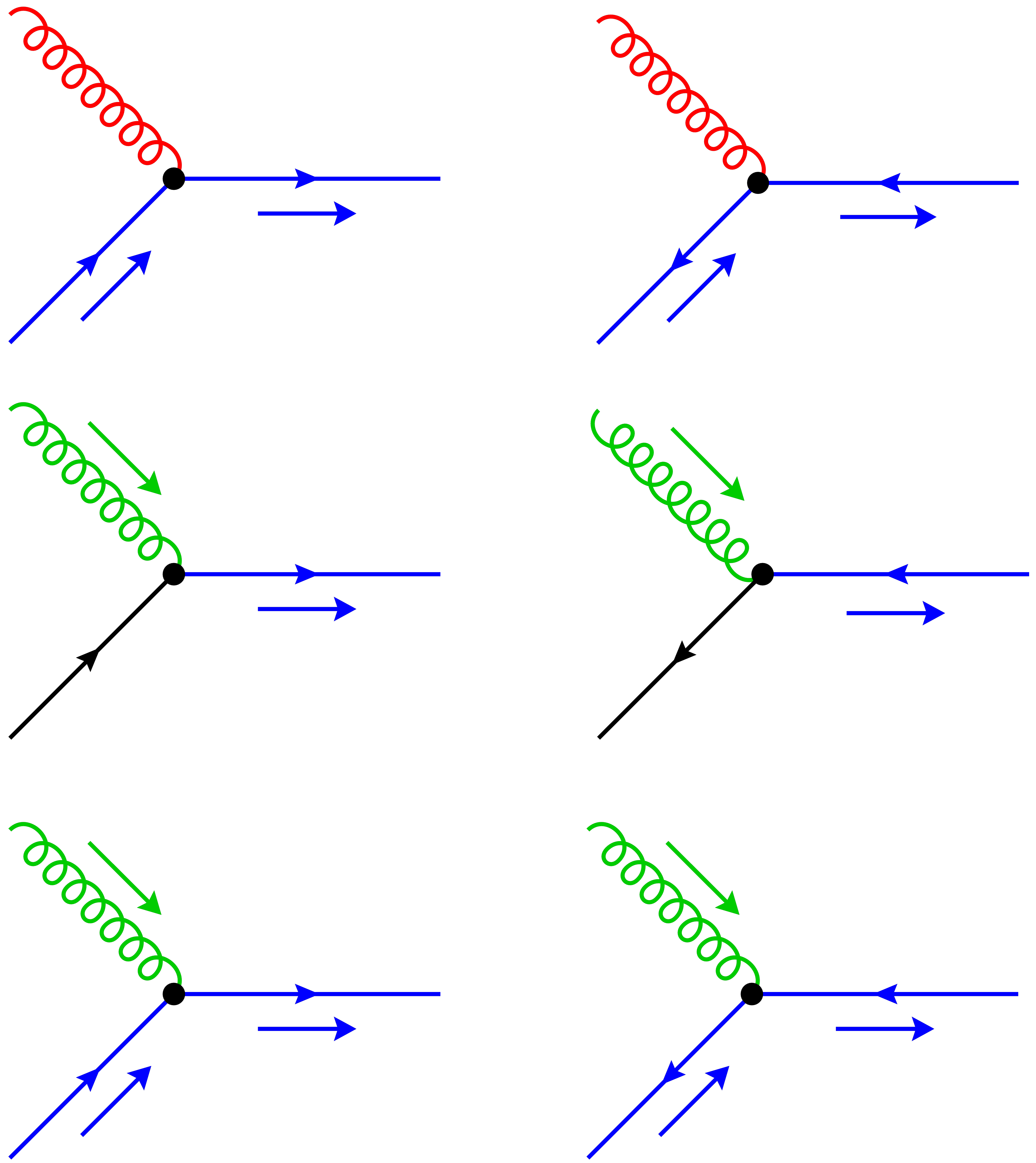
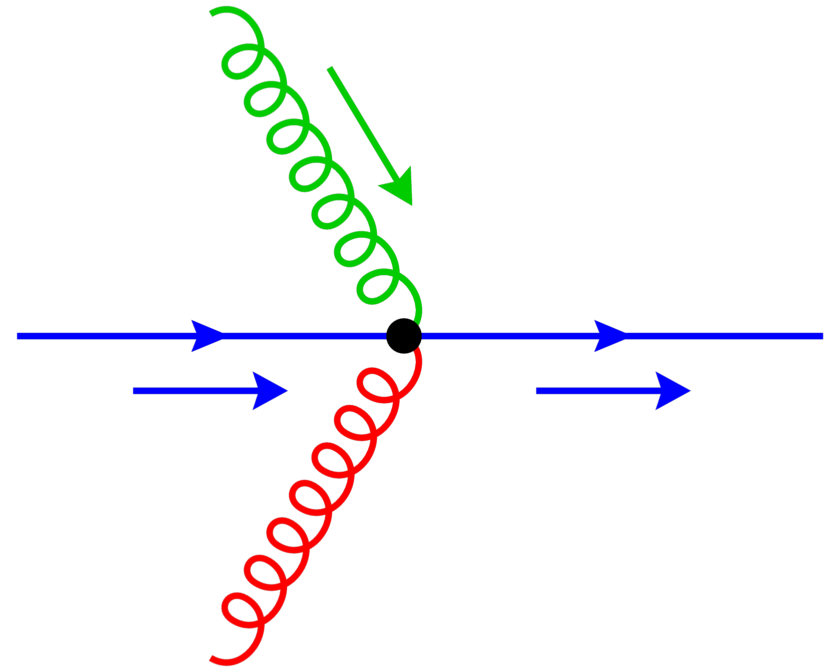


$$-\mathrm{i}g f^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

+ 4-gluon vertex

Perturbative approach





etc.

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⇒ renormalization unaffected!

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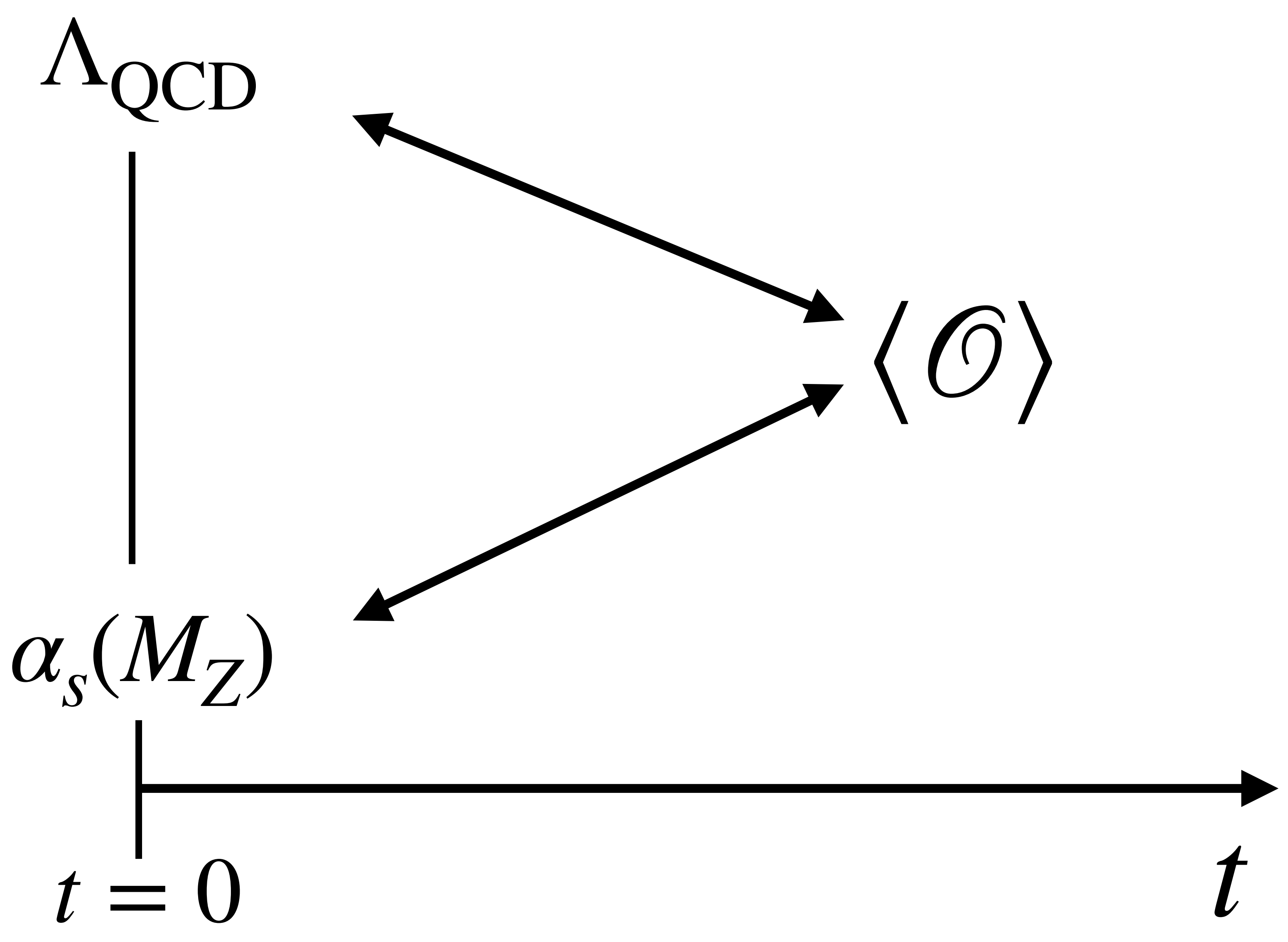

Let's calculate

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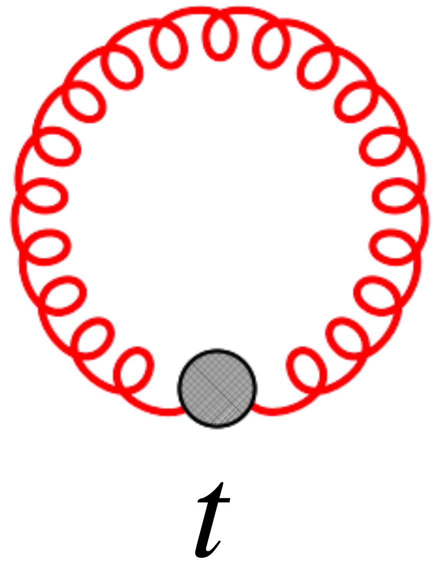

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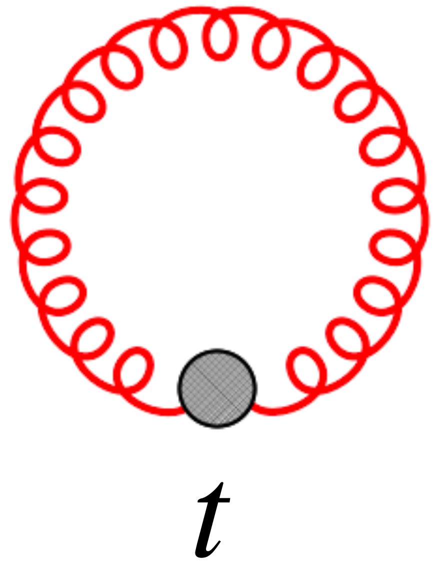

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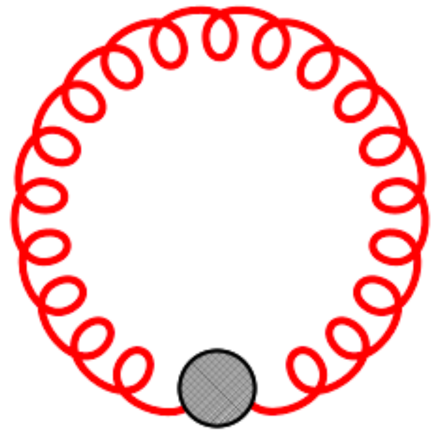

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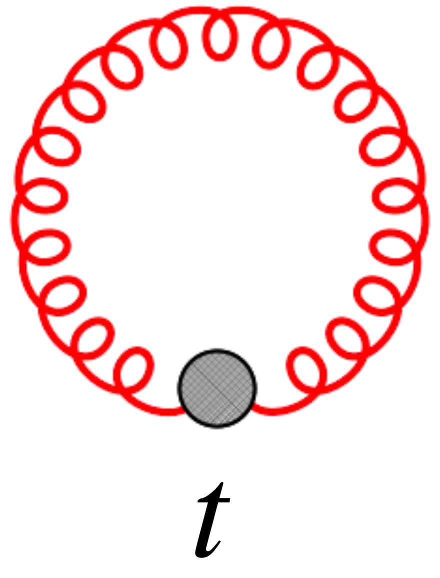

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explicitly:

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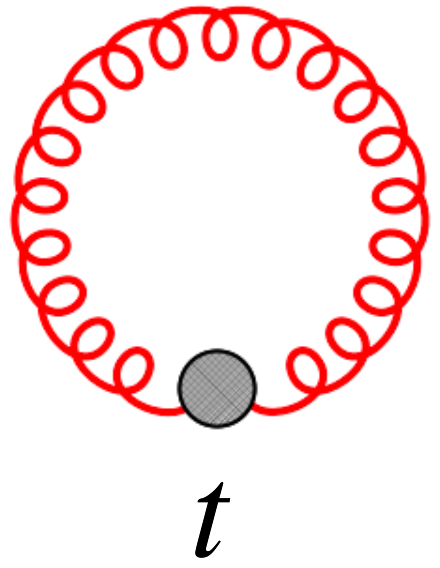

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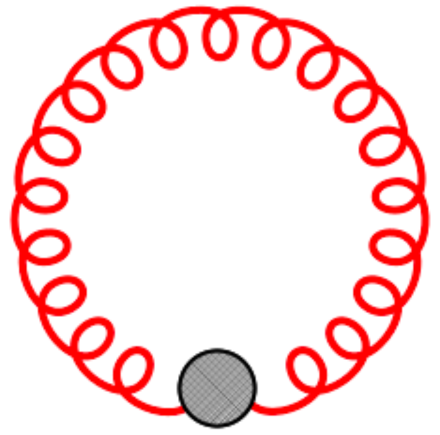
LO: 
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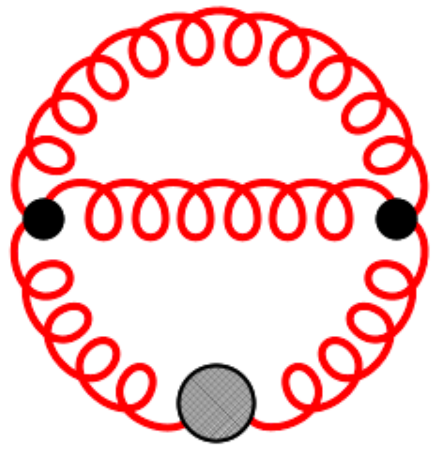
→ measure α_s on the lattice?

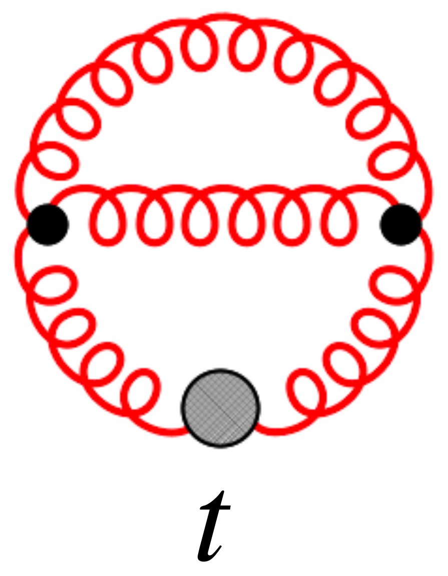
$$\alpha_s = \alpha_s(\mu)$$

Higher orders



$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

Higher orders



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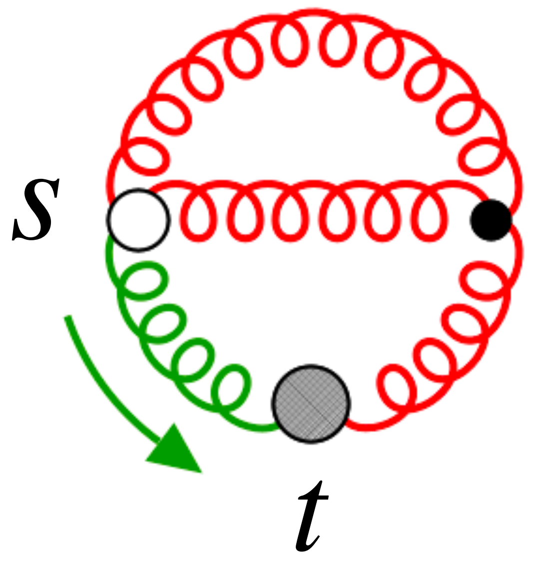

$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

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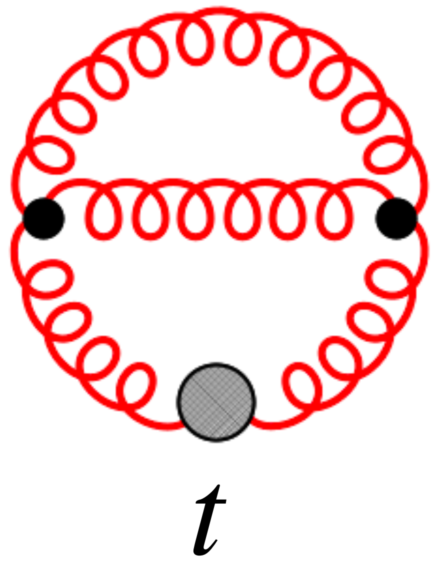

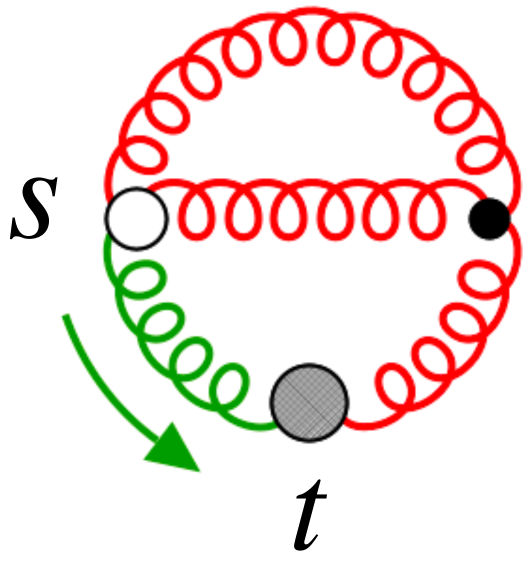

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- generalized loop integrals

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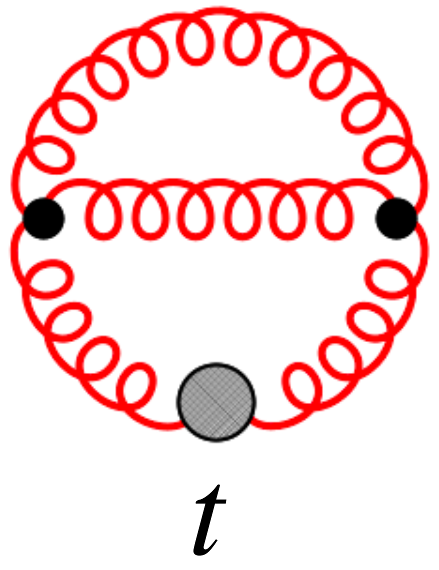

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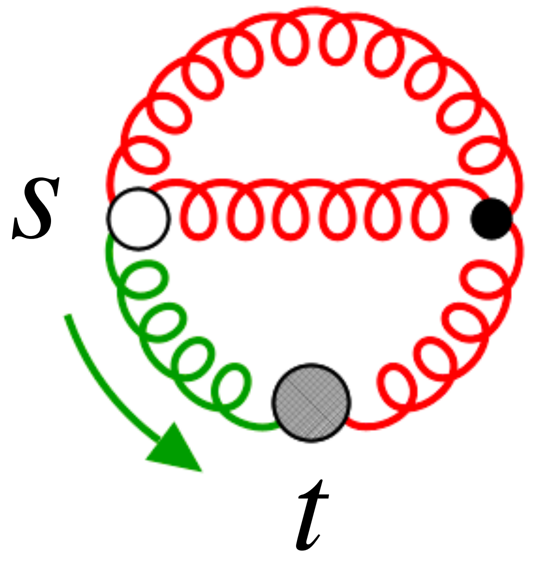
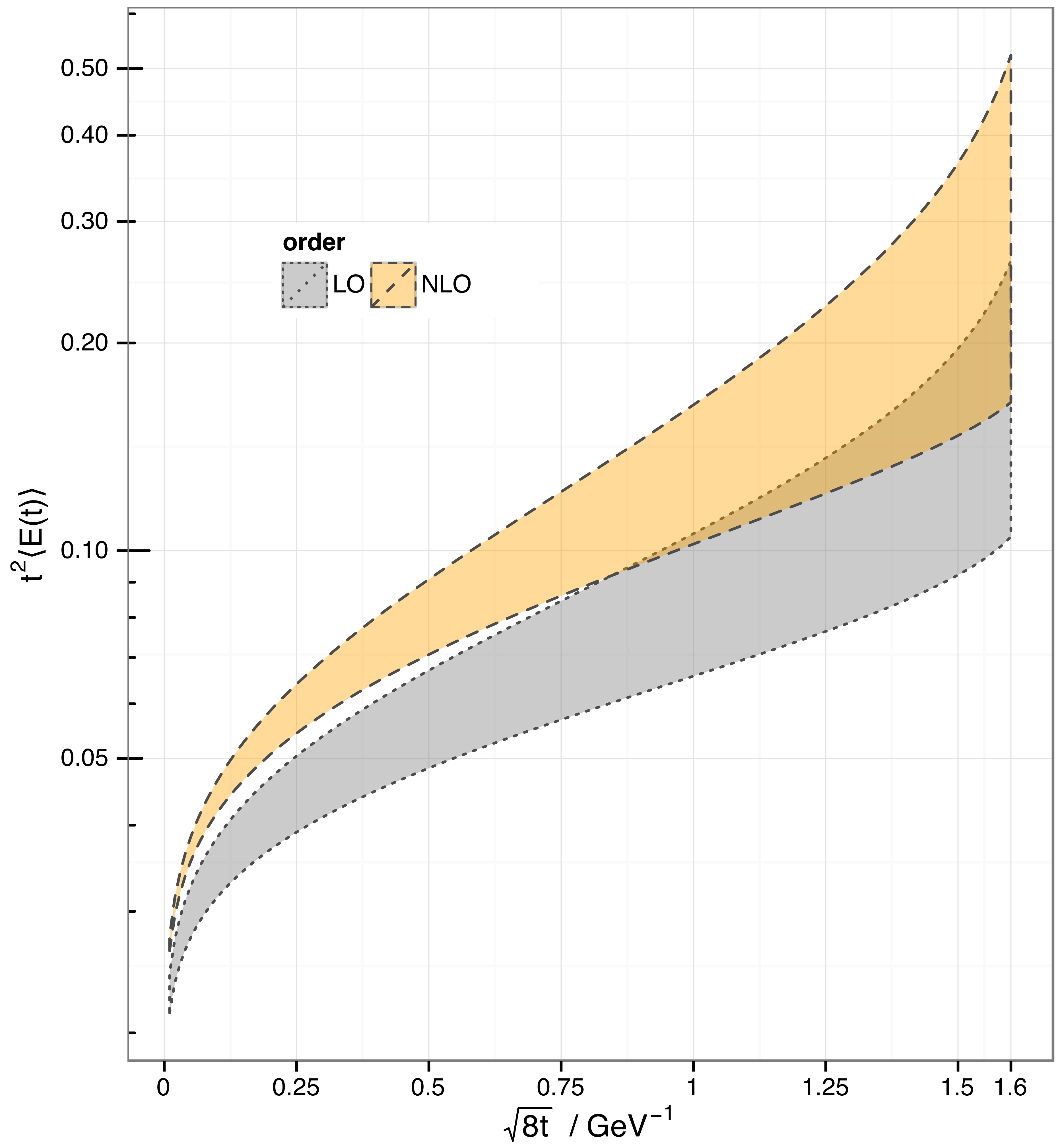


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\, k^2\, (p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

Three-loop calculation

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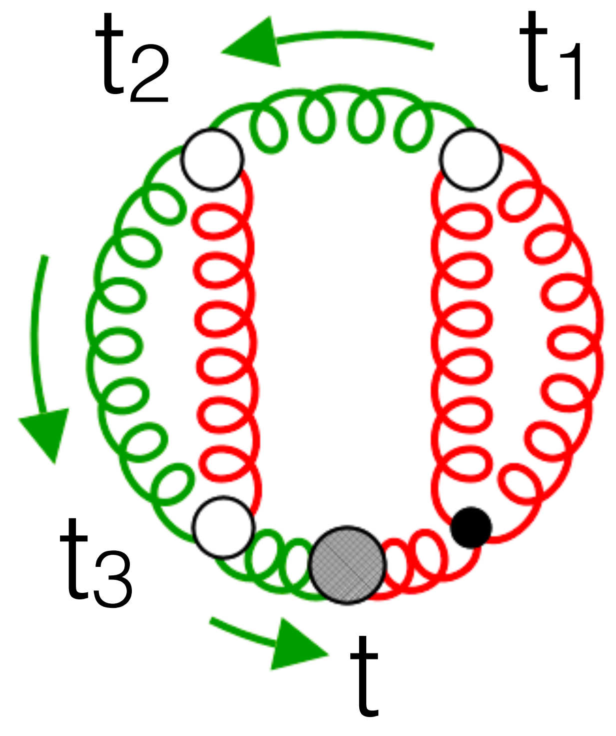
The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



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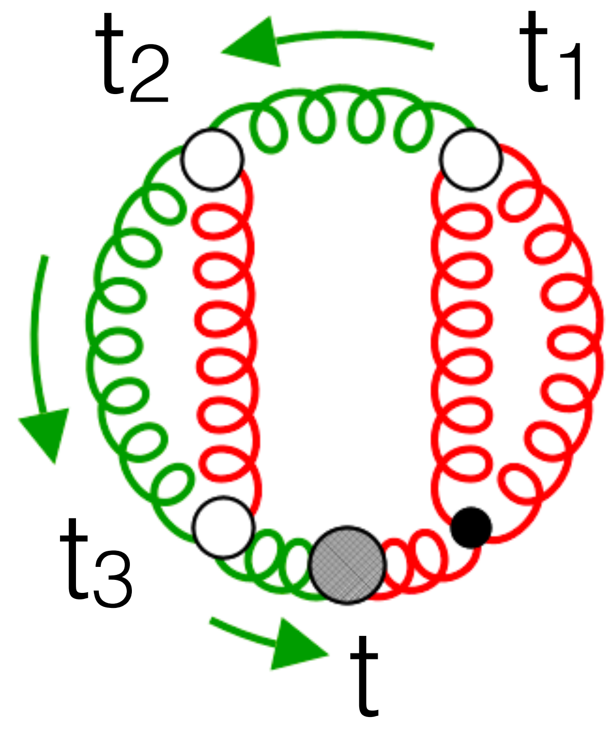


The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals

Artz, RH, Lange, Neumann, Prausa ’19

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

Diagram generation: `qgraf` Nogueira 1993

Diagram analyzation: `q2e/exp` RH, Seidensticker, Steinhauser 1997

$\to$ `tapir/exp` Gerlach, Herren, Lang 2022

Algebraic manipulations: `FORM` Vermaseren < 2000

Reduction to masters: `Kira` $\otimes$ `FireFly`

Chetyrkin, Tkachov 1981
Laporta 2000

Usovitsch, Uwer, Maierhöfer 2017 $\otimes$ Klappert, Klein, Lange 2019

Sector Decomposition:

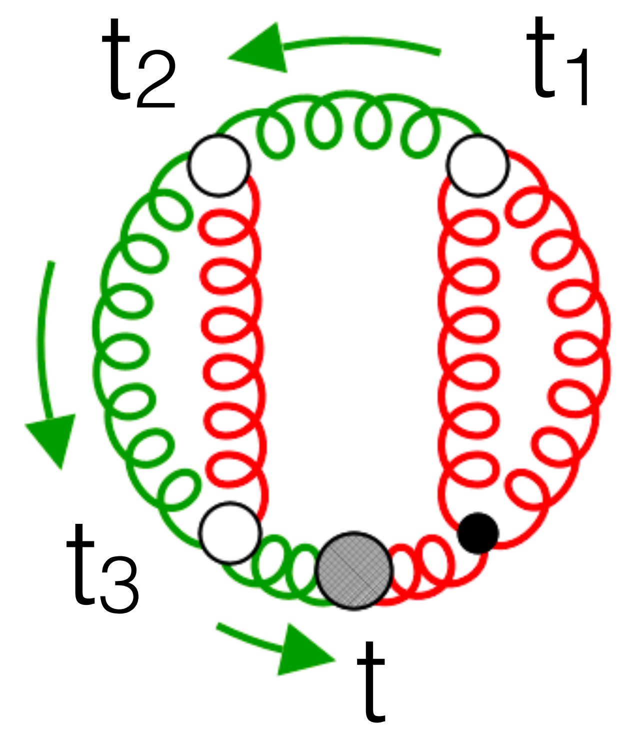
Binoth, Heinrich 2002

$$\int d^Dk \int d^Dp \int_0^t ds \frac{e^{-tp^2-s(k-p)^2}}{k^2p^2(k-p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \ldots$$

Three-loop calculation

$$I(\{c_1,\cdots,c_f\},\{a_1(u),\cdots,a_6(u)\},\{b_1,\cdots,b_6\}) =$$

$$= \left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2)^{b_1}\,\cdots\,(p_6^2)^{b_6}}$$



$$u_i = t_i/t$$

IbP identities:

$$\frac{\partial}{\partial p_i}\cdot p_j\, I(c,a,b) = D\,\delta_{ij}\, I(c,a,b) + \sum I(c',a,b')$$

$$\frac{\partial}{\partial u_i} I(c,a,b) = I(c',a(u=1),b') - I(c',a(u=0),b')$$

Huge systems of linear equations, solved by “master integrals”.

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

Diagram generation: `qgraf` Nogueira 1993

Diagram analyzation: `q2e/exp` RH, Seidensticker, Steinhauser 1997

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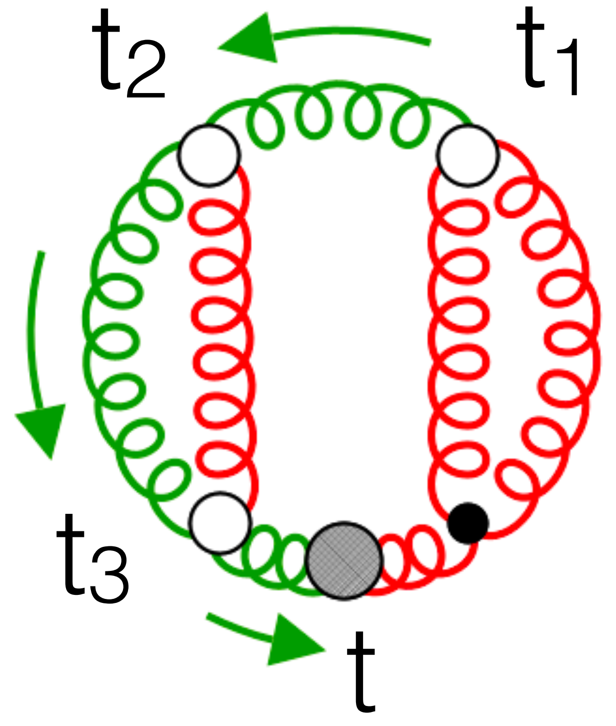
Numerical evaluation

RH, Neumann (2016)



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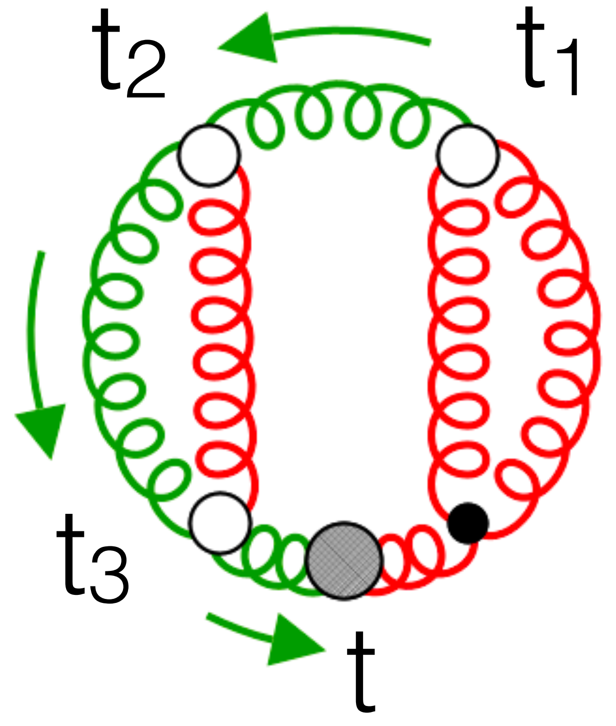

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Schwinger parameters:

$$\frac{1}{(p^2)^b} \sim \int_0^\infty dx\, x^{b-1}\, e^{-xp^2}$$

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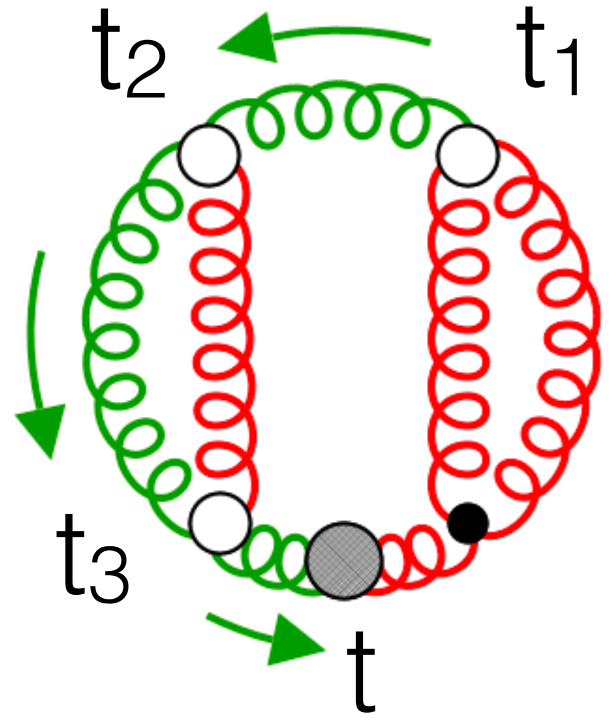
RH, Neumann (2016)



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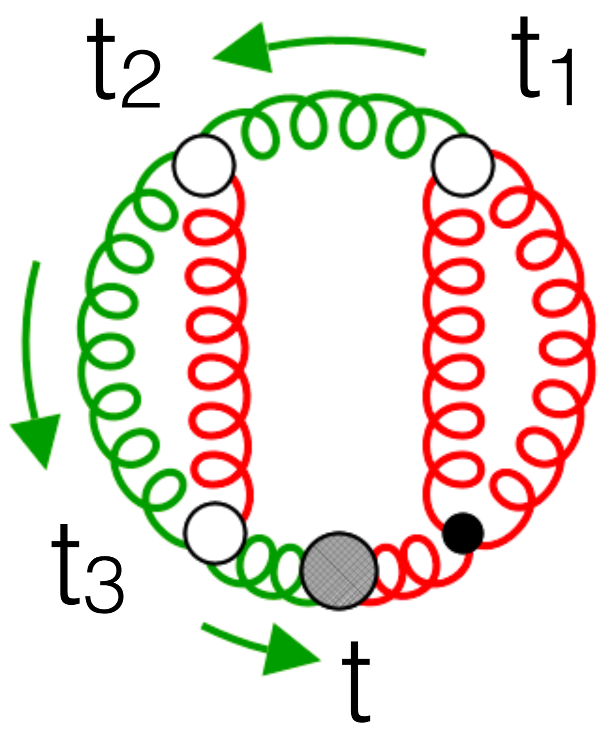

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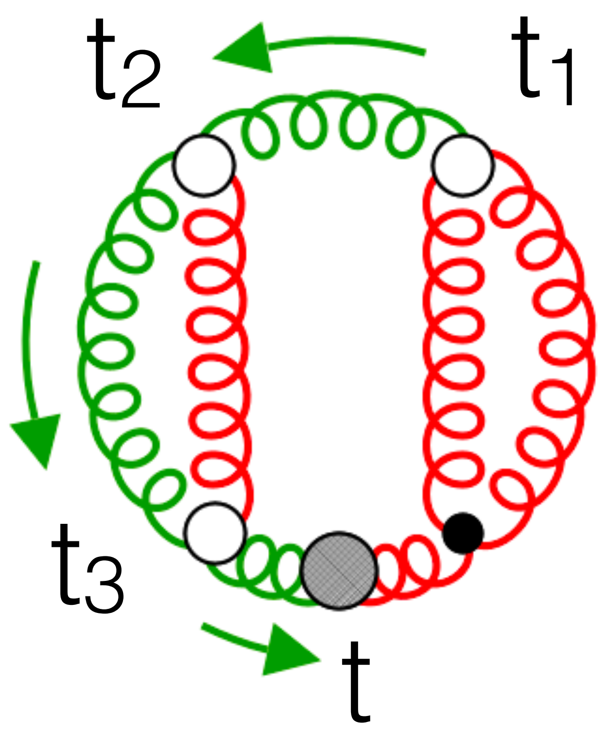

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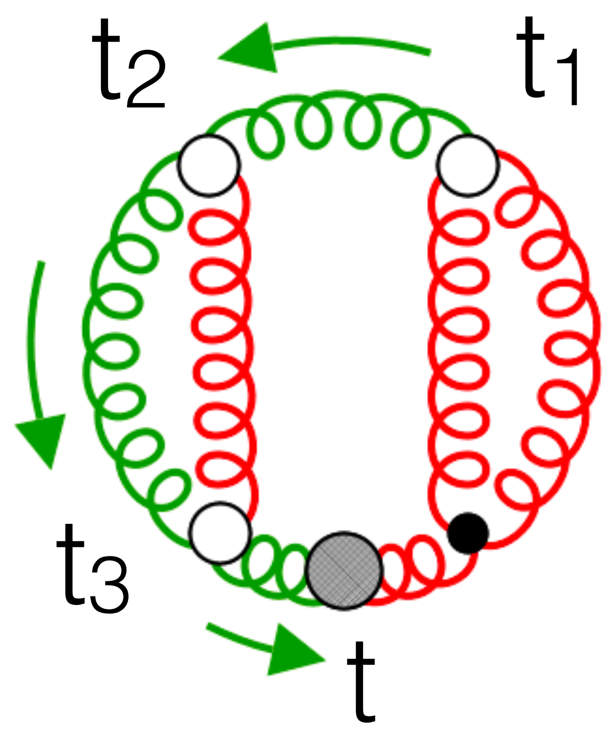

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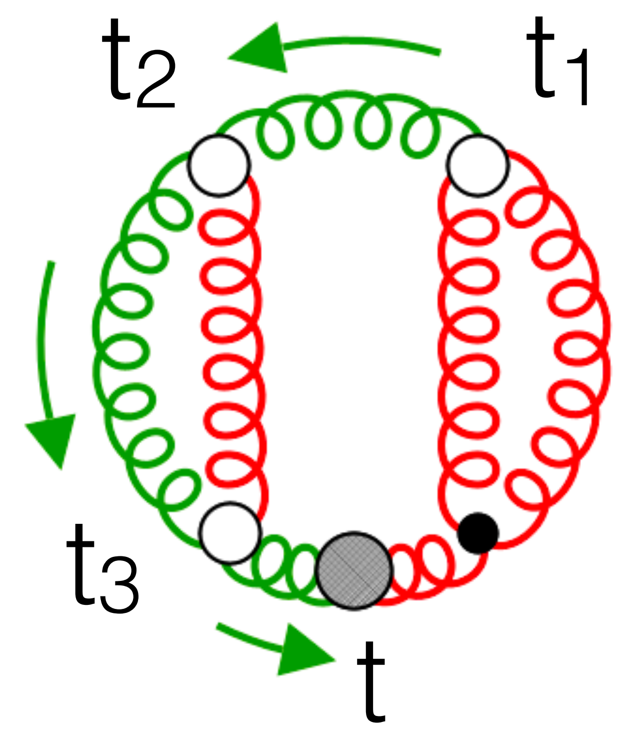

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→ sector decomposition
Binoth, Heinrich (2000)

Implementation

$$I(\{c_1, \cdots, c_f\}, \{a_1(u), \cdots, a_6(u)\}, \{b_1, \cdots, b_6\})$$

$$c_1 = c_2 = 0$$

$$a_1 = u_1 u_2\,, \quad a_2 = u_2\,, \quad a_3 = u_2 - u_1 u_2$$

$$a_4 = 1\,, \quad a_5 = 1 + u_1 u_2\,, \quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

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`ftint` RH, Nellopoulos, Olsson, Wesle (in prep)

(based on `pySecDec`)

Heinrich, Magerya, Kerner, Jones, …

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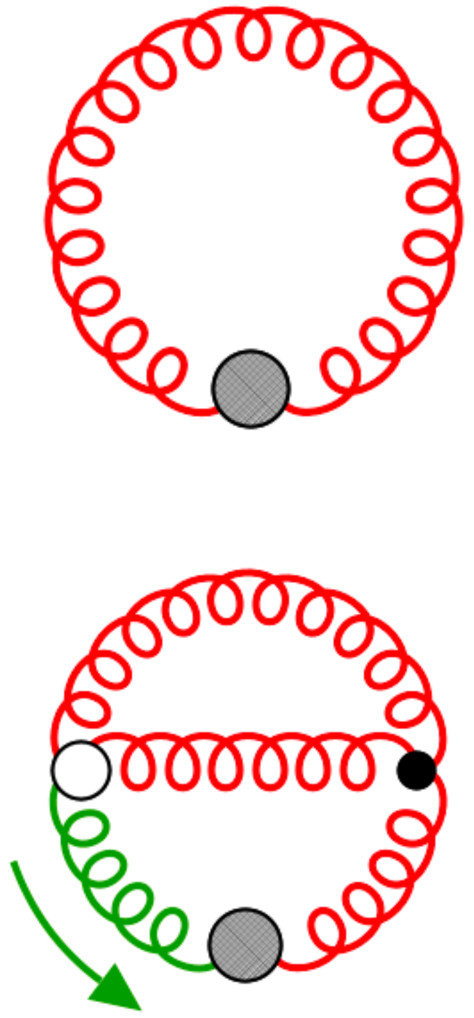
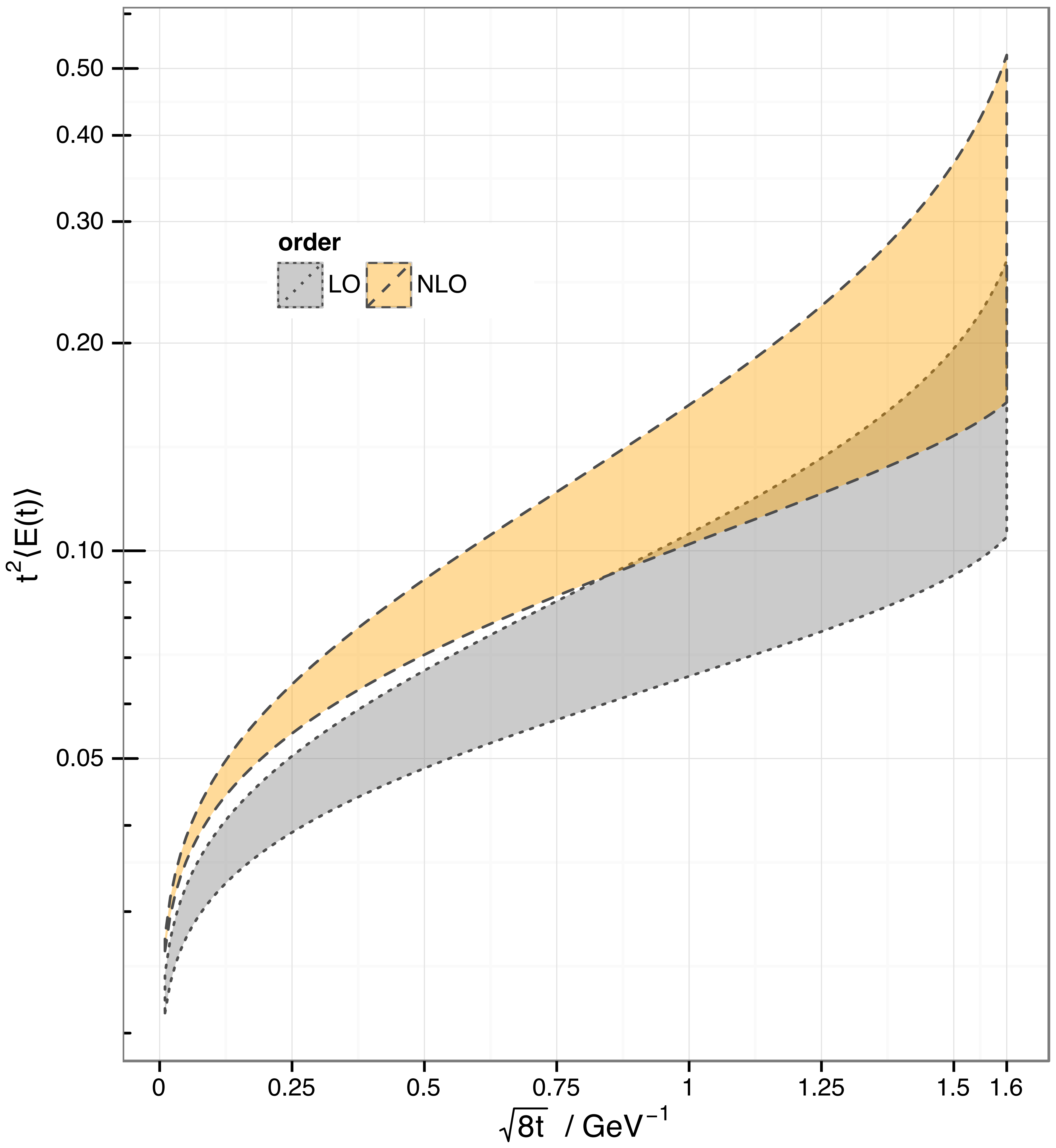
```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}},{1,0,0,1,0,0}] -> (
    +eps^-1*(+8.333333333333343*10^-02+0.0000000000000000*10^+00*I)
    +eps^-1*(+1.4433895444086145*10^-15+0.0000000000000000*10^+00*I)*plusminus
    +eps^0*(+3.0238270284562663*10^-01+0.0000000000000000*10^+00*I)
    +eps^0*(+1.6918362746499228*10^-08+0.0000000000000000*10^+00*I)*plusminus
    +eps^1*(+6.5531010458012129*10^-01+0.0000000000000000*10^+00*I)
    +eps^1*(+3.7857260802916662*10^-08+0.0000000000000000*10^+00*I)*plusminus
  ),
```

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010





$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

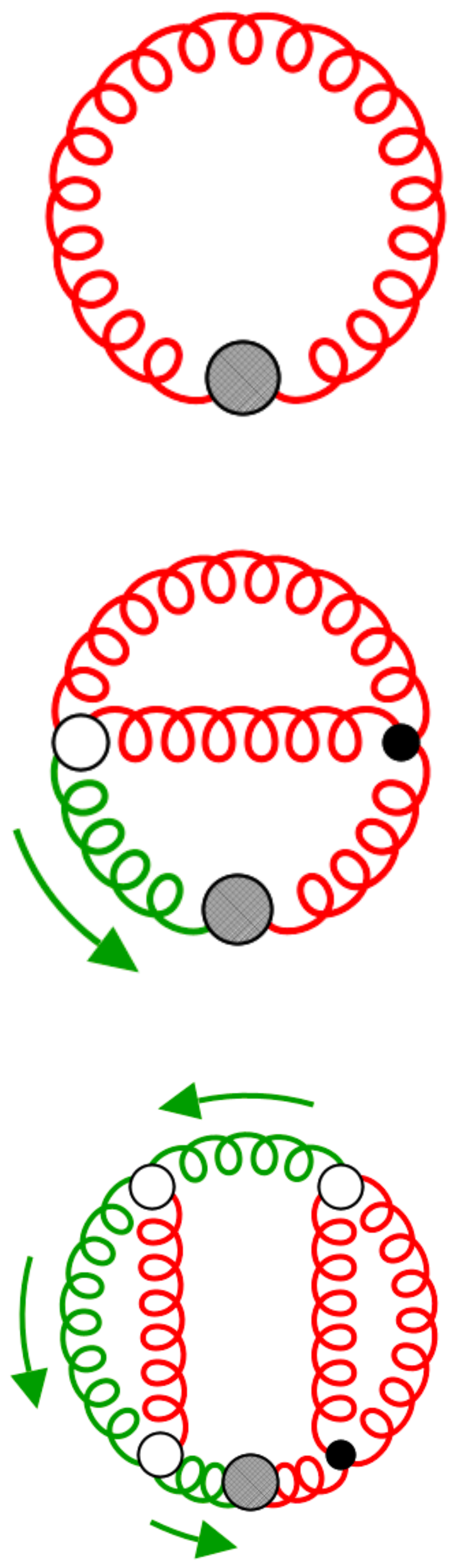
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

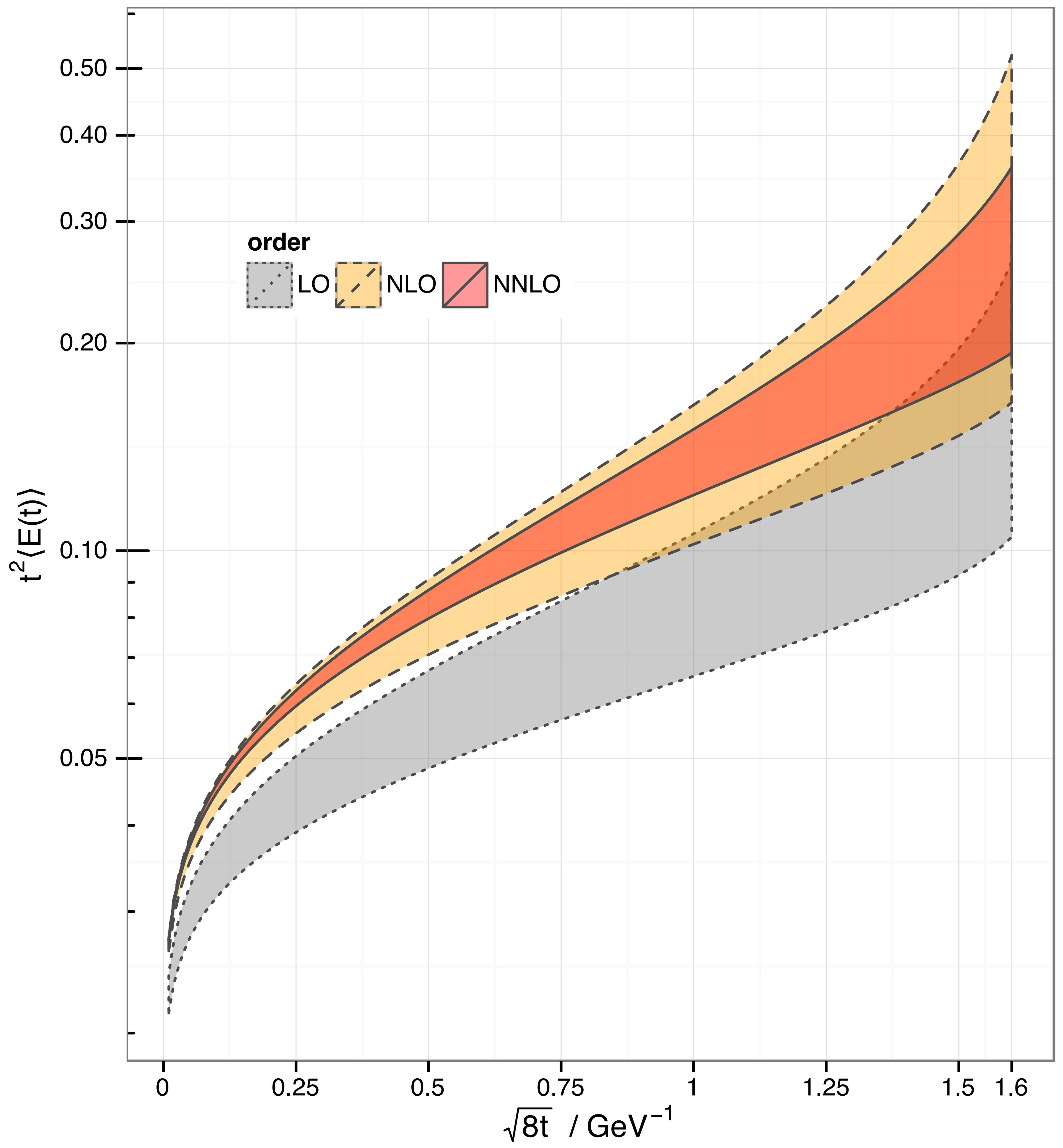
resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann 2016





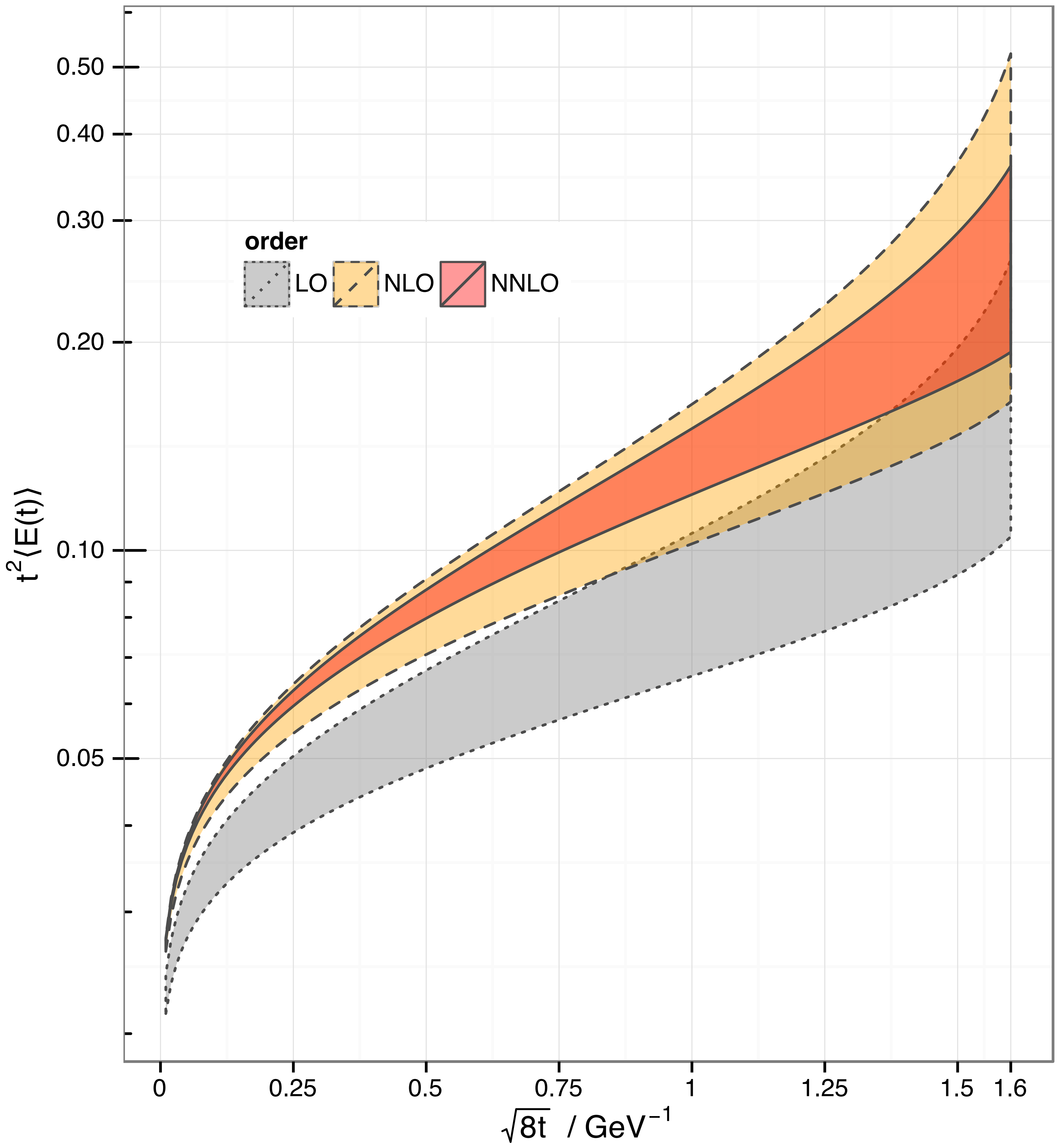
resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

Derive $\alpha_s(m_Z)$

	$t^2\langle E(t)\rangle \cdot 10^4$							
q_8	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$



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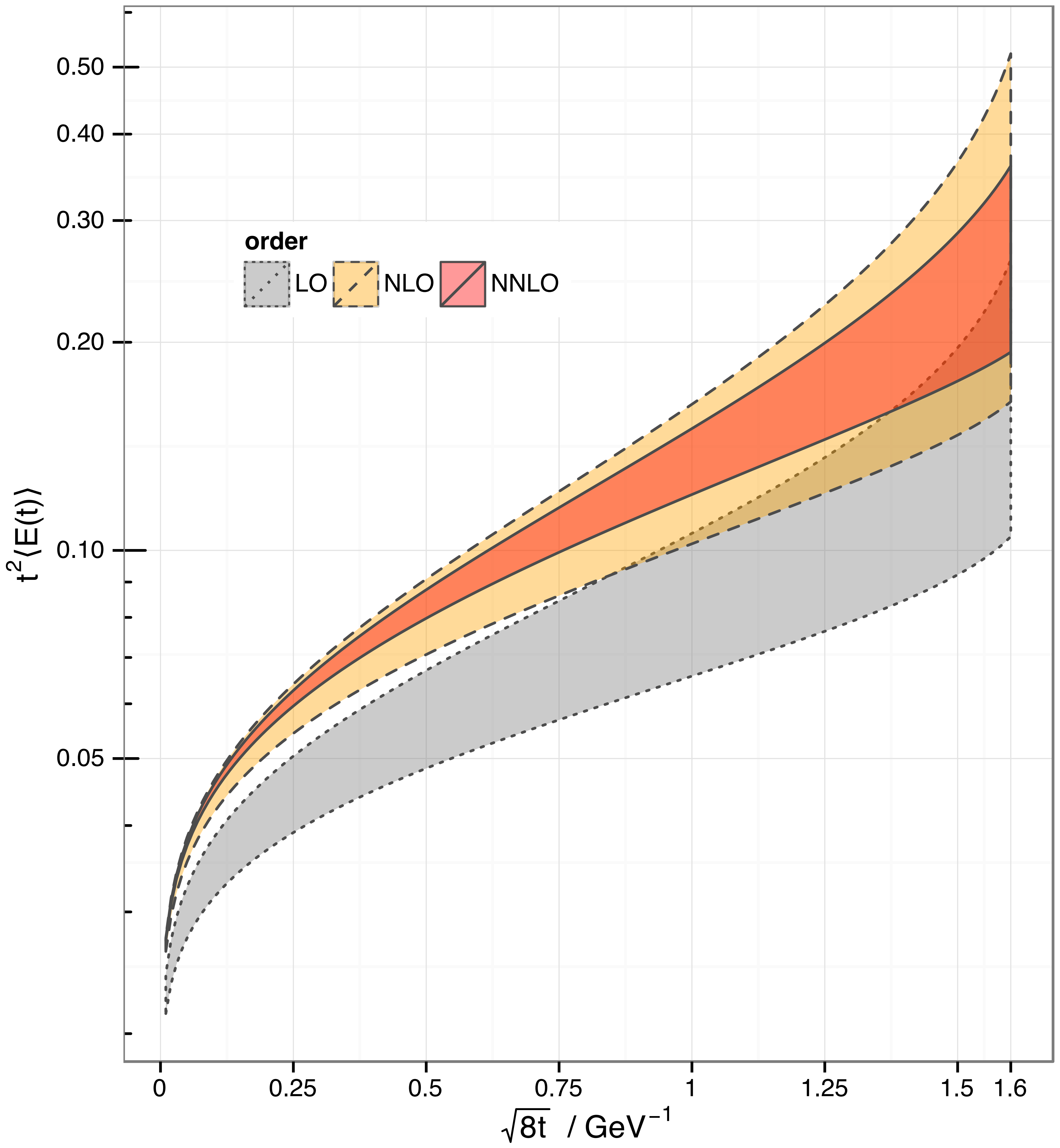

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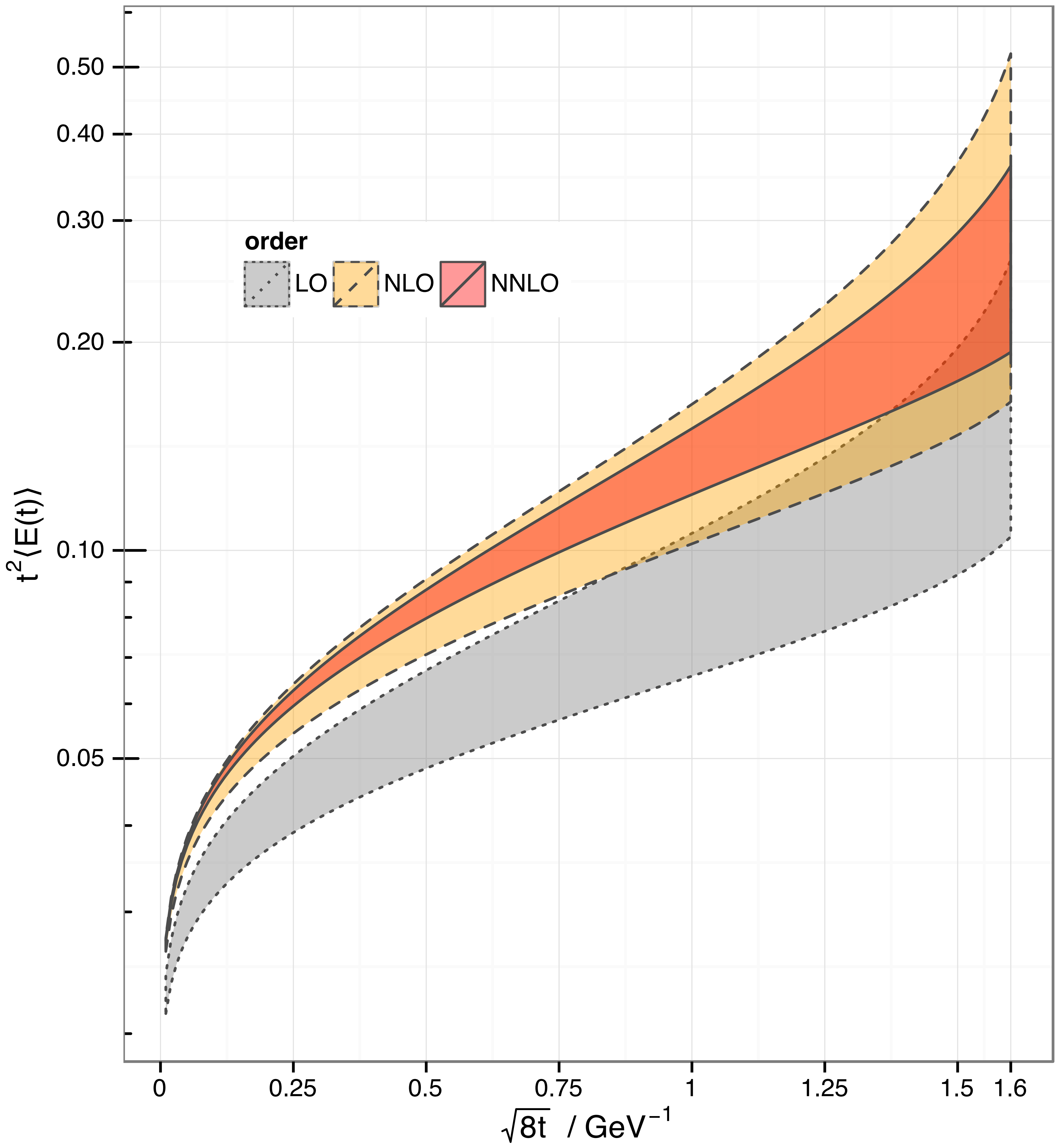

$$\mu^2 \frac{d}{d\mu^2}\hat{a}_s(\mu^2) = \hat{\beta}(\hat{a}_s)$$

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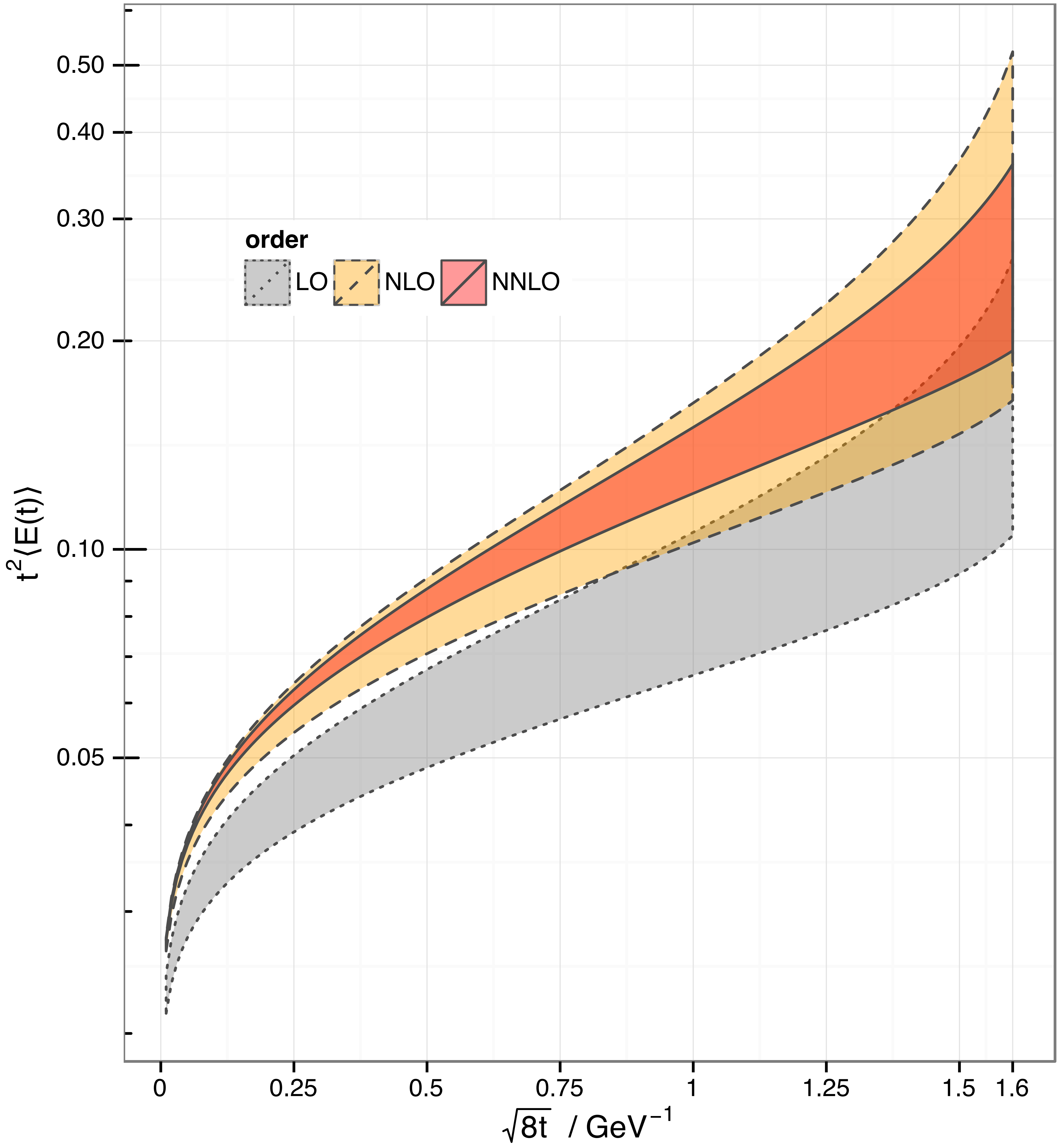

$$\mu^2\frac{d}{d\mu^2}\hat{a}_s(\mu^2) = \hat{\beta}(\hat{a}_s)$$

$$= -\,\hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$

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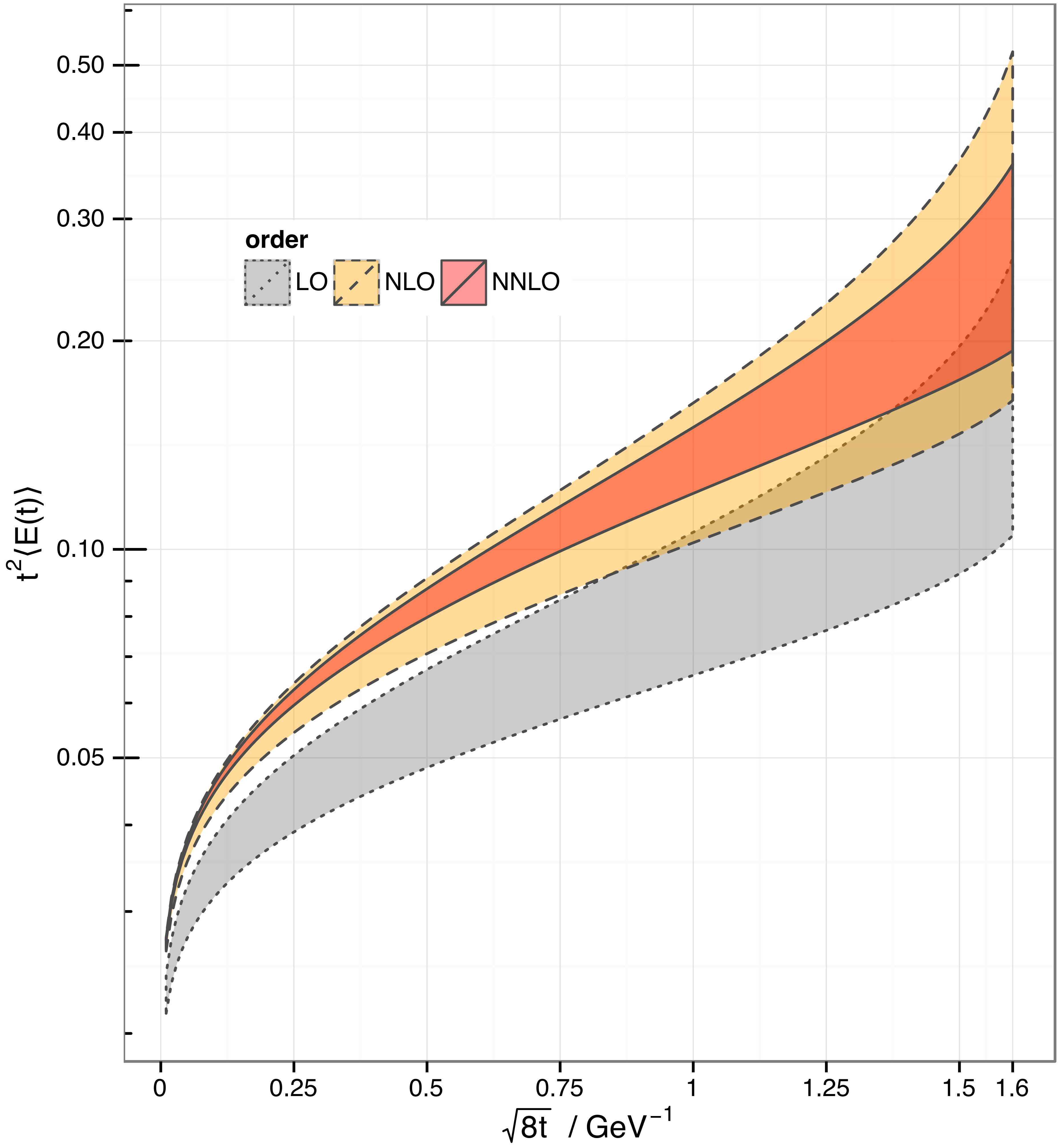

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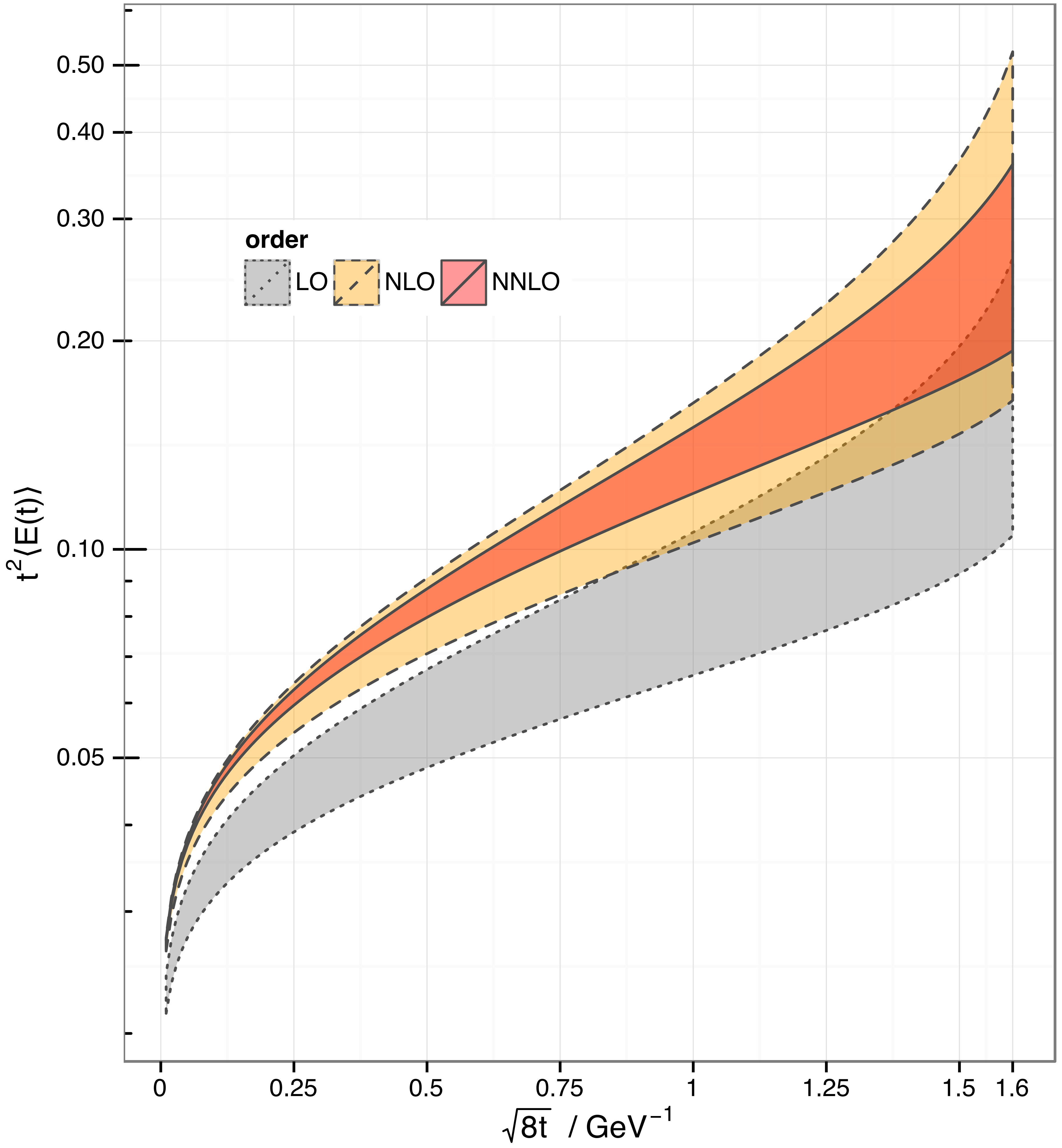

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GF specific
depends on k_2

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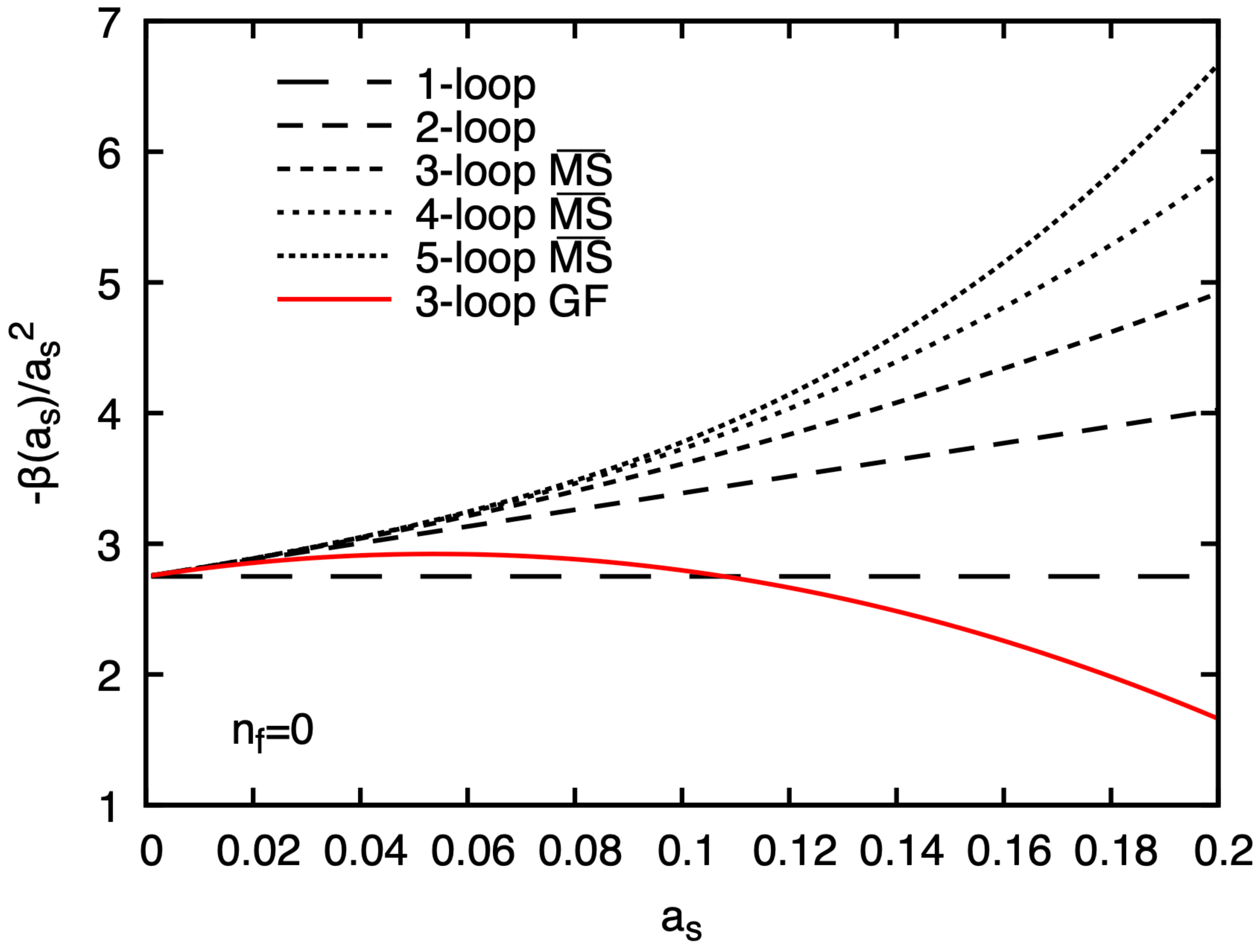

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universal

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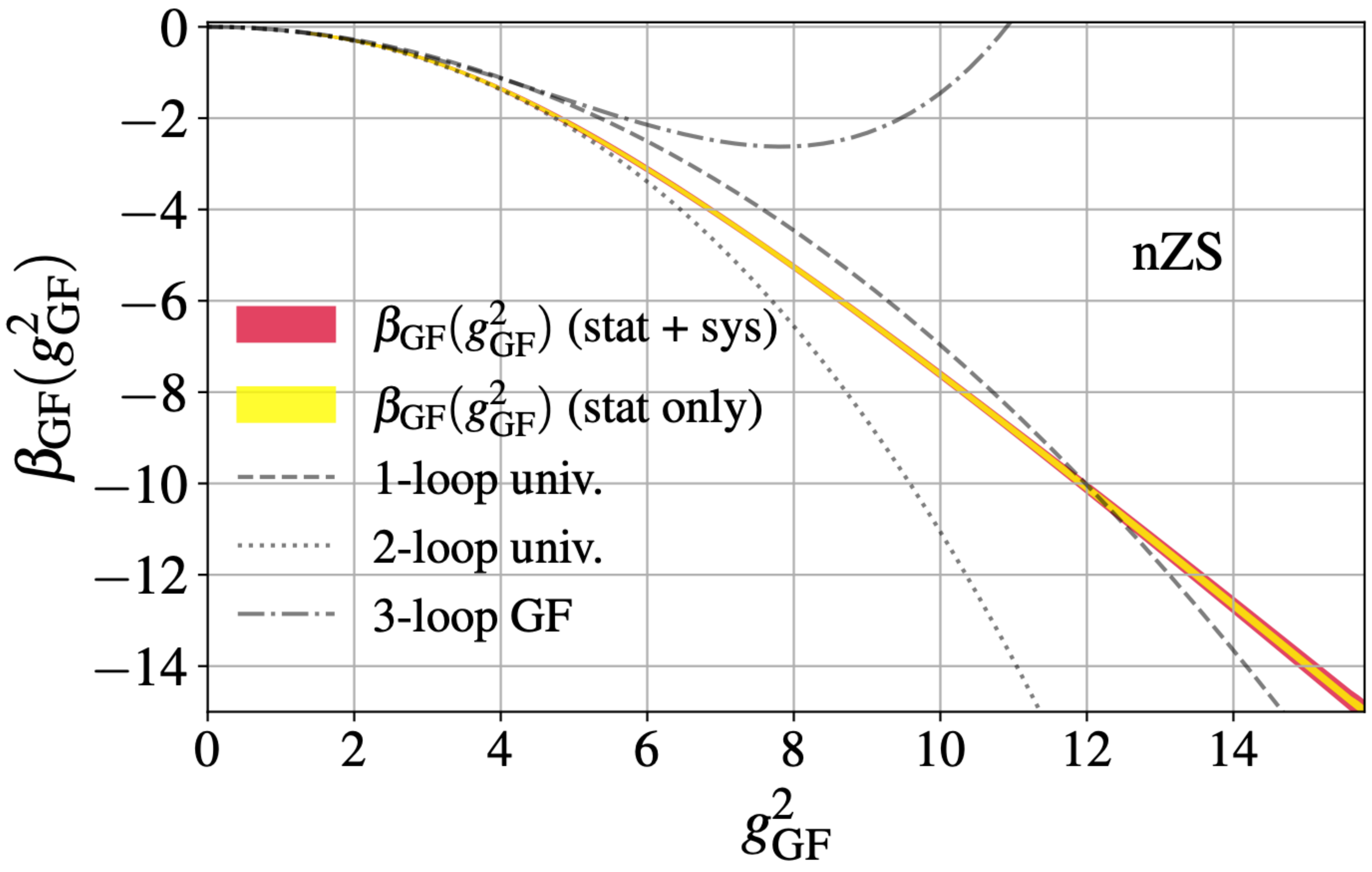
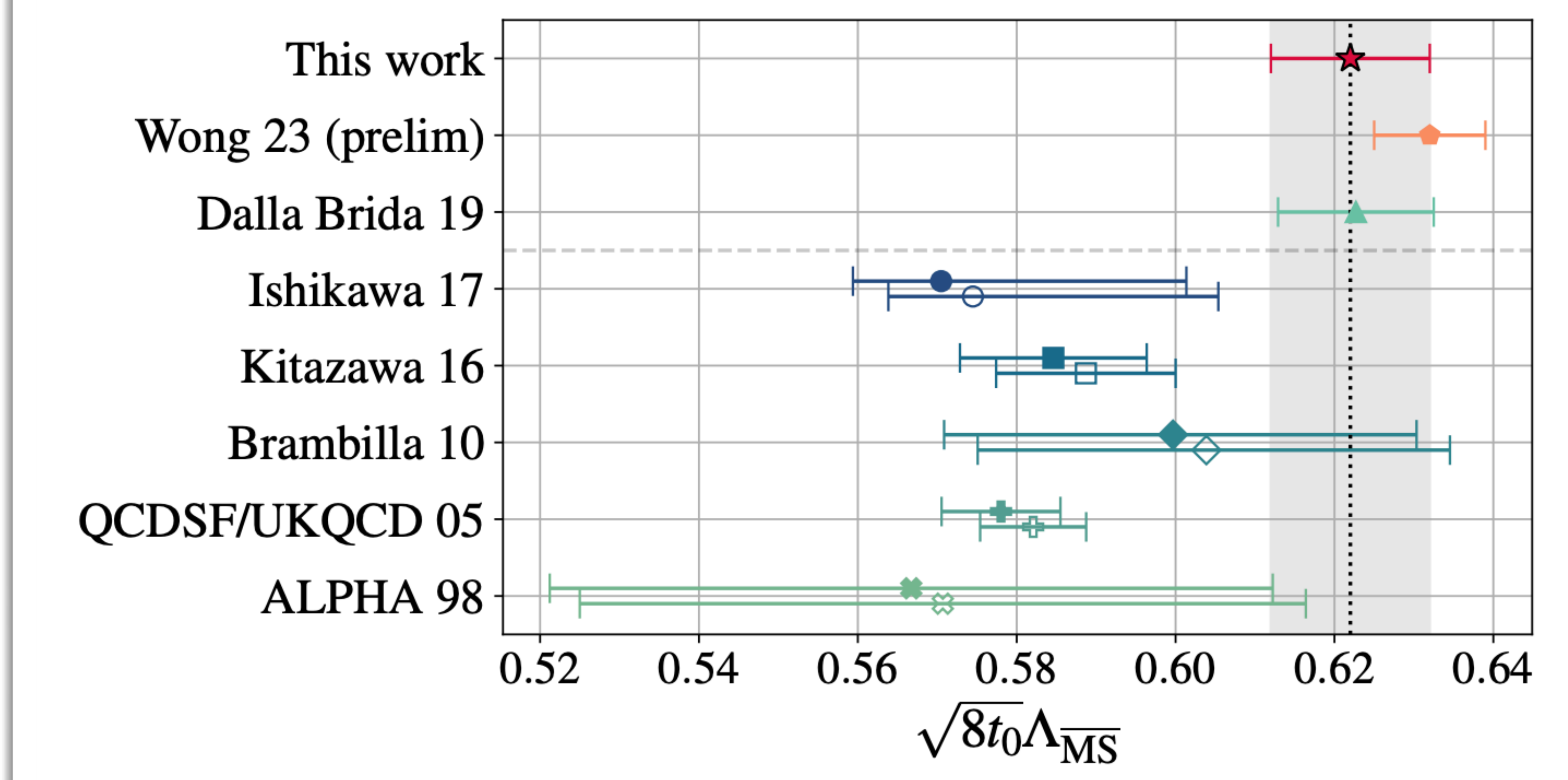
Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel 2023

see also C.H. Wong et al.



A common problem

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory → C_n

lattice → $\langle \mathcal{O}_n \rangle$

match renormalization schemes?

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match renormalization schemes?

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

perturbation theory → $\tilde{C}_n(t)$

lattice → $\langle \tilde{\mathcal{O}}_n(t) \rangle$

gradient flow renormalization

Small flow-time expansion

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

Small flow-time expansion

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

small flow-time expansion:
Lüscher, Weisz '11

$$\tilde{\mathcal{O}}_n(t) \stackrel{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

$$\tilde{C}_n(t) \stackrel{t\to 0}{\to} \sum_m C_m\, \zeta^{-1}_{mn}(t)$$

⇒ need $\zeta_{nm}(t)$ for small t ⇒ perturbation theory

Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \stackrel{t\to 0}{\to} \sum_m \zeta_{nm}(t) \, \langle \mathcal{O}_m \rangle$$

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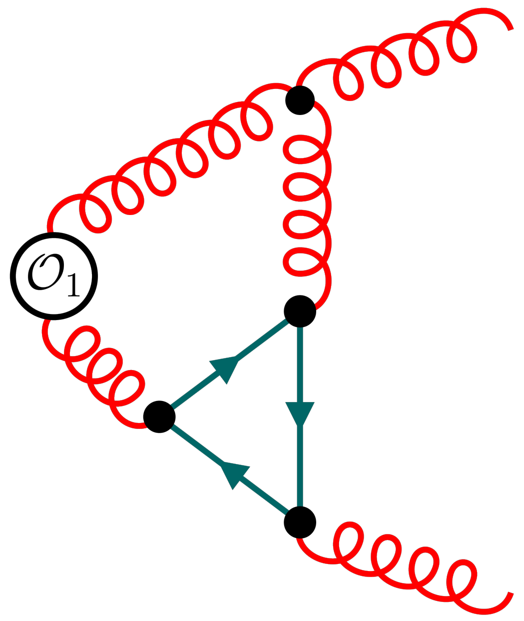

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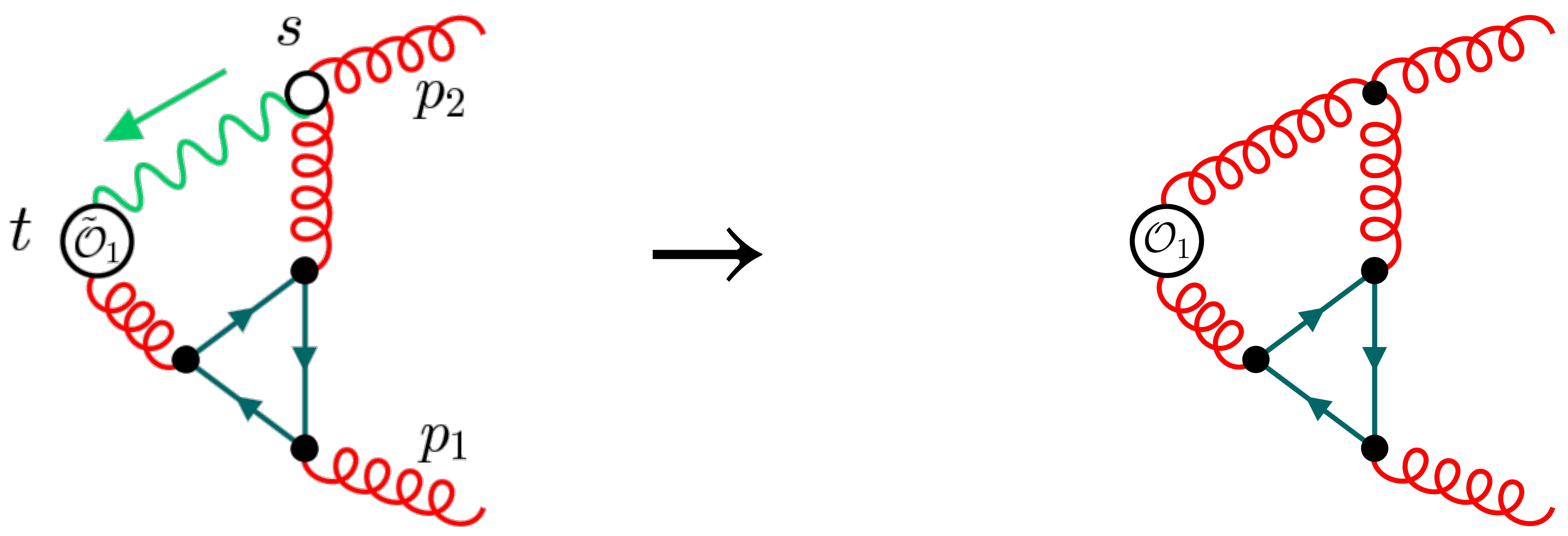

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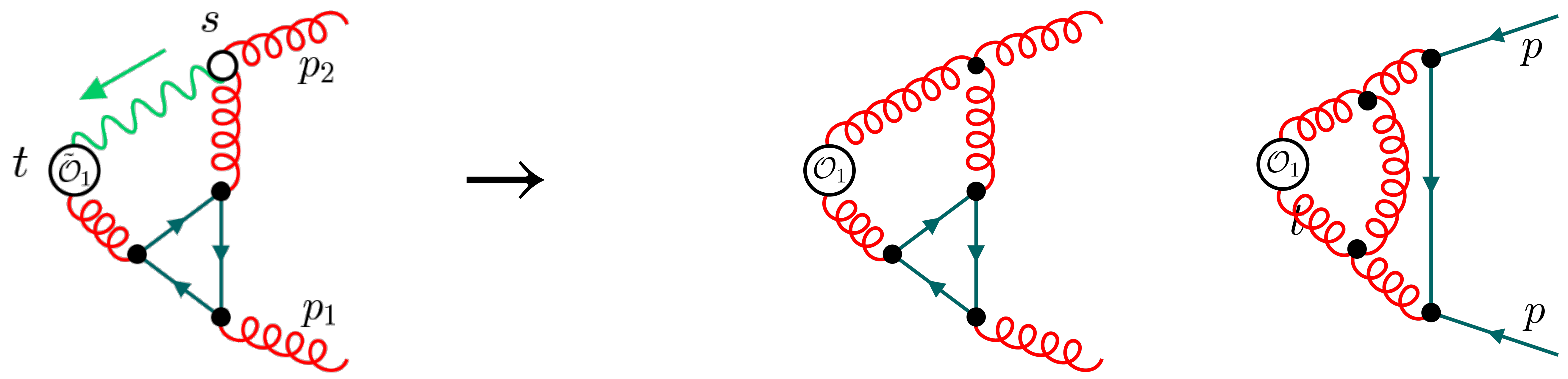

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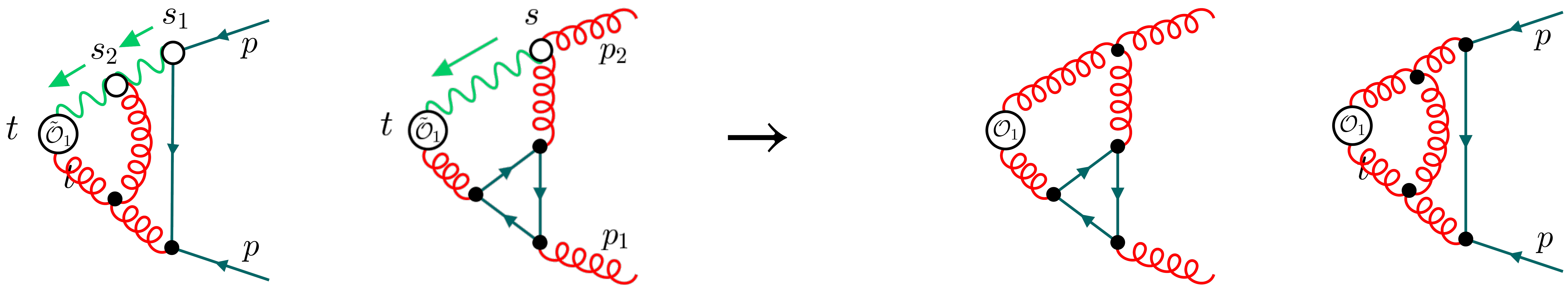

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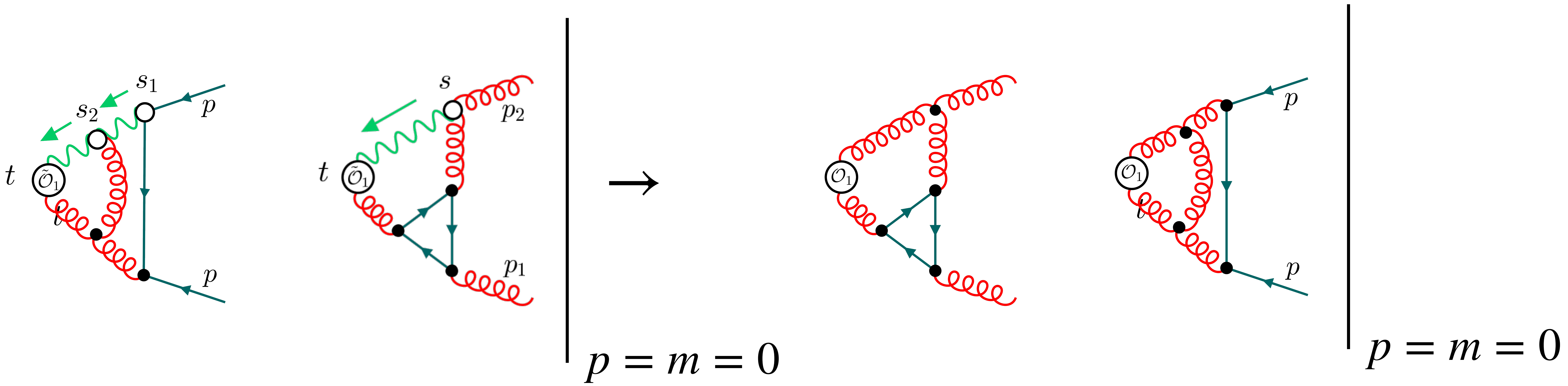

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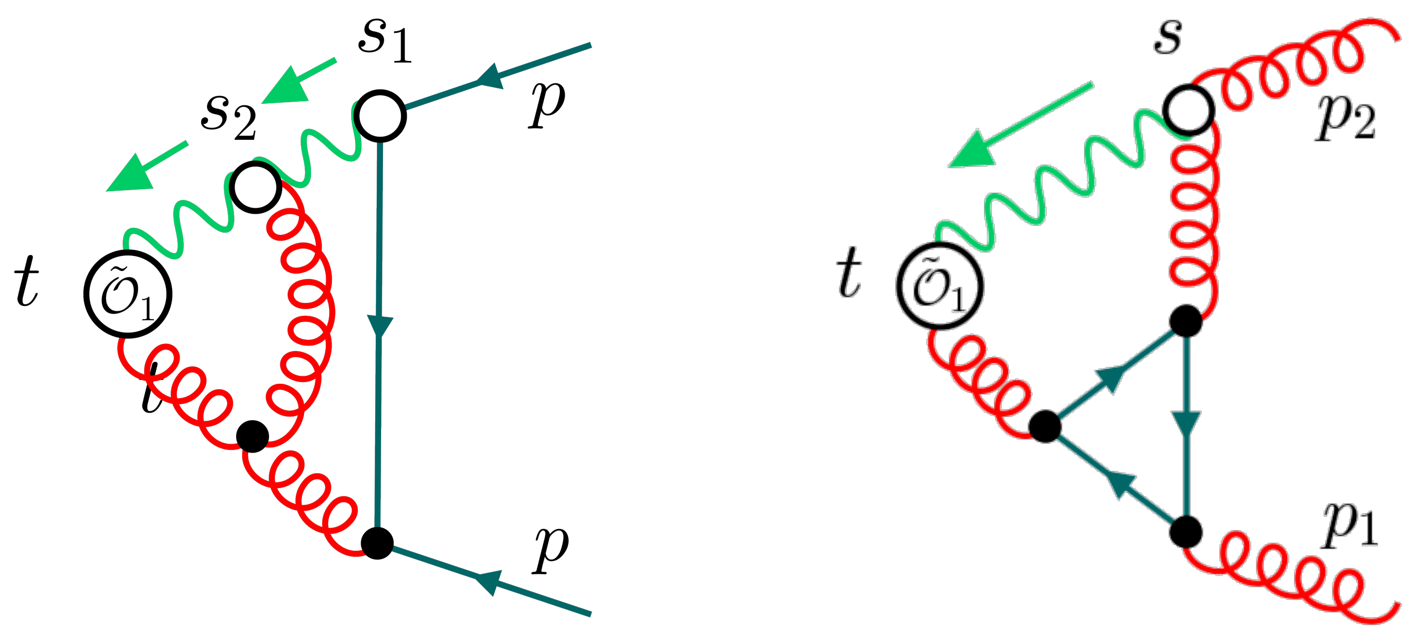
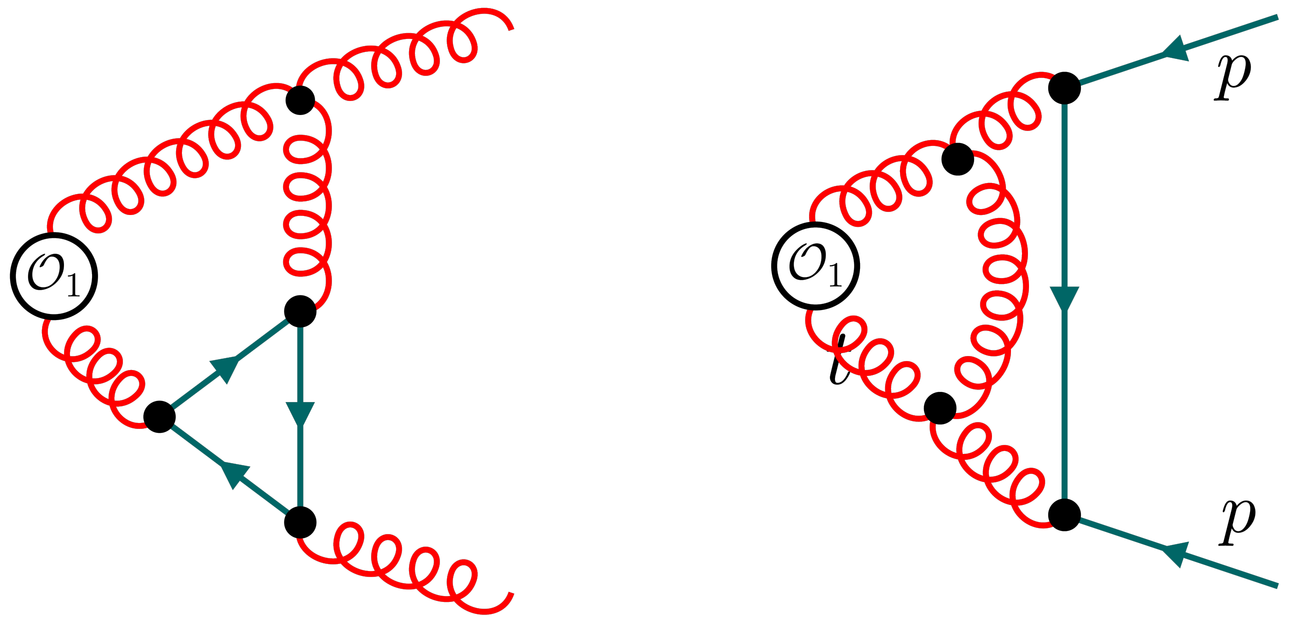


$p = m = 0$

$\rightarrow$



$p = m = 0$

only tree-level diagrams survive on r.h.s.

Gorishnii, Larin, Tkachov '83

Ex. 1: QCD energy-momentum tensor Suzuki, Makino '13, '14

$$T_{\mu\nu} = \sum_n C_n \mathcal{O}_{n,\mu\nu}$$

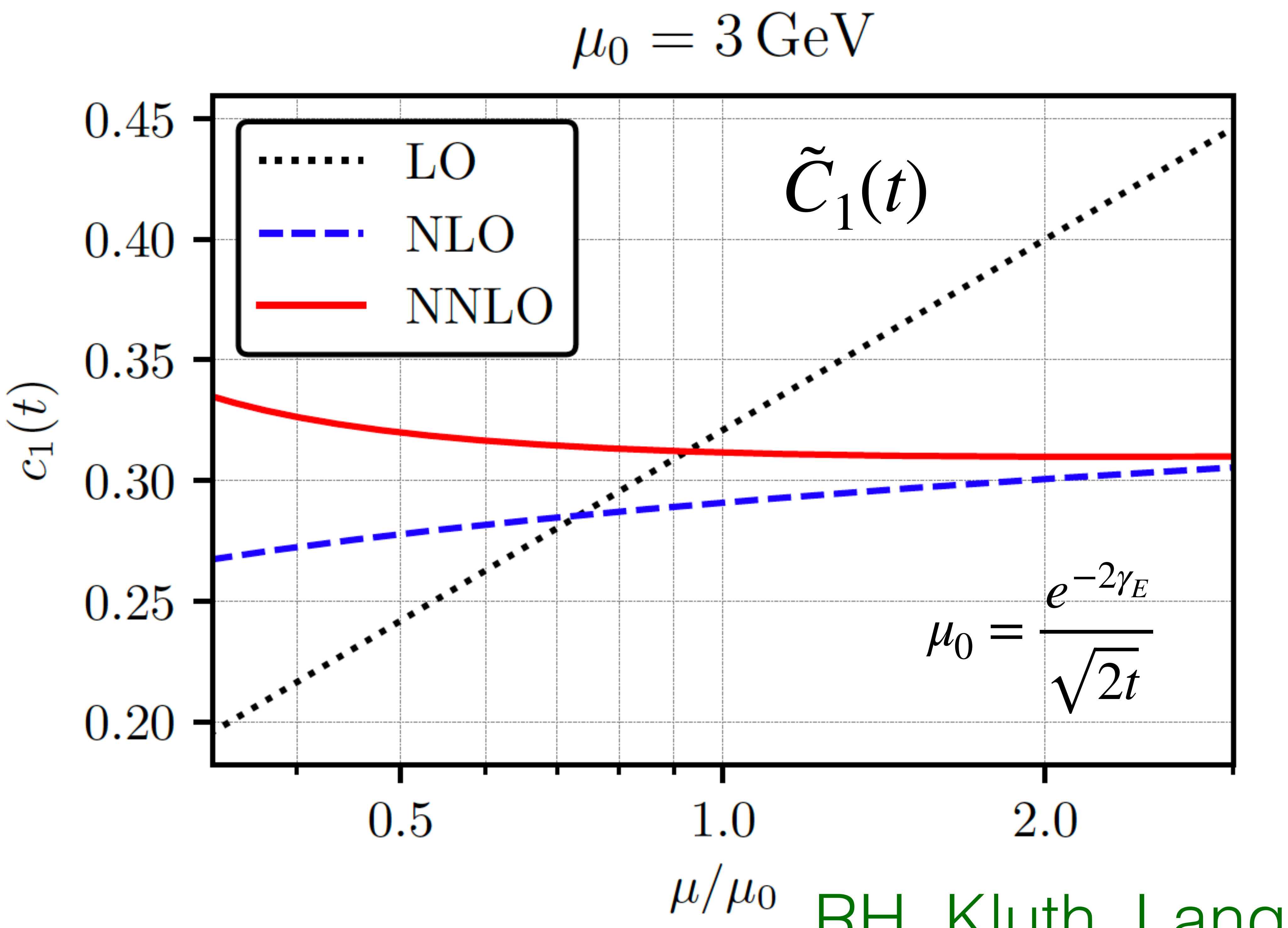
$$T_{\mu\nu} = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

$$\mathcal{O}_{1,\mu\nu} = \frac{1}{g_0^2} F^a_{\mu\rho}F^a_{\nu\rho} \qquad C_1 \equiv 1$$

$$\mathcal{O}_{2,\mu\nu} = \frac{\delta_{\mu\nu}}{g_0^2} F^a_{\rho\sigma}F^a_{\rho\sigma} \qquad C_2 \equiv -\frac{1}{4}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi \qquad C_3 \equiv \frac{1}{4}$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi \qquad C_4 \equiv 0$$



RH, Kluth, Lange '18

application: see WHOT collaboration

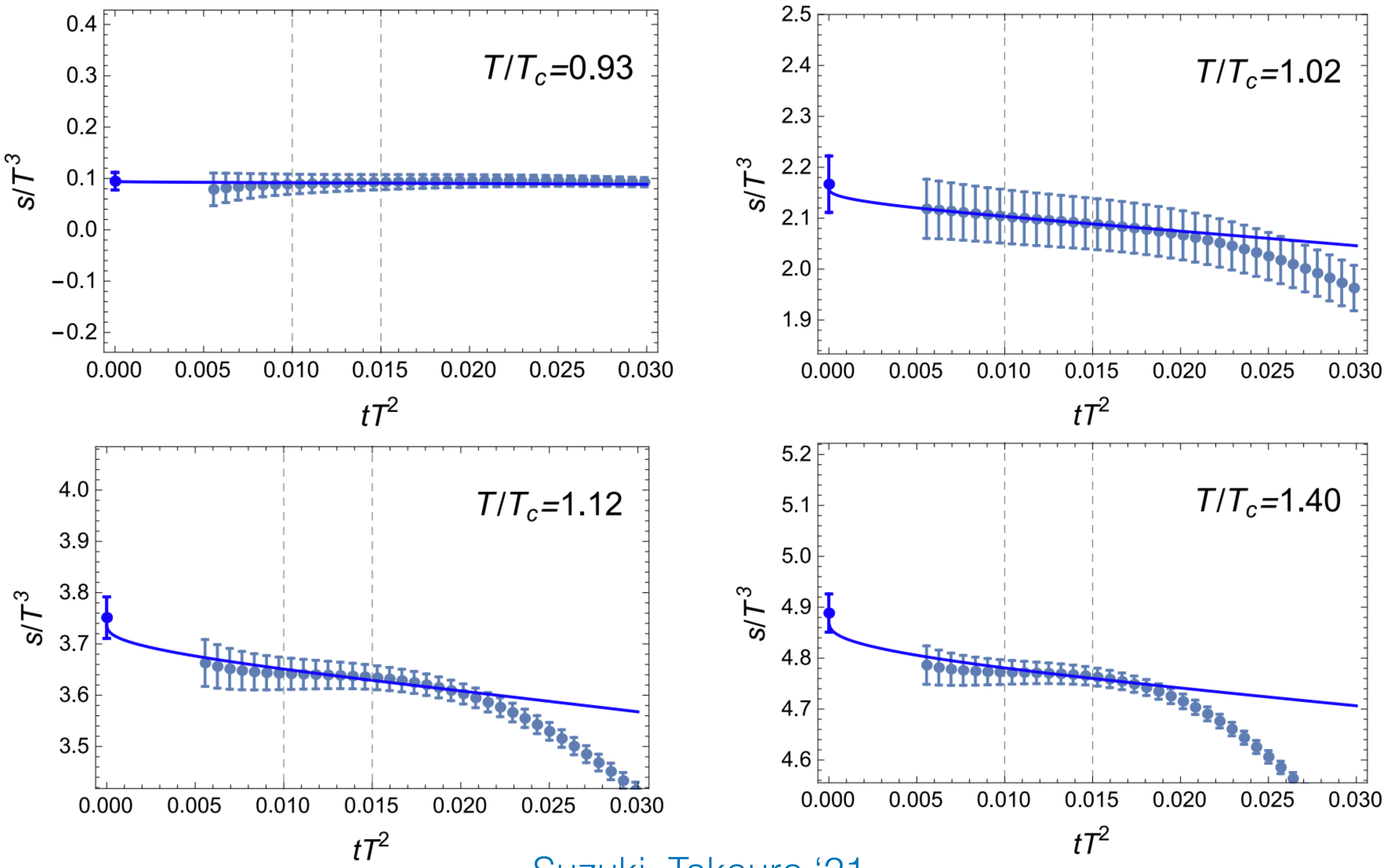
Application

Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

$$T_{\mu\nu}(x) = \sum_{n=1}^{4} c_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t,x)$$

NLO



Suzuki, Takaura ‘21

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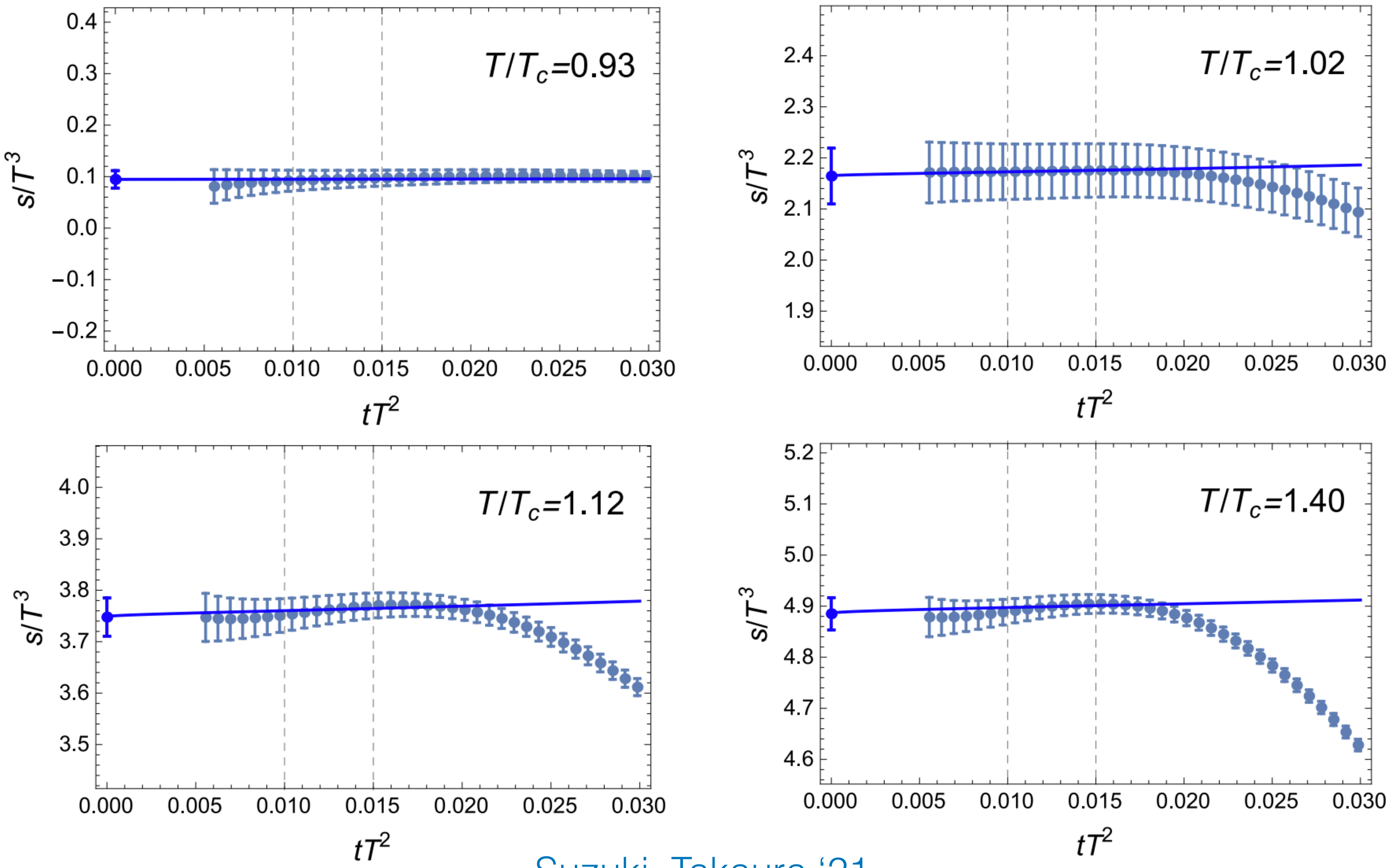

Suzuki, Takaura '21

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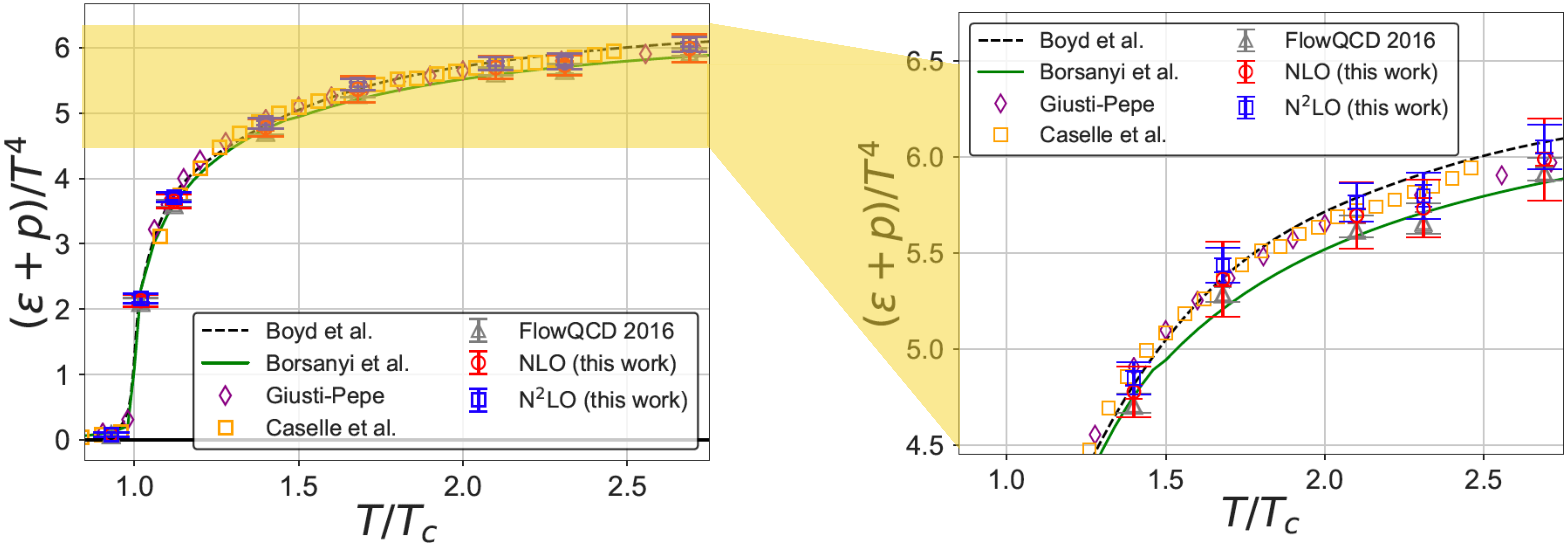
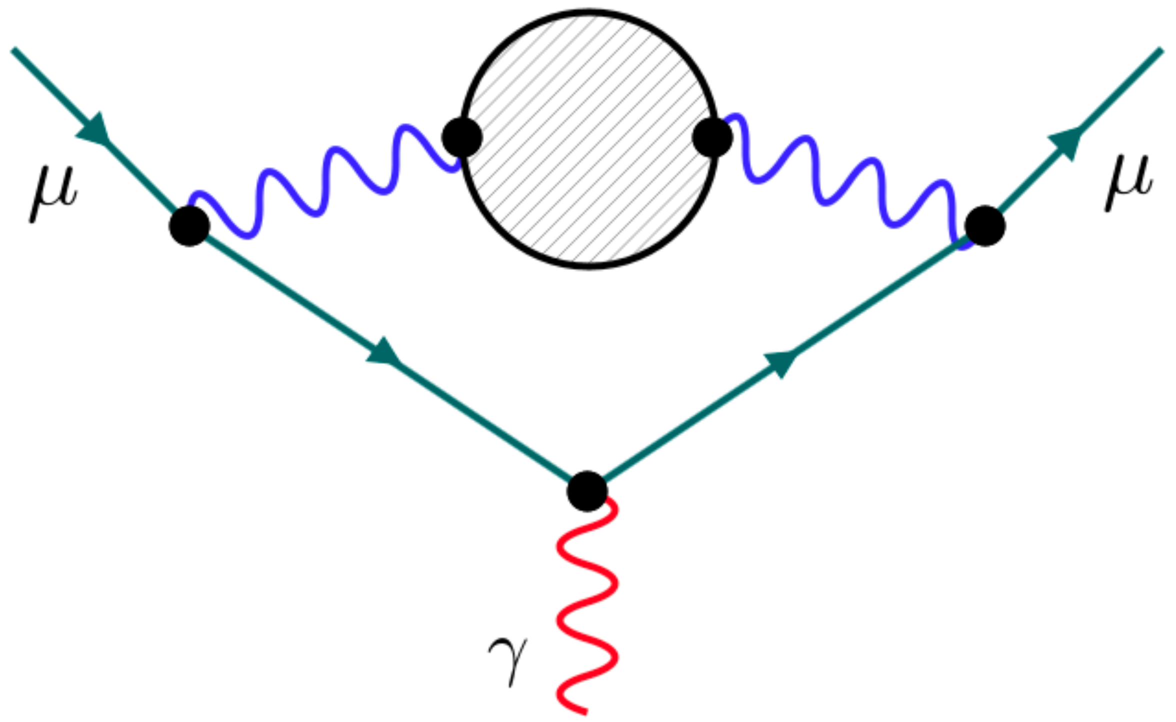


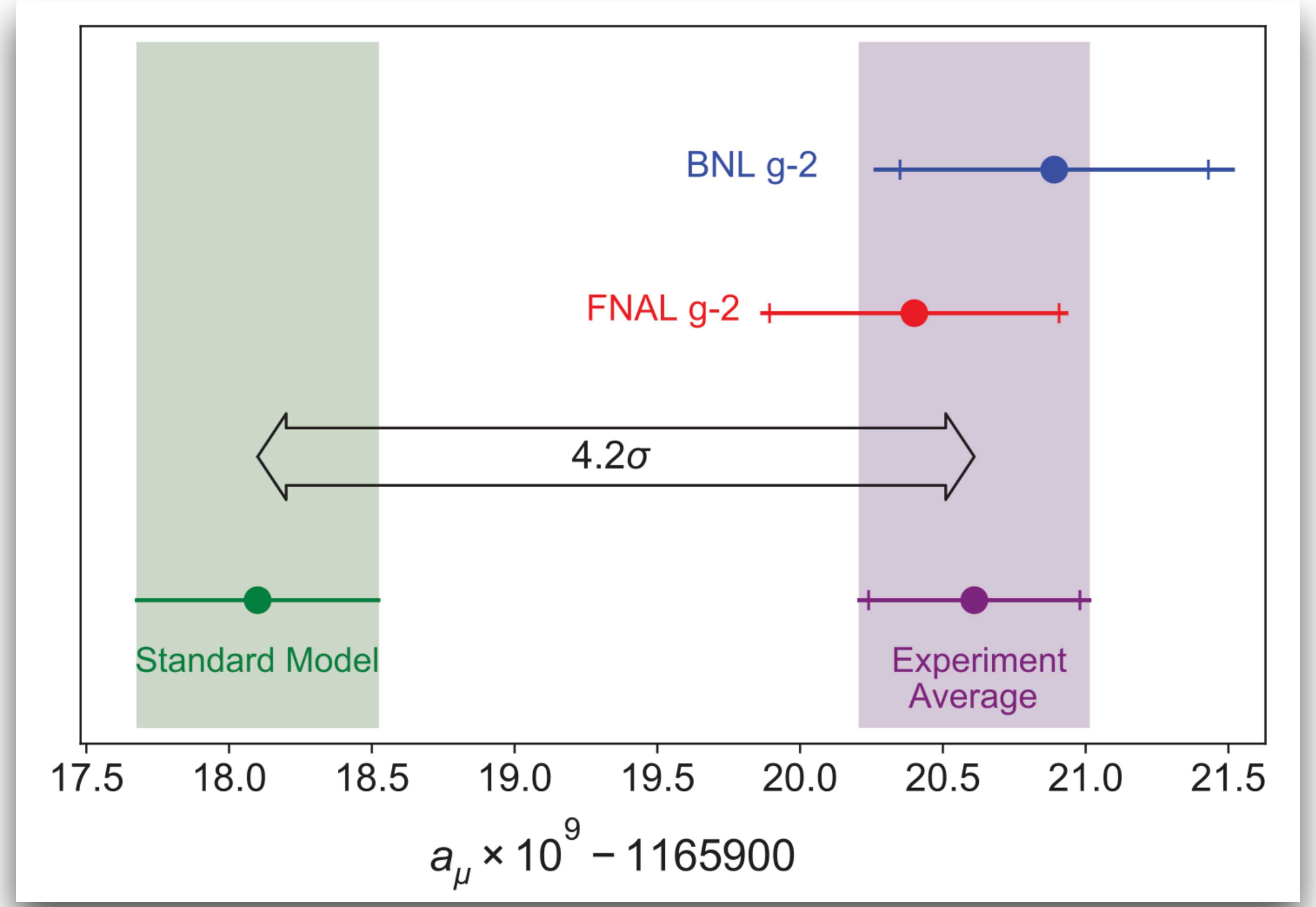
Iritani, Kitazawa, Suzuki, Takaura 2019

Ex. 2: Hadronic vacuum polarization

$$\int d^4x\, e^{iQx} \langle T\, j(x)\, j(0) \rangle$$

contribution to $(g-2)_\mu$





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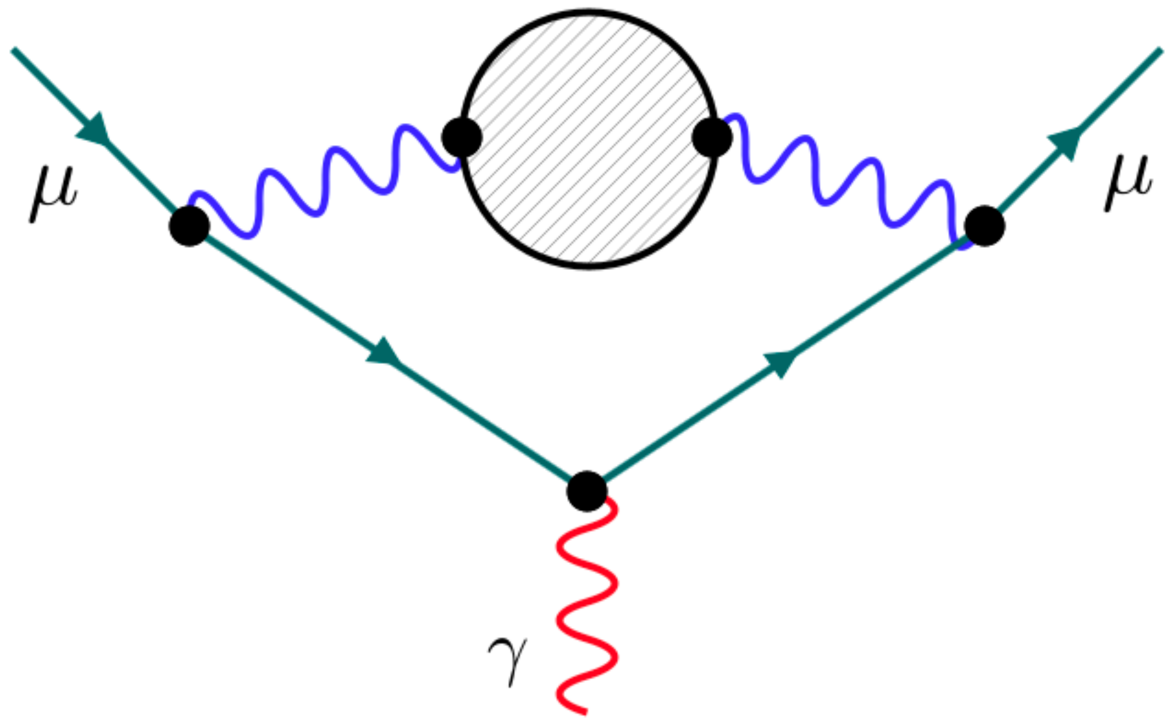
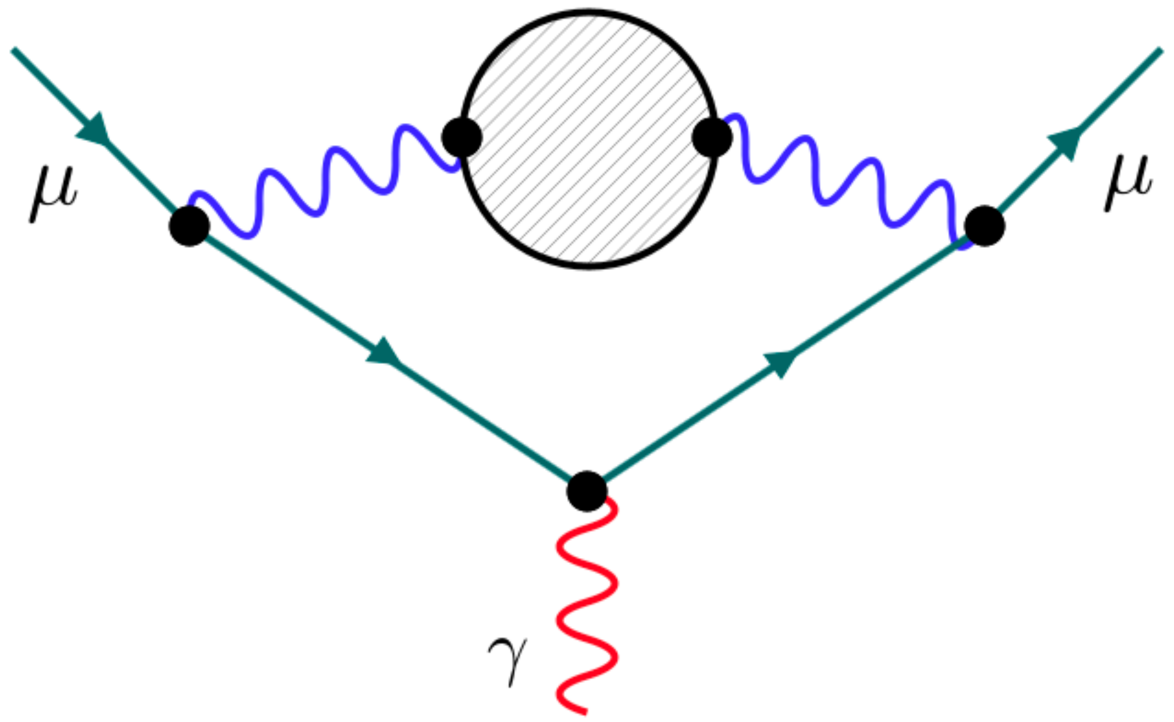


Ex. 2: Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T j(x) j(0)\rangle \to \sum_n C_n(Q)\langle \mathcal{O}_n\rangle$$

contribution to $(g-2)_\mu$



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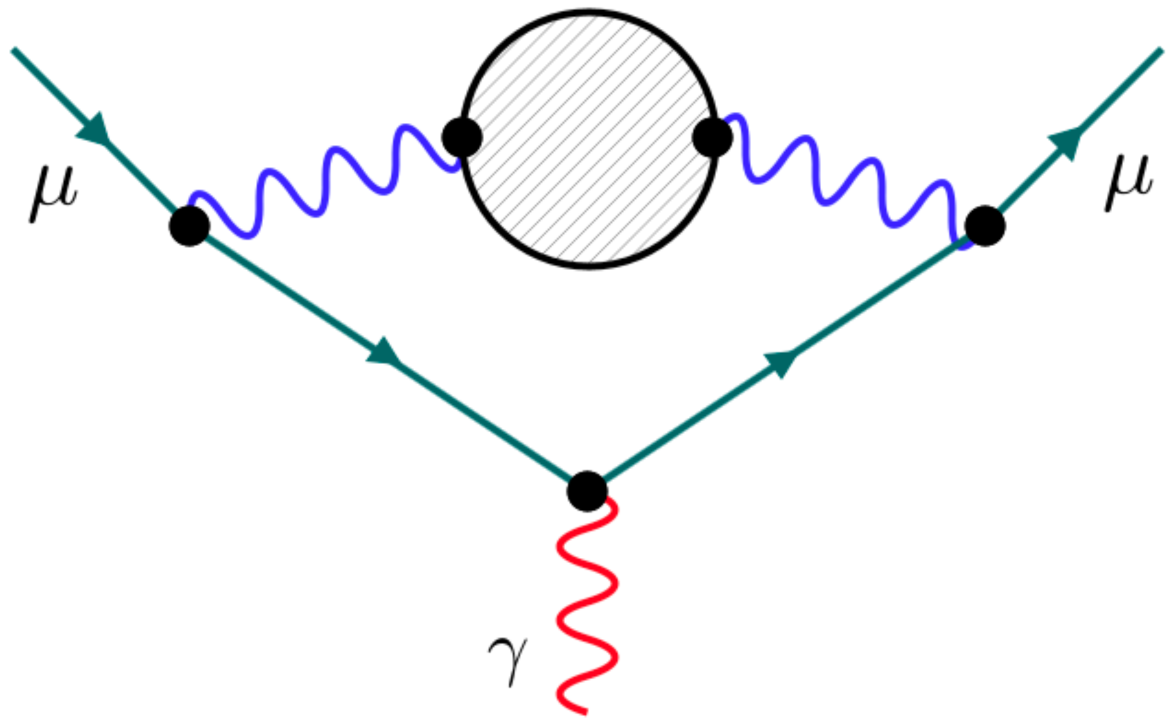
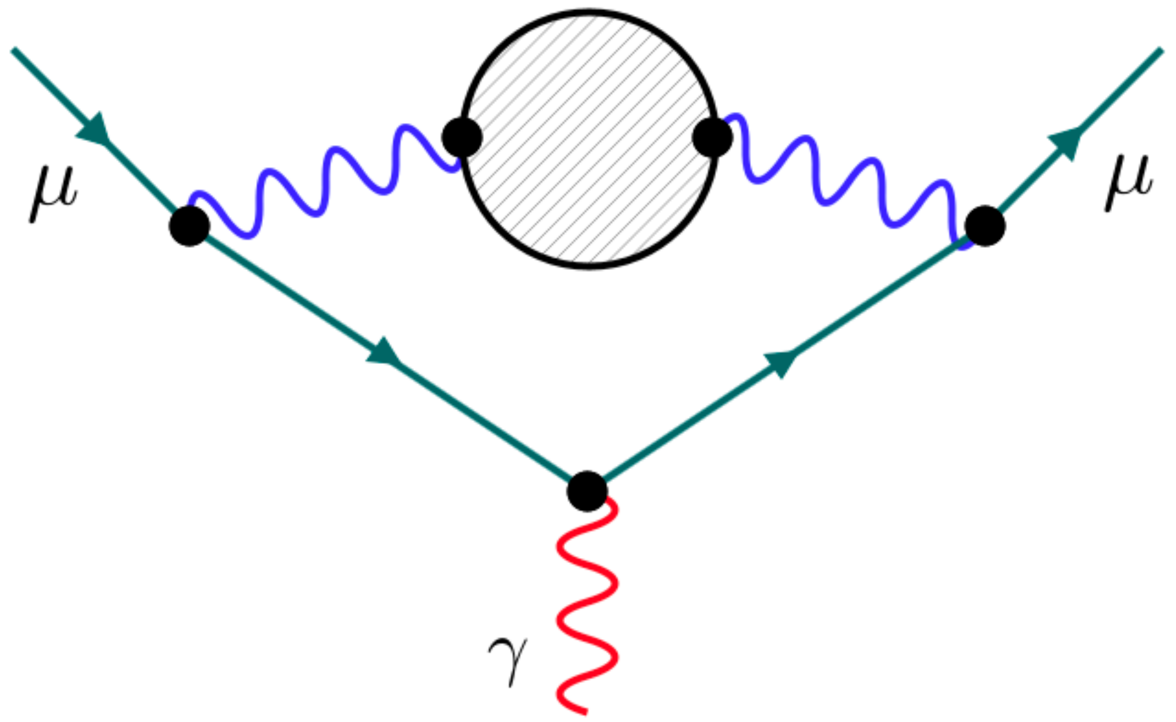


Known to high orders in perturbation theory

Ex. 2: Hadronic vacuum polarization

$$\int d^4x\, e^{iQx} \langle T j(x) j(0) \rangle \to \sum_n C_n(Q) \langle \mathcal{O}_n \rangle \;=\; \sum_n \tilde{C}_n(Q, t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

contribution to $(g-2)_\mu$



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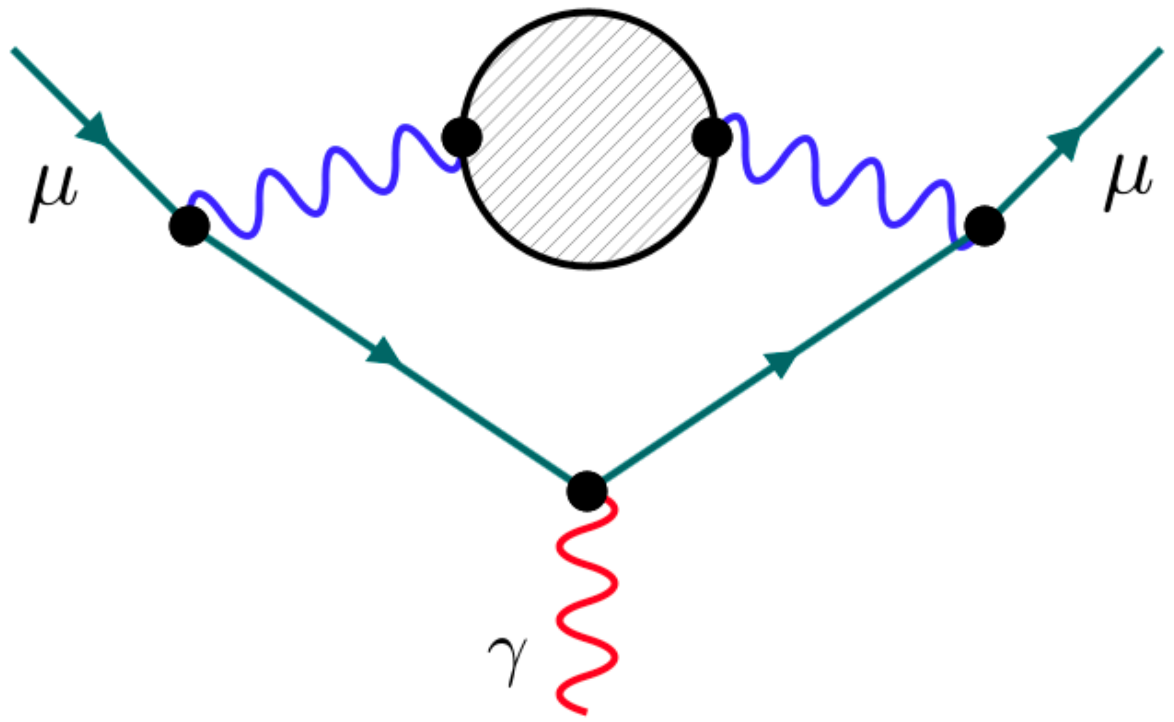

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$\mathcal{O}_1 = 1 \qquad \mathcal{O}_2 = m^2 \qquad \mathcal{O}_3 = m^4$

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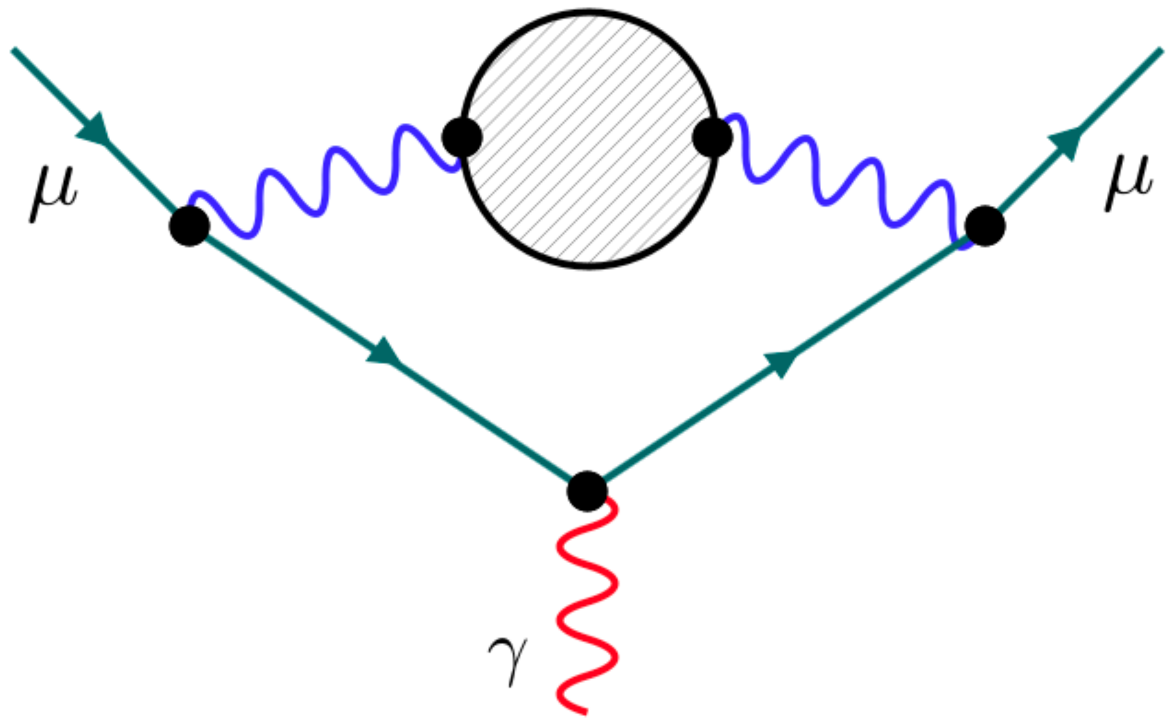

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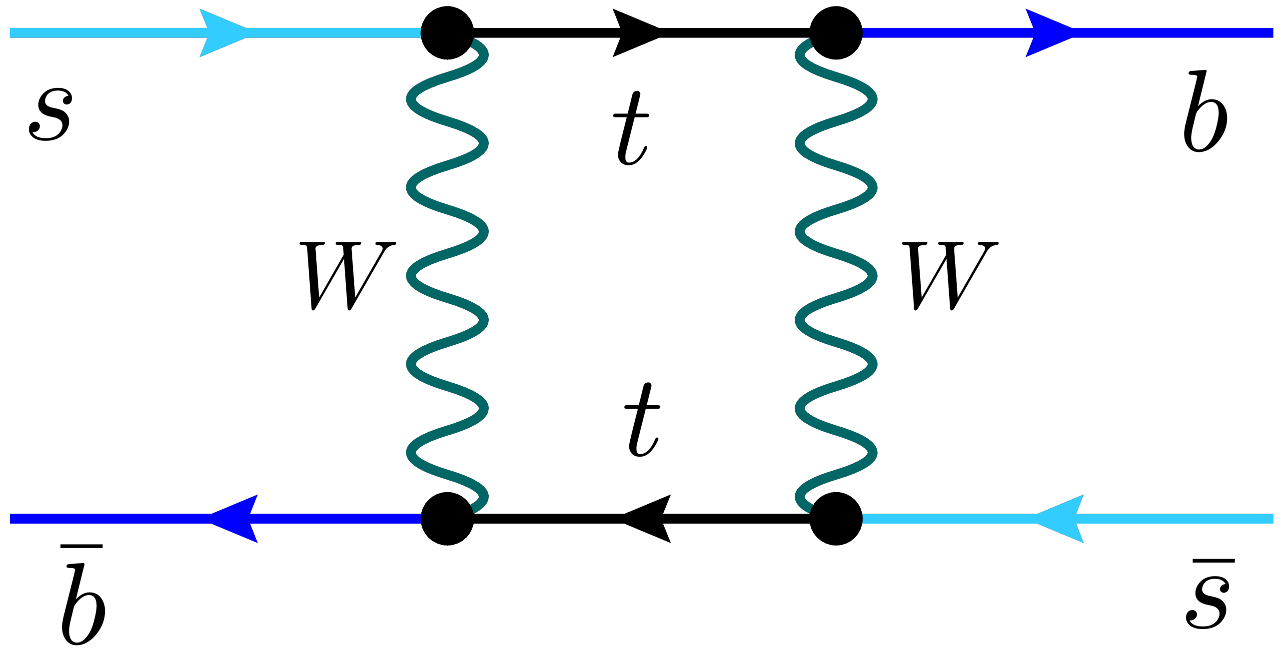
$\tilde{C}_n(Q,t)$ to NNLO

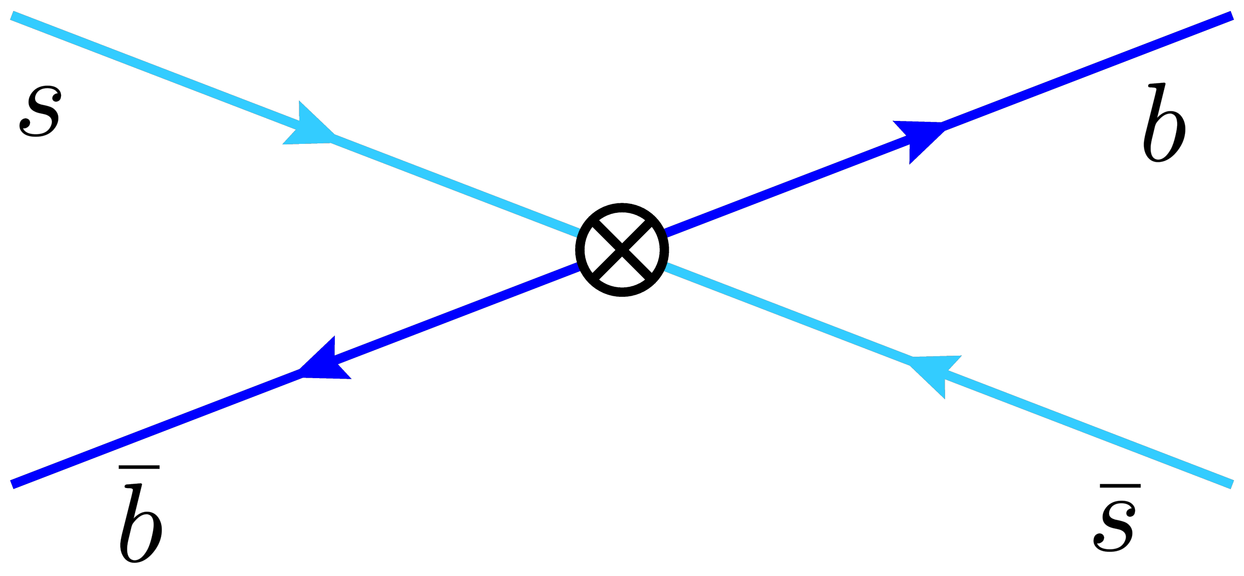
RH, Lange, Neumann ‘20

Flavor physics

meson mixing:



$$M_W, m_t \to \infty$$



$$\mathcal{O}_1 = (\bar{b}\gamma_L^\mu s)(\bar{b}\gamma_\mu^L s)$$

Flavor physics

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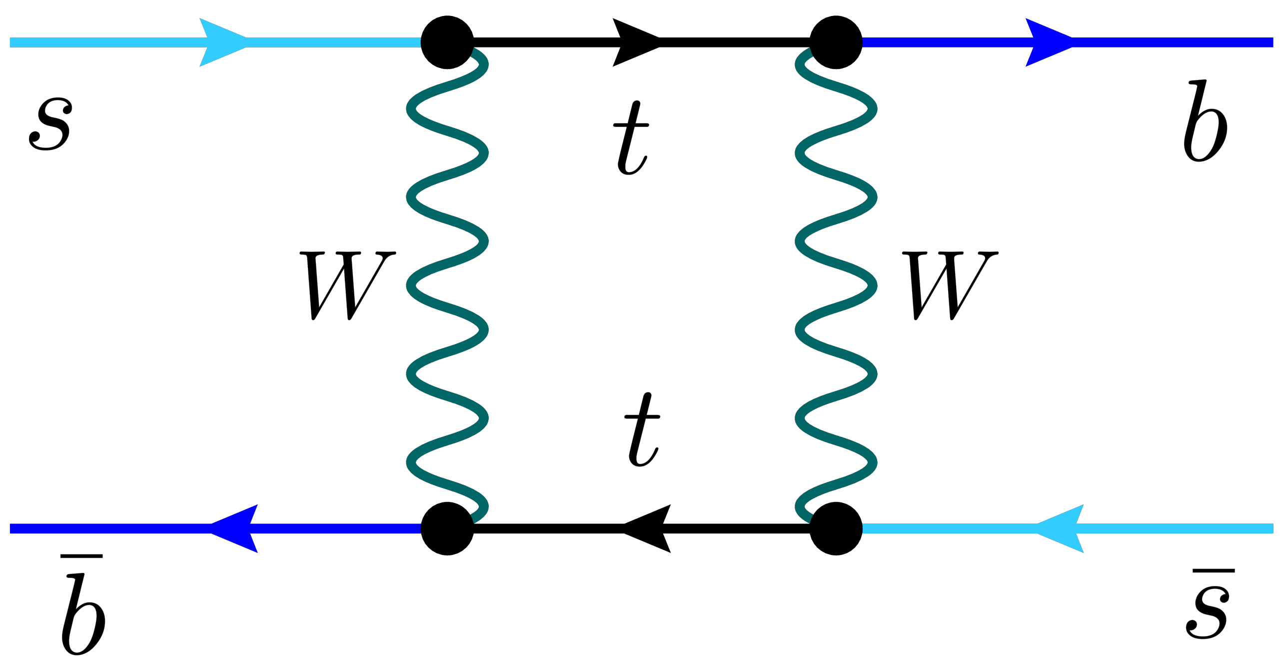

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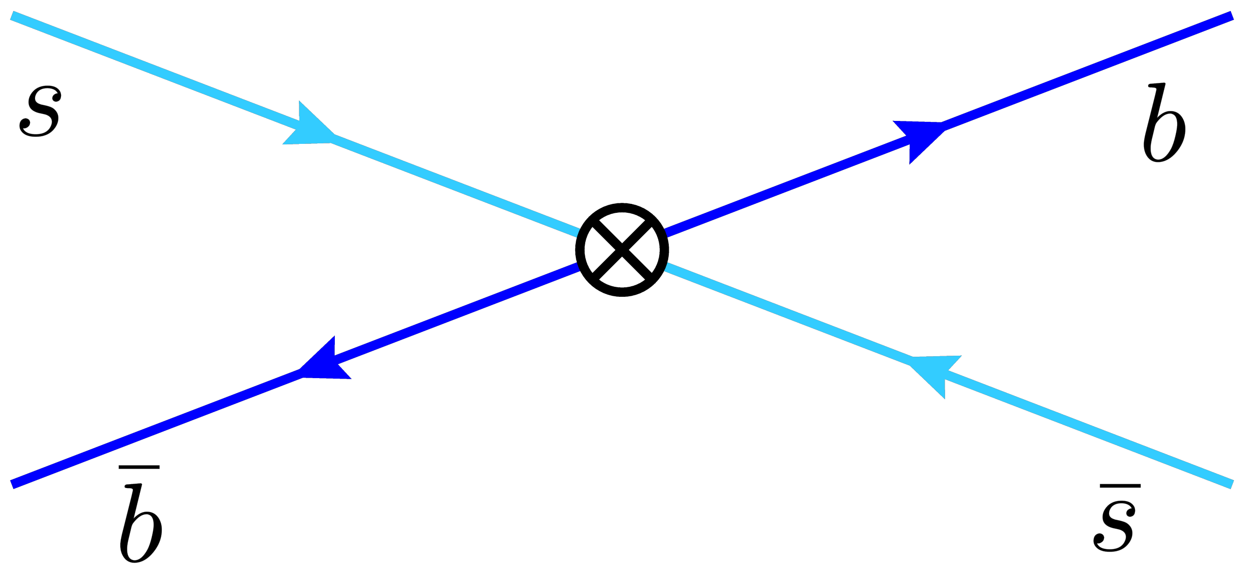

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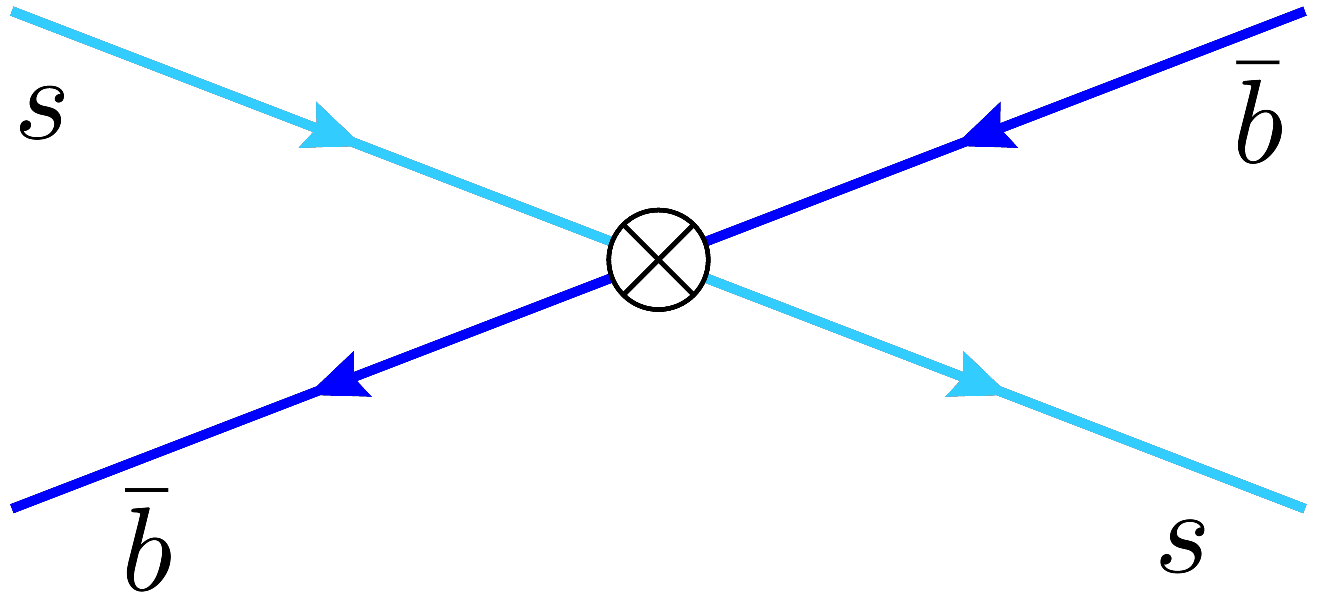
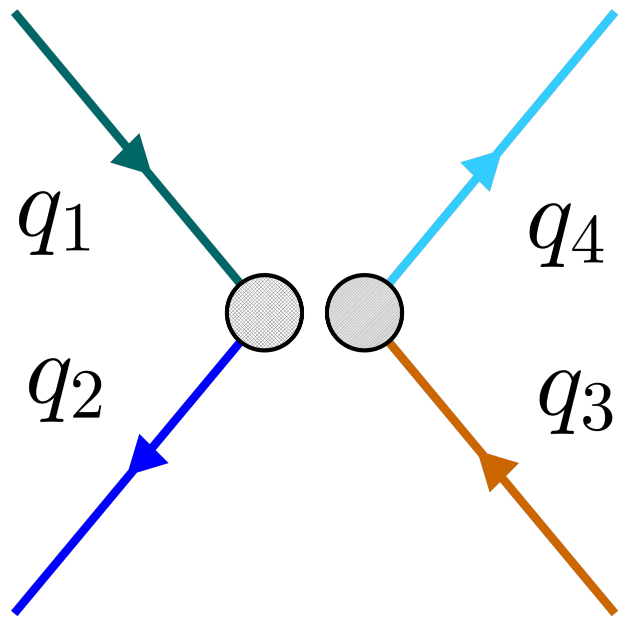
meson decay:



$$\mathcal{Q}_1 = (\bar{b}\gamma^\mu_L s)(\bar{s}\gamma^L_\mu b)$$

$$\mathcal{T}_1 = (\bar{b}\gamma^\mu_L T s)(\bar{s}\gamma^L_\mu T b)$$

Flavor physics



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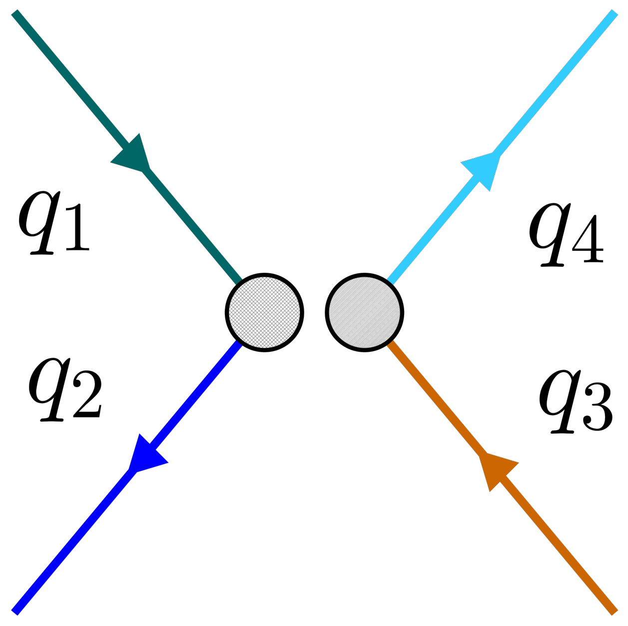
$$H = \sum_n C_n \mathcal{O}_n = \sum_n \tilde{C}_n(t) \tilde{\mathcal{O}}_n(t)$$
$$\tilde{\mathcal{O}}_n(t) = \sum_{nm} \zeta_{mk}(t) \mathcal{O}_k$$

Flavor physics



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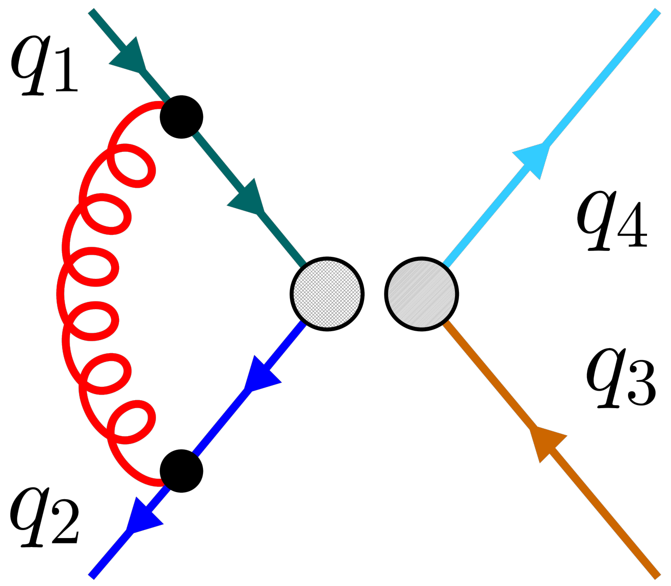
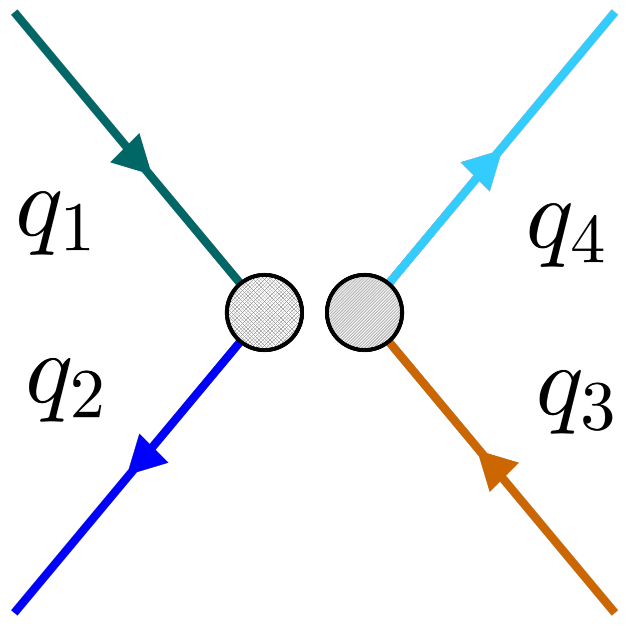
In $D = 4 - 2\epsilon$



Flavor physics



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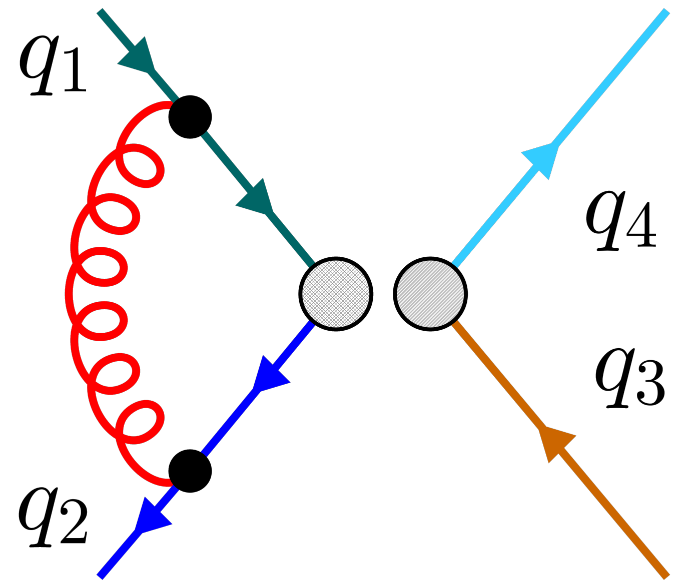
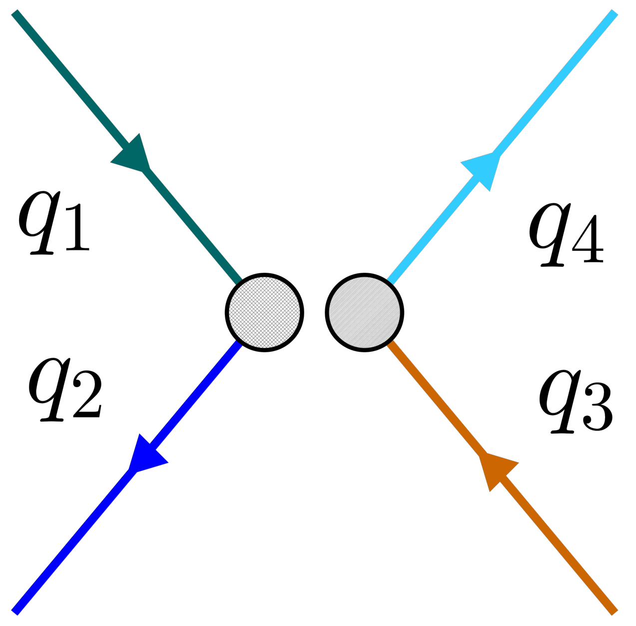


UV divergences $\sim \mathcal{O}_1\,,\ \mathcal{O}_2$

Flavor physics



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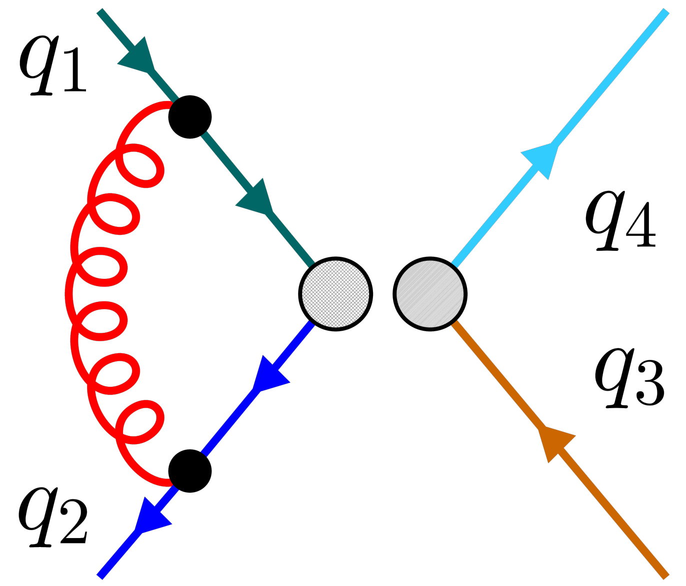
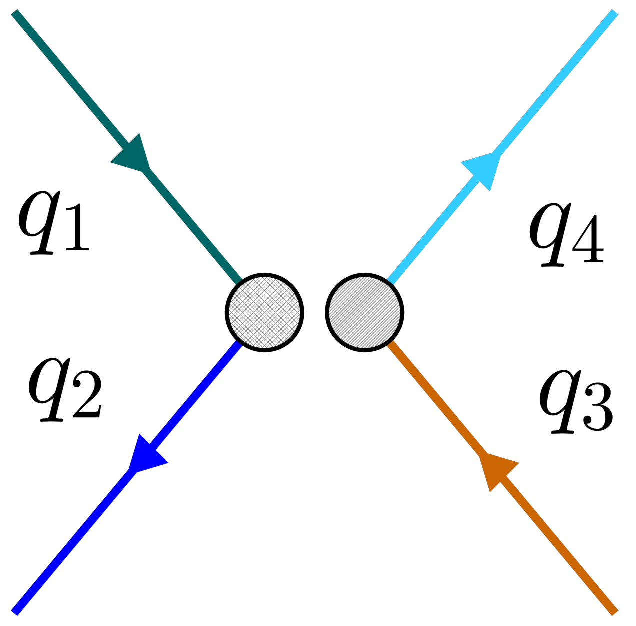


UV divergences $\sim \mathcal{O}_1, \mathcal{O}_2$

and $\sim \mathcal{E}_1^{(1)}, \mathcal{E}_2^{(1)}$

Flavor physics



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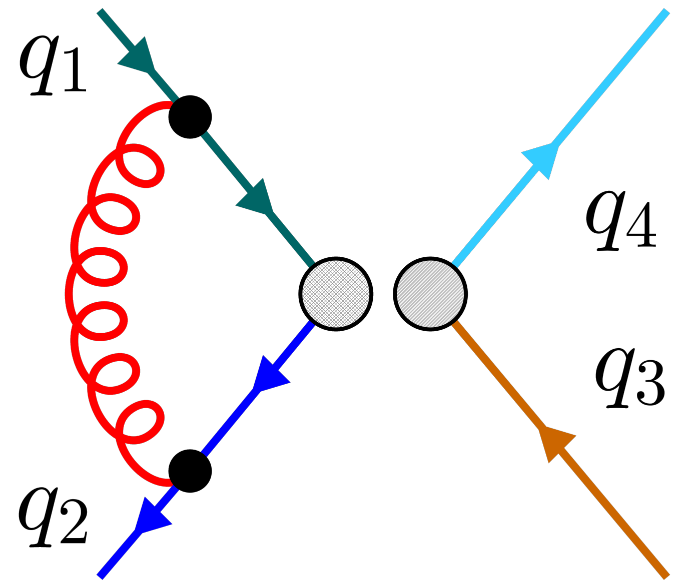

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$$\mathcal{E}_1^{(1)} = (\bar{q}_1\gamma_\mu\gamma_\rho\gamma_\sigma^L T q_2)(\bar{q}_3\gamma^\mu\gamma^\rho\gamma_L^\sigma T q_4) - 16\mathcal{O}_1$$
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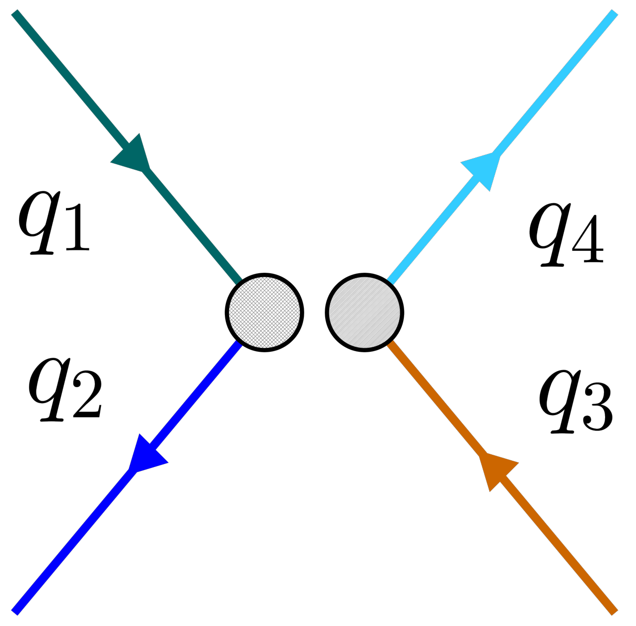
Flavor physics



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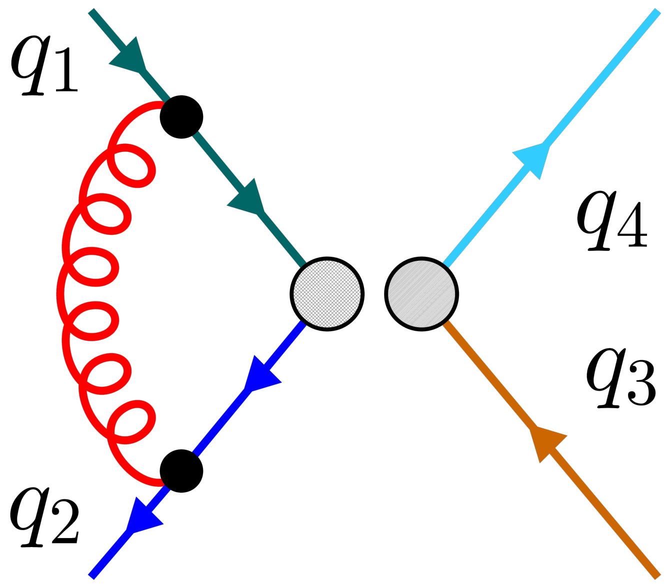

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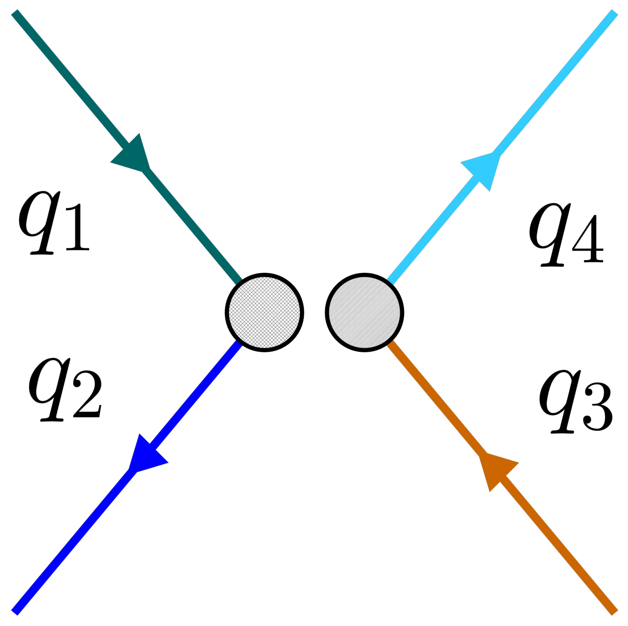
In $D = 4$: $\mathcal{E}_i^{(n)} = 0$ evanescent operators

Chetyrkin, Misiak, Münz ‘98

Flavor physics



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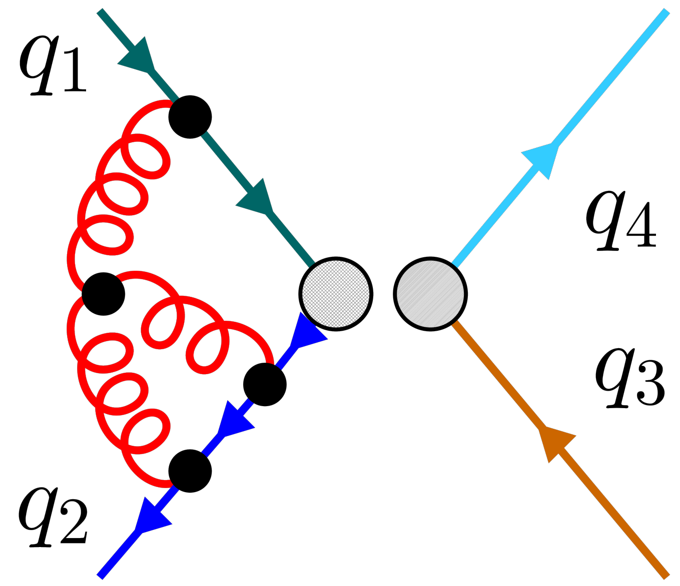

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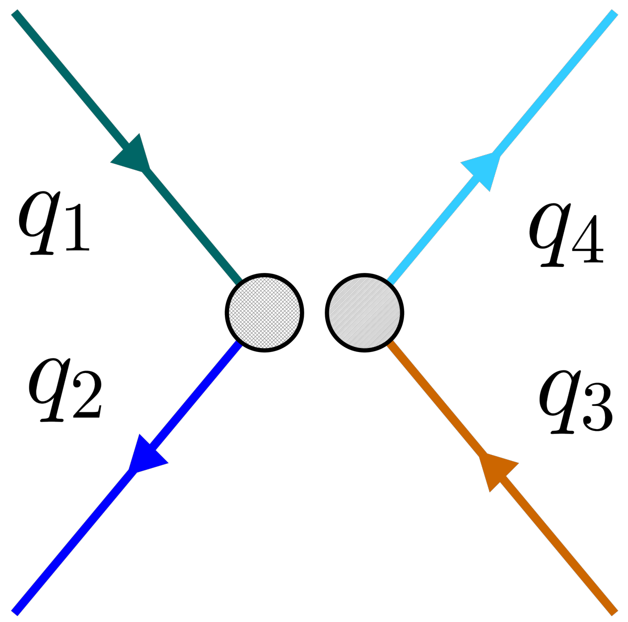
Chetyrkin, Misiak, Münz ‘98

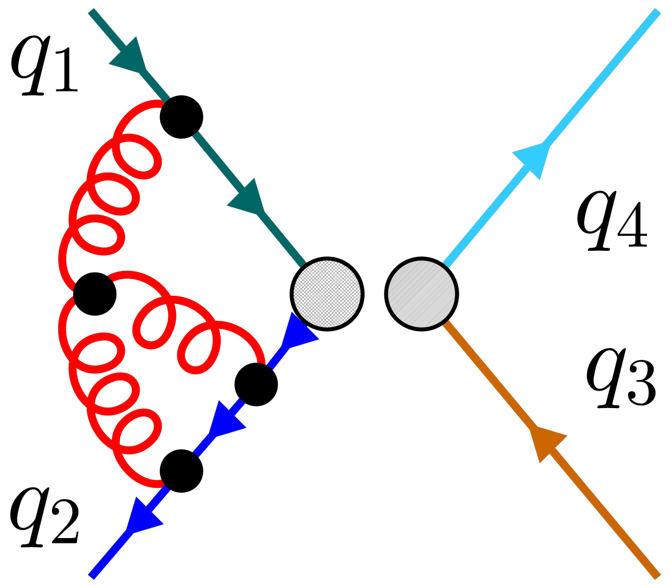
Flavor physics



$$\mathcal{O}_1 = (\bar{q}_1\gamma_\mu^L T q_2)(\bar{q}_3\gamma_L^\mu T q_4)$$
$$\mathcal{O}_2 = (\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_L^\mu q_4)$$

$$H = \sum_n C_n\mathcal{O}_n = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_n(t)$$
$$\tilde{\mathcal{O}}_n(t) = \sum_{nm} \zeta_{mk}(t)\mathcal{O}_k$$

In $D = 4 - 2\epsilon$



UV divergences $\sim \mathcal{O}_1\,,\ \mathcal{O}_2$

and $\sim \mathcal{E}_1^{(1)},\ \mathcal{E}_2^{(1)}$

and $\sim \mathcal{E}_1^{(2)},\ \mathcal{E}_2^{(2)}$

$$\mathcal{E}_1^{(1)} = (\bar{q}_1\gamma_\mu\gamma_\rho\gamma_\sigma^L T q_2)(\bar{q}_3\gamma^\mu\gamma^\rho\gamma_L^\sigma T q_4) - 16\mathcal{O}_1$$
$$\mathcal{E}_2^{(1)} = (\bar{q}_1\gamma_\mu\gamma_\rho\gamma_\sigma^L q_2)(\bar{q}_3\gamma^\mu\gamma^\rho\gamma_L^\sigma q_4) - 16\mathcal{O}_2$$

$$\mathcal{E}^{(2)} = (\bar{q}_1\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_L^\tau q_2)\cdots$$

In $D = 4$: $\mathcal{E}_i^{(n)} = 0$ evanescent operators

Chetyrkin, Misiak, Münz '98

Results

$$\zeta_{11}(t) = 1 + a_s\left[-\frac{1}{6} - \frac{8}{3}\ln 2 - 2\ln 3 - \frac{1}{2}L_{\mu t}\right] + a_s^2\left[-\frac{10669}{576} + \frac{1}{2}c_{\chi,\mathrm{A}} + \frac{2}{9}c_{\chi,\mathrm{F}}\right.$$
$$- \frac{181}{96}\zeta_2 - \frac{235}{8}\ln 2 + \frac{1537}{48}\ln 3 + \frac{14}{3}\ln^2 2 + \frac{8}{3}\ln 2\ln 3 + \ln^2 3$$
$$+ \frac{829}{48}\mathrm{Li}_2(1/4) + n_\mathrm{f}\left(\frac{305}{432} + \frac{1}{12}c_{\chi,\mathrm{R}} + \frac{5}{24}\zeta_2\right) + L_{\mu t}\Bigg(-\frac{61}{96} - 6\ln 2$$
$$\left.- \frac{9}{2}\ln 3 + n_\mathrm{f}\left(\frac{1}{12} + \frac{4}{9}\ln 2 + \frac{1}{3}\ln 3\right)\Bigg) + L_{\mu t}^2\left(-\frac{5}{16} + \frac{1}{24}n_\mathrm{f}\right)\right],$$

$$\zeta_{12}(t) = a_s\left(\frac{5}{6} + \frac{1}{3}L_{\mu t}\right) + a_s^2\Bigg(\frac{773}{432} + \frac{67}{432}\zeta_2 + \frac{163}{108}\ln 2 - \frac{115}{24}\ln 3 + \frac{3}{8}\mathrm{Li}_2(1/4)$$
$$+ n_\mathrm{f}\left(-\frac{145}{1296} - \frac{1}{36}\zeta_2\right) + L_{\mu t}\left(\frac{205}{144} - \frac{8}{9}\ln 2 - \frac{2}{3}\ln 3 - \frac{5}{54}n_\mathrm{f}\right) + L_{\mu t}^2\Big(\frac{3}{8}$$
$$- \frac{1}{36}n_\mathrm{f}\Big)\Bigg),$$

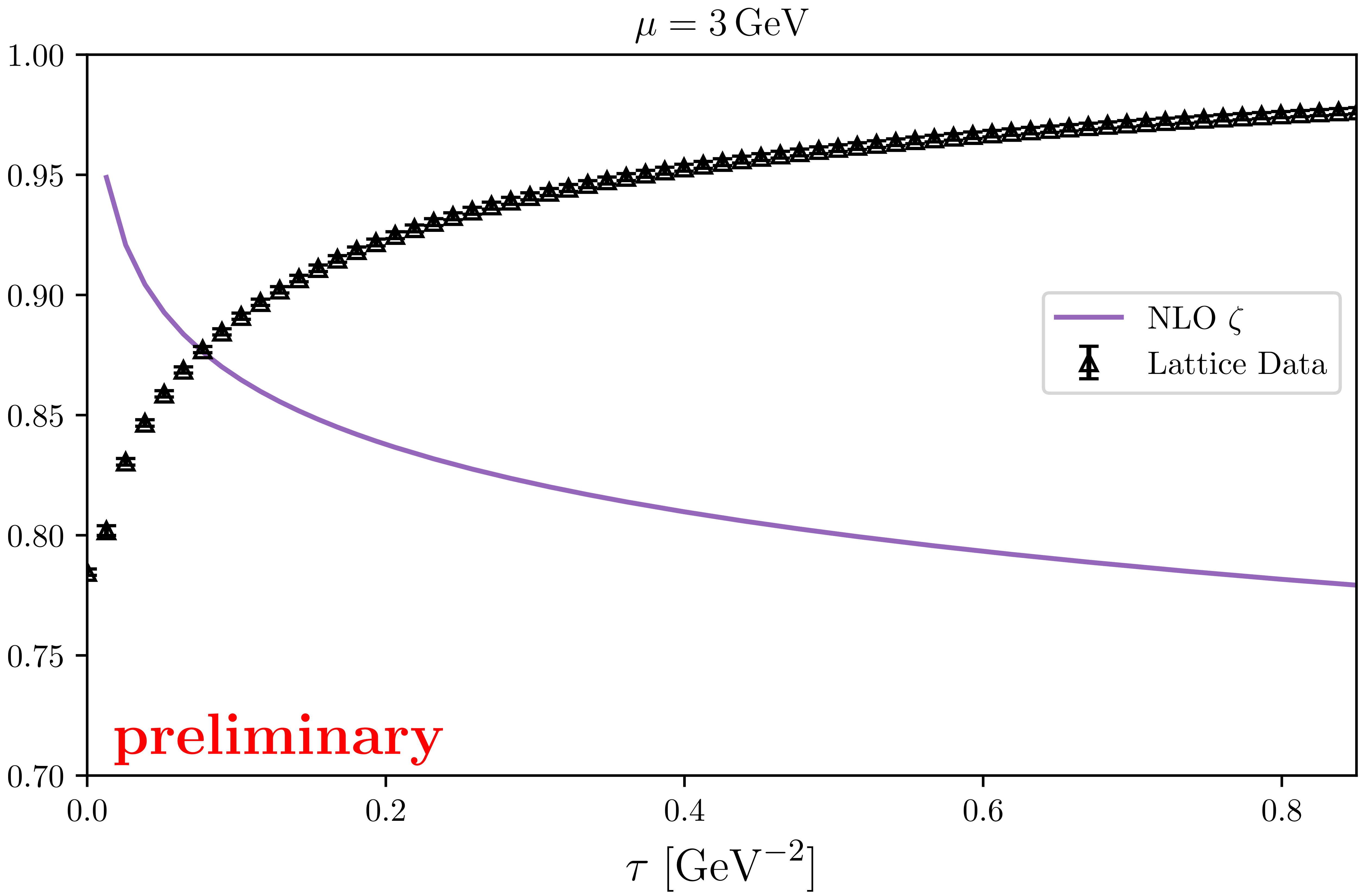
$$\zeta_{21}(t) = a_s\left(\frac{15}{4} + \frac{3}{2}L_{\mu t}\right) + a_s^2\Bigg(\frac{1043}{96} + \frac{67}{96}\zeta_2 + \frac{163}{24}\ln 2 - \frac{345}{16}\ln 3$$
$$+ \frac{27}{16}\mathrm{Li}_2(1/4) + n_\mathrm{f}\left(-\frac{145}{288} - \frac{1}{8}\zeta_2\right) + L_{\mu t}\left(\frac{241}{32} - 4\ln 2 - 3\ln 3 - \frac{5}{12}n_\mathrm{f}\right)$$

RH, Lange 2022

Flavor physics

first studies: $\zeta^{-1}(t)\,\langle D_s\,|\,\tilde{\mathcal{O}}_1(t)\,|\,\bar{D}_s\rangle$ Black, RH, Lange, Rago, Shindler, Witzel (2023)

Collaborative Research Center TRR 257
Particle Physics after the Higgs Discovery



Flavor physics

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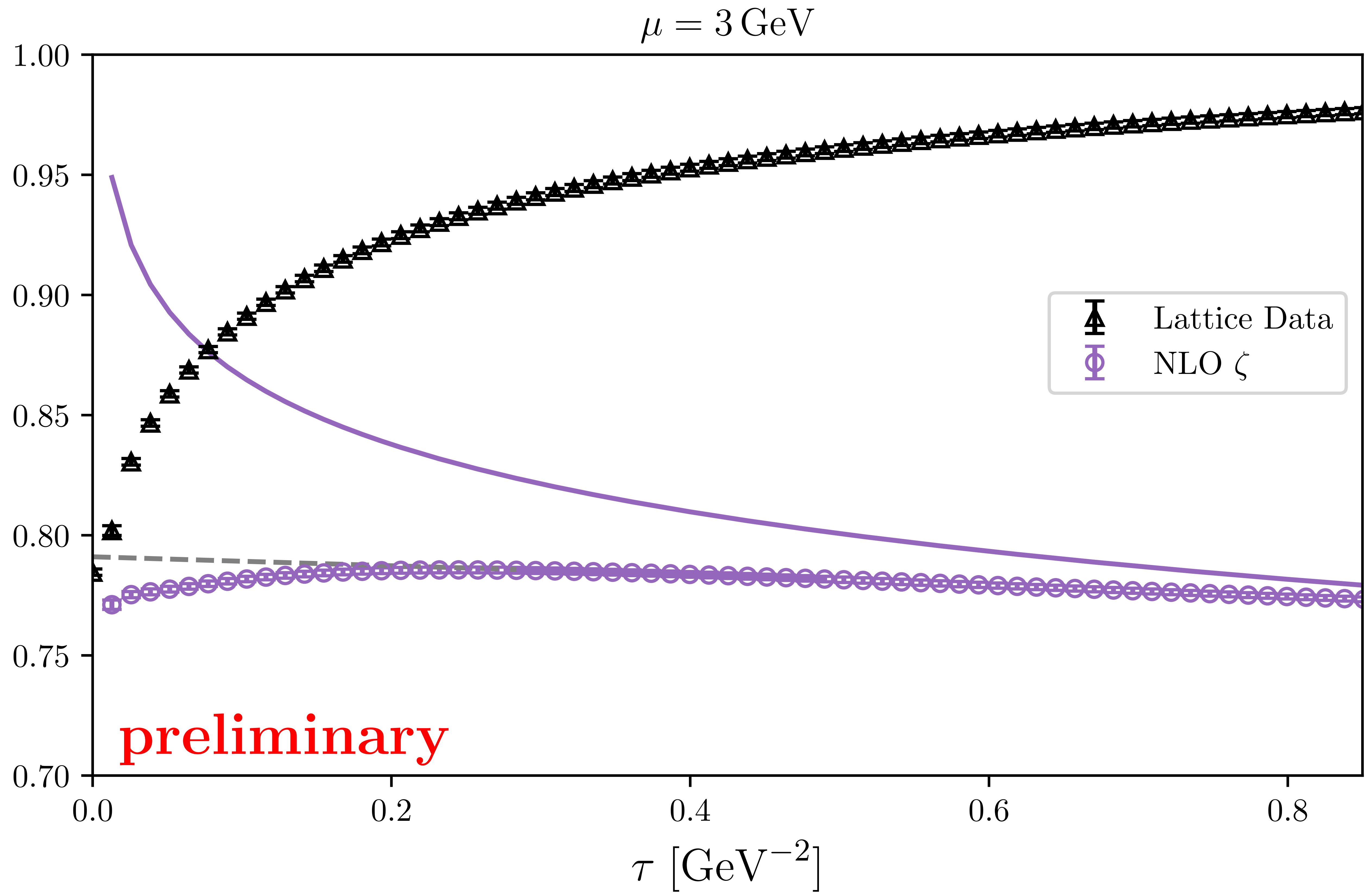

Flavor physics

first studies: $\zeta^{-1}(t) \langle D_s | \tilde{\mathcal{O}}_1(t) | \bar{D}_s \rangle$ Black, RH, Lange, Rago, Shindler, Witzel (2023)

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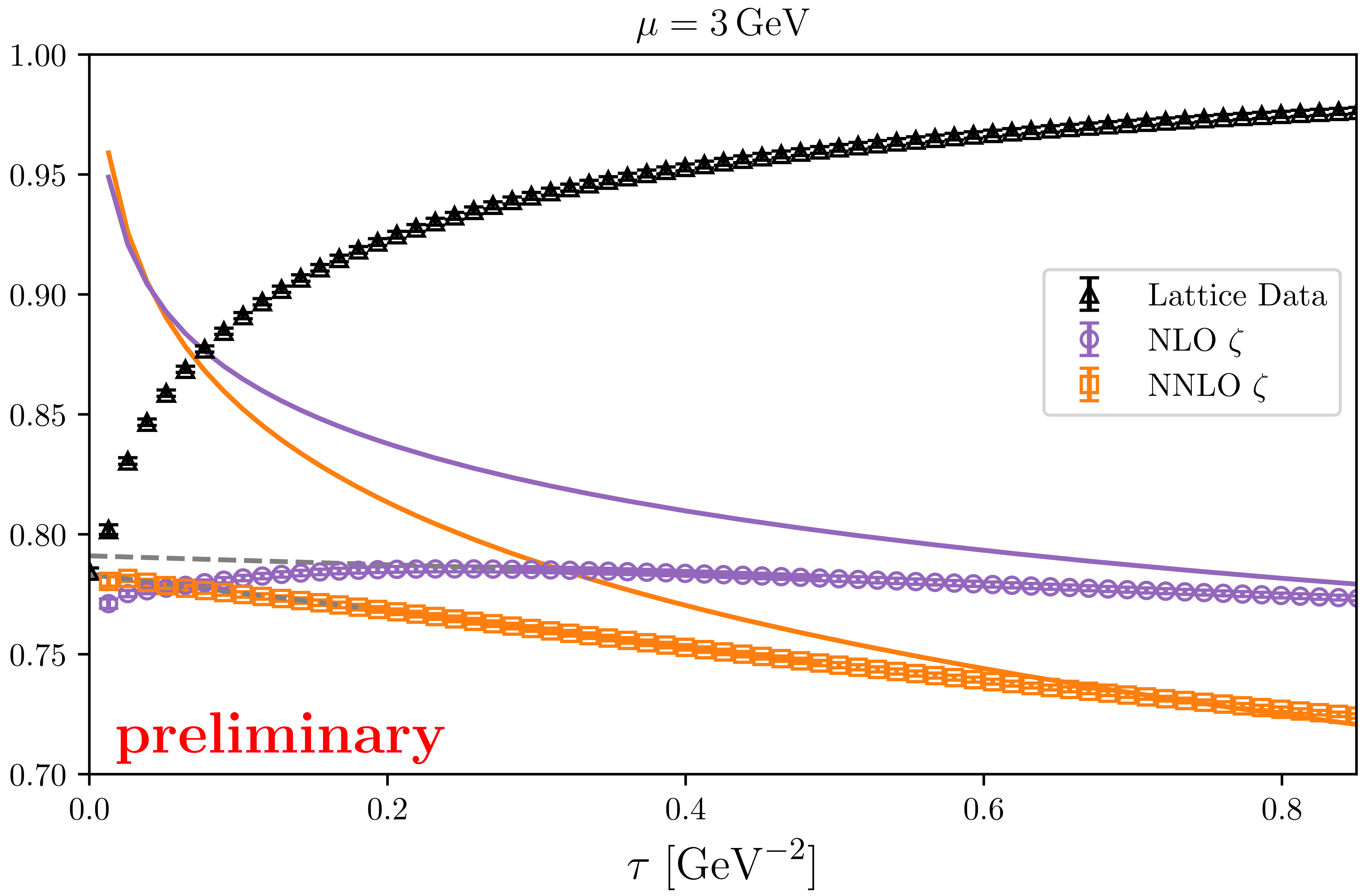

Flow-time evolution

Consider effective Hamiltonian in MSbar:

$$H = \sum_n C_n(\mu, M_W)\, \langle \mathcal{O}_n \rangle(\mu, \Lambda_{\mathrm{QCD}})$$

Resum logarithms through RGE: $$\mu^2 \frac{d}{d\mu^2} \mathcal{O} = \gamma\, \mathcal{O}$$

Flow-time evolution

Consider effective Hamiltonian in MSbar:

$$H = \sum_n C_n(\mu, M_W)\, \langle \mathcal{O}_n \rangle(\mu, \Lambda_{\rm QCD})$$

Resum logarithms through RGE:

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O} = \gamma\, \mathcal{O}$$

here:

$$H = \sum_n \tilde{C}_n(t, M_W)\, \langle \tilde{\mathcal{O}}_n \rangle(t, \Lambda_{\rm QCD})$$

Flow-time evolution

$t \to 0:$ $$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m$$

matrix notation: $$\tilde{\mathcal{O}}(t) = \zeta(t)\,\mathcal{O}$$

$$t\frac{\partial}{\partial t}\tilde{\mathcal{O}}(t) = \left(t\frac{\partial}{\partial t}\zeta(t)\right)\mathcal{O} = \left(t\frac{\partial}{\partial t}\zeta(t)\right)\zeta^{-1}(t)\,\tilde{\mathcal{O}}(t)$$

$$t\frac{\partial}{\partial t}\tilde{\mathcal{O}}(t) = \tilde{\gamma}(t)\tilde{\mathcal{O}}(t) \quad \text{with} \quad \tilde{\gamma}(t) = \left(t\frac{\partial}{\partial t}\zeta(t)\right)\zeta^{-1}(t)$$

RH, Lange, Neumann ‘20

Conclusions and Outlook

- Gradient flow provides ideal basis for combining lattice and perturbation theory
- Many perturbative tools can be adapted
- Several proofs of principle already available
- Full potential still to be explored

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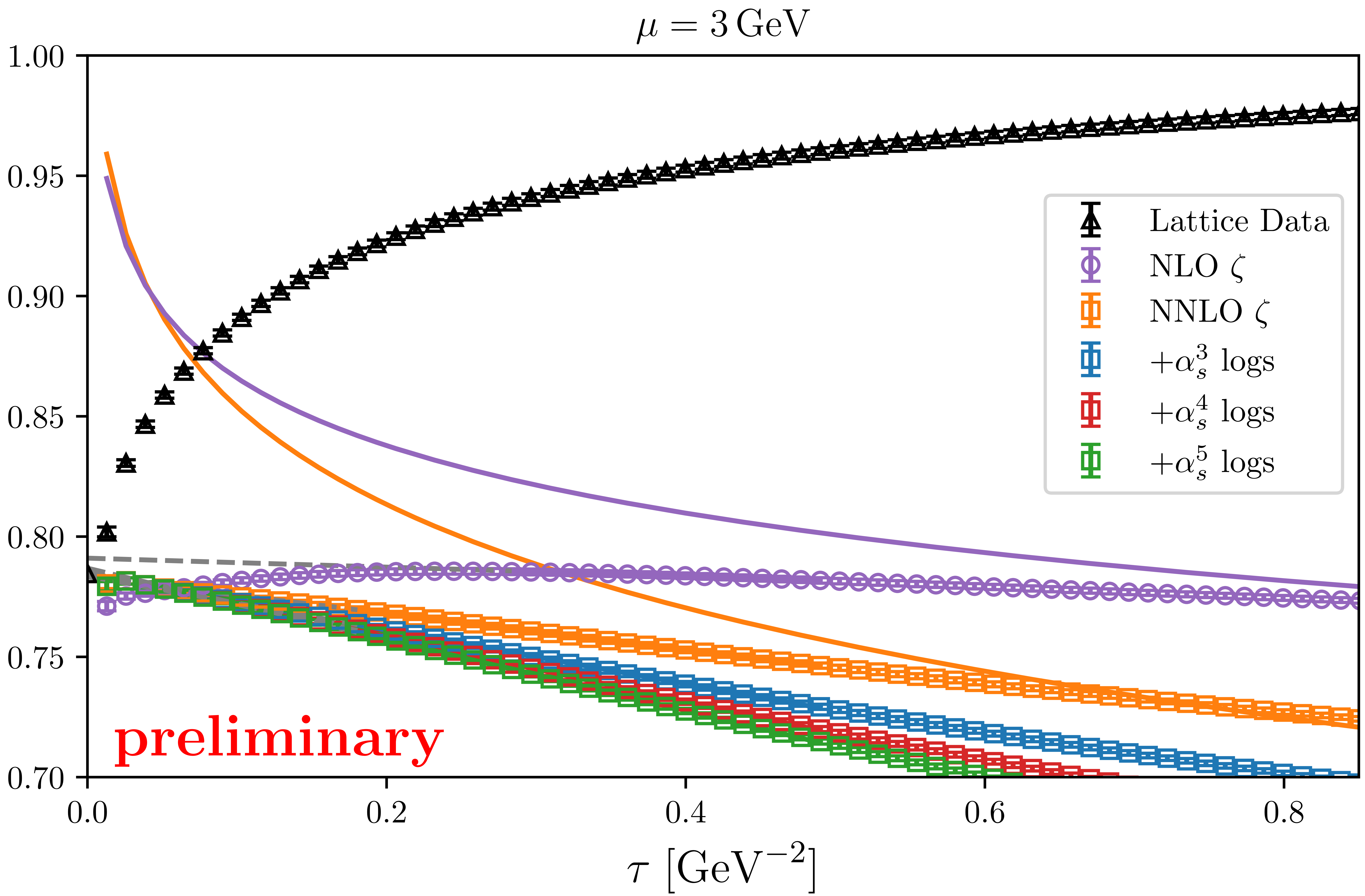

$\mu = 3\,\mathrm{GeV}$
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.0
0.2
0.4
0.6
0.8
$\tau\ [\mathrm{GeV}^{-2}]$
Lattice Data
NLO ζ
NNLO ζ
$+\alpha_s^3$ logs
$+\alpha_s^4$ logs
$+\alpha_s^5$ logs
preliminary